SCHOOL MATHEMATICS 2

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School Mathematics 2

1974 Edition

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TO THE STUDENT

Welcome to another year of mathematics!

This mathematics book was written in a manner that may be different from other mathematics that you have used. You will have the chance to take an active part in learning mathematics. You will be given the opportunity to carry out interesting investigations of mathematical ideas. You will discuss your results with your classmates. You will solve problems and develop your skills in mathematics. You will find interesting mathematical recreations, challenging Think problems, and special projects.

While it is important for you to develop computational skills in mathematics, it is also important for you to know why various rules or procedures work and when to use them. Therefore, you will need to study carefully, work out examples for yourself, and ask questions when you do not understand.

We hope that you find your study of mathematics this year interesting and rewarding.

Good Luck

The Authors

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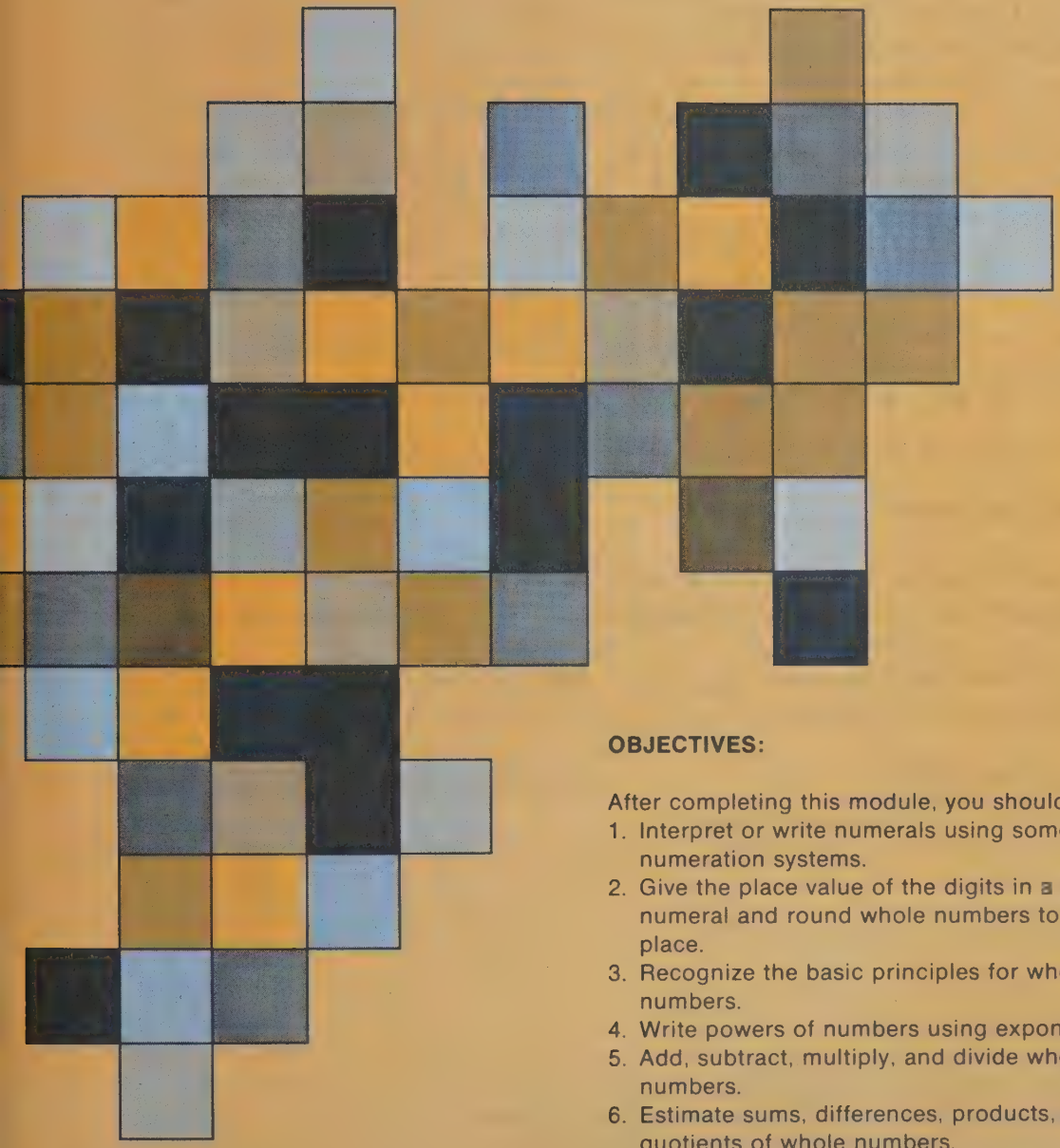
UNIT E: *Integers*

MODULE 1: *Numerals, Numbers, and Operations*



OBJECTIVES:

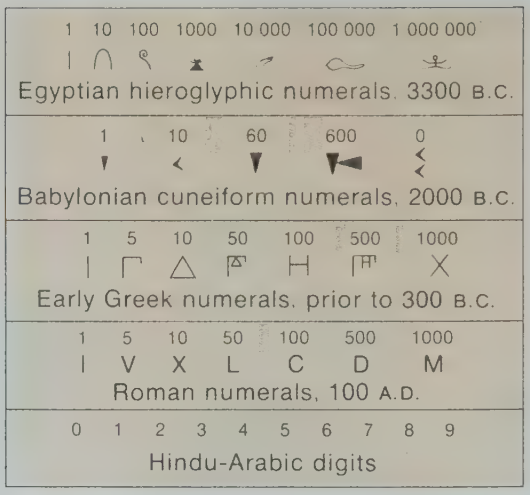
After completing this module, you should be able to:

1. Interpret or write numerals using some ancient numeration systems.
2. Give the place value of the digits in a decimal numeral and round whole numbers to a given place.
3. Recognize the basic principles for whole numbers.
4. Write powers of numbers using exponents.
5. Add, subtract, multiply, and divide whole numbers.
6. Estimate sums, differences, products, and quotients of whole numbers.
7. Write a base-two numeral for a base-ten numeral or convert a base-two numeral to a base-ten numeral.
8. Perform calculations using base-two numerals.

Ancient Numeration Systems

Investigating the Ideas

Symbols for numbers are called **numerals**. Some ancient numerals as well as the present-day Hindu-Arabic numerals are given in the table at the right.


1 10 100 1000 10 000 100 000 1 000 000
Egyptian hieroglyphic numerals, 3300 B.C.
1 10 60 600 0
Babylonian cuneiform numerals, 2000 B.C.
1 5 10 50 100 500 1000
Early Greek numerals, prior to 300 B.C.
1 5 10 50 100 500 1000
I V X L C D M
Roman numerals, 100 A.D.
0 1 2 3 4 5 6 7 8 9
Hindu-Arabic digits

How many of these questions can you answer by writing the numeral indicated?

A What is your age in years? (Egyptian)

B What is your height in centimetres? (Babylonian)

C What is the number of people in your classroom? (Greek)

D What is the year of your birth? (Roman)

Discussing the Ideas

1. The Egyptian numeration system was an **additive system**. The numbers for the symbols were simply added together. For 23, they might have written III∩∩ or ∩∩III or ∩∩ . What are some other numerals you might have written for your age for question **A** above?

2. The Babylonians wrote in clay with a **stylus**. The figure shows how they might have represented 172. (For the numbers less than 60, the values of the numerals were added.) What Babylonian symbol would represent 215?

3. The Greek system did not use place value either. Like the Egyptian system, it was an additive system. What number would HH△△△ΓII represent?

4. The early Roman system was also an additive system. Later it also had some **subtractive** phases. Thus nine, once written as VIIII, became IX. What is the Roman numeral for the present year?

1. Write the Hindu-Arabic numeral for each Egyptian numeral.

A

B

C

D

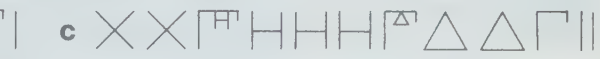
E

F

2. Write the Hindu-Arabic numeral for each Babylonian numeral.

A **B** **C** **D**

3. Write the Hindu-Arabic numeral for each Greek numeral.

A **B** **C** 

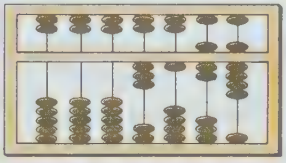
4. Write the Hindu-Arabic numeral for each Roman numeral.

A V	**D** X	**G** LX	**J** CD	**M** MCLIX
B VI	**E** XI	**H** XL	**K** MC	**N** CLXXVIII
C IV	**F** IX	**I** DC	**L** CM	**O** MCMLXX

5. Write the numeral 247 using

A Egyptian hieroglyphics. **B** Greek numerals. **C** Roman numerals.

6. The Romans often used an **abacus** to calculate and then recorded their answers using numerals. The answer to a problem is shown on the abacus. Record the answer using Roman numerals.

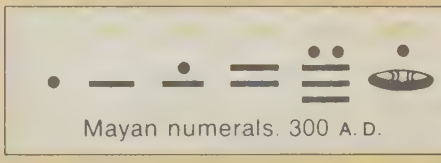


think

The Mayan Indians of Yucatan used a system based on 20 and wrote their numerals vertically.

The diagram at the right shows how the Mayans utilized place value. Because they were interested in astronomy and used 360 for the number of days in a year, they used 18 · 20 in the third place instead of 20 · 20. The symbol was used as a zero to show an empty place.

This Mayan numeral represents a famous historical date. What is it?



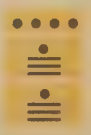
Mayan numerals. 300 A.D.

$\cdot 18 \cdot 20 \cdot 20 = 7200$

$\cdot 18 \cdot 20 = 2160$

$\cdot 20 = 60$

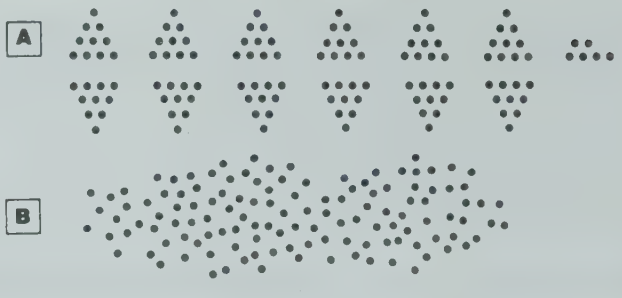
$= 12$



Investigating the Ideas

We use **whole numbers** {0, 1, 2, 3, 4, 5, 6, . . .} when we count. We use Hindu-Arabic digits to record our counting. Counting is the most basic aspect of mathematics.

Can you count and record the number of dots that are in set A and in set B?

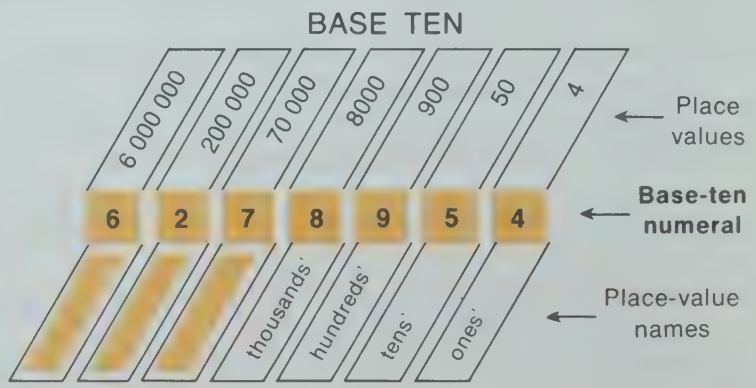


Discussing the Ideas

1. A How many dots are in set A?
   B Why are they easy to count?

2. A How many dots did you find in set B?
   B Can you describe a way to make the counting easier?

3. Our system of writing numerals to represent numbers is called the **decimal system**. The word decimal comes from the Latin word **decem** which means **ten**. By using ten as a base, the idea of **place value**, and the ten Hindu-Arabic digits, we can write numerals for every whole number.



   A Give the names of the next three places beyond the thousands' place in the base-ten diagram above.
   B In the numeral in the diagram, the three digits 2, 7, and 8 tell how many thousands and thus make up the thousands' **period**. Give the names of as many periods beyond the thousands' period as you can.
   C Read the base-ten numeral shown in the diagram.

1. Give the missing number for each ▮.

A In 6▮87, the 2 means 200.

B In 72▮64, the 9 means ▮.

C In 8▮760, the 4 means ▮.

D In ▮84 219, the 6 means ▮.

E In 6▮728 134, the 5 means ▮.

2. Write a base-ten numeral for each sum.

A $800 + 70 + 4$

B $3000 + 600 + 90 + 1$

C $20\,000 + 8000 + 400 + 30 + 7$

D $100\,000 + 100 + 1$

E $60\,000 + 7000 + 80$

F $3\,000\,000 + 700\,000$

3. The names for the first 12 periods are given in the table.

PERIOD	NAME	PERIOD	NAME	PERIOD	NAME
1st	units, tens, hundreds	5th	trillions	9th	septillions
2nd	thousands	6th	quadrillions	10th	octillions
3rd	millions	7th	quintillions	11th	nonillions
4th	billions	8th	sextillions	12th	decillions

Use the information in the table to write the Hindu-Arabic numeral for each number name.

EXAMPLE: 3 trillion ANSWER: 3 000 000 000 000

A 6 quadrillion

B 73 billion

C 276 septillion

D 1 decillion

E 439 octillion

F 89 million

4. The value of the gross national product in the United States for a certain year was one **trillion dollars**. Write a base-ten numeral for the amount.

think

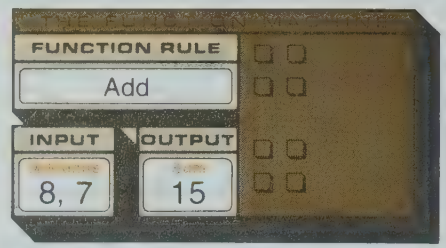
Suppose that in a certain newspaper the total number of digits used to number the pages of the newspaper was 111. How many pages were in the newspaper?

Investigating the Ideas

The **function machines** will help you to review the operations of addition and multiplication for whole numbers.

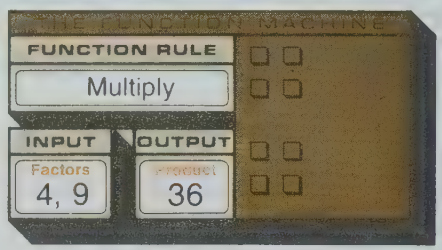
For each input pair (addends), this **addition** function machine produces a unique output number (sum).

$8 + 7 = 15$



For each input pair (factors), this **multiplication** function machine produces a unique output number (product).

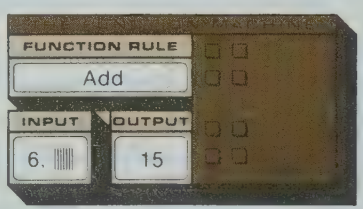
$4 \cdot 9 = 36$



How many different pairs of addends can you find to have a sum of 15? How many different pairs of factors can you find to have a product of 36?

Discussing the Ideas

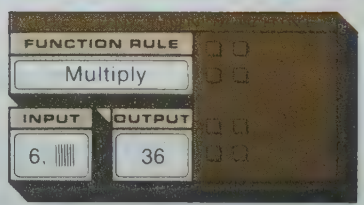
1. What is the missing addend for this function machine?



2. You can think of **subtraction** as an operation in which you know the sum (S) and one addend (A) and find the other addend. What is the missing addend n?

$15 - 6 = n$; $? + 6 = 15$

3. What is the missing factor for this function machine?



4. You can think of **division** as an operation in which you know the product (P) and one factor (F) and find the other factor (F). What is the missing factor n?

$36 \div 6 = n$; $? \cdot 6 = 36$

1. These function tables show a way of recording input and output numbers for function machines. Complete the tables.

Function Rule: Add

	input	output
A	(8, 6)	
B	(9, 4)	
C	(19, 6)	
D	(50, 30)	
E	(99, 101)	
F	(324, 976)	

Function Rule: Add

	input	output
G	(, 5)	13
H	(7,)	12
I	(, 16)	23
J	(14,)	14
K	(, 48)	56
L	(186,)	324

Function Rule: Multiply

	input	output
M	(5, 7)	
N	(9, 8)	
O	(6, 0)	
P	(9,)	9
Q	(, 40)	120
R	(47,)	376

2. Think about finding the missing addend or factor. Then solve each subtraction and division equation.

A $? + 7 = 16$
$16 - 7 = n$

B $? \cdot 9 = 63$
$63 \div 9 = r$

C $? + 29 = 32$
$32 - 29 = c$

D $270 \div 30 = t$

E $105 - 98 = d$

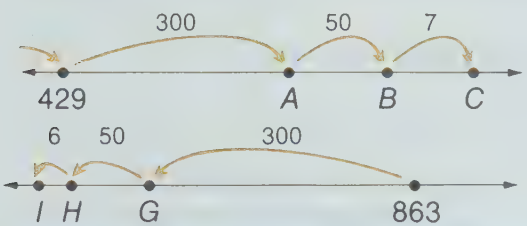
F $3600 \div 40 = w$

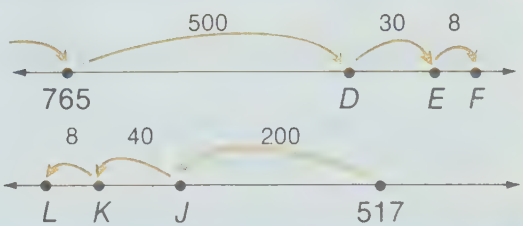
G $1000 - 498 = x$

H $10^3 \div 10^2 = s$

I $2268 \div 36 = y$

3. Give the number for the point above each letter.





Think

Which of these figures can be traced without lifting your pencil from the paper or retracing?

A B C D E F

There is a way to answer this question without tracing. Can you discover it?

A Review of Basic Principles

The basic principles for addition and multiplication can be used to explain the computational rules used for the whole numbers. In the table below, the letters a, b, and c refer to any whole number.

Addition	Multiplication
Commutative principle: $a + b = b + a$	Commutative principle: $a \cdot b = b \cdot a$
Associative principle: $a + (b + c) = (a + b) + c$	Associative principle: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Zero principle: $a + 0 = a$	One principle: $a \cdot 1 = a$
Distributive principle: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	

Answer each question below. Then check your answers with the inverted answers at the bottom of the page.

1. Does the commutative principle for addition deal with **order** of addends or with **grouping** of addends?

2. Does the associative principle for multiplication deal with **order** of factors or with **grouping** of factors?

3. The number zero is called the **identity element** for addition. What is the identity element for multiplication?

4. Which principle involves both addition and multiplication?

5. If the answer to **A** is correct, what is the answer to **B**?

6. If the answer to **C** is correct, what is the product for **D**?

A $47 + 86 + 92 + 13 = 238$
B $92 + 47 + 13 + 86 =$ ▮▮▮
C $(42 \cdot 69) \cdot 37 = 107\,226$
D $(69 \cdot 37) \cdot 42 =$ ▮▮▮

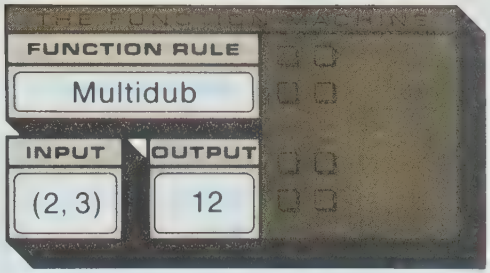
7. Give the number for n and the name of the principle illustrated.

A $89 + n = 46 + 89$
B $97 \cdot 42 = n \cdot 97$
C $n + 0 = 12$
D $9 \cdot (8 \cdot 10) = (9 \cdot 8) \cdot n$
E $57 \cdot 1 = n$
F $(8 \cdot 30) + (8 \cdot 4) = 8 \cdot n$

ANSWERS: 1. order 2. grouping 3. 1 4. distributive 5. 238
6. 107 226 7. **A** 46, commutative (+); **B** 42, commutative (×); **C** 12, zero; **D** 10, associative (×); **E** 57, one; **F** 34, distributive

Some Special Function Machine Rules

Each function table below shows an interesting operation made up by various students. Complete each table.



1.

Function Rule: Multidub	Input	Output
	(2, 3)	12
A	(3, 4)	
B	(5, 12)	
C	(10, 10)	

2.

Function Rule: Doubleadd	Input	Output
	(2, 5)	14
A	(4, 5)	
B	(10, 7)	
C	(25, 20)	

3.

Function Rule: Advide 2	Input	Output
	(8, 4)	6
A	(5, 7)	
B	(4, 36)	
C	(17, 39)	

4.

Function Rule: Dubfirst, add	Input	Output
	(2, 1)	5
	(10, 9)	29
A	(50, 10)	
B	(37, 8)	
C	(98, 4)	

5.

Function Rule: Greaterplus diff	Input	Output
	(3, 2)	4
	(100, 2)	198
A	(8, 5)	
B	(25, 20)	
C	(500, 200)	

6.

Function Rule: Admul 3	Input	Output
	(2, 3)	15
	(1, 7)	24
A	(3, 4)	
B	(9, 6)	
C	(3, 10)	

★ **7.** Study the function table. Then give a name for the function rule.

Function Rule: _____	Input	Output
	(1, 1)	2
	(1, 2)	9
	(2, 2)	16
	(2, 3)	35

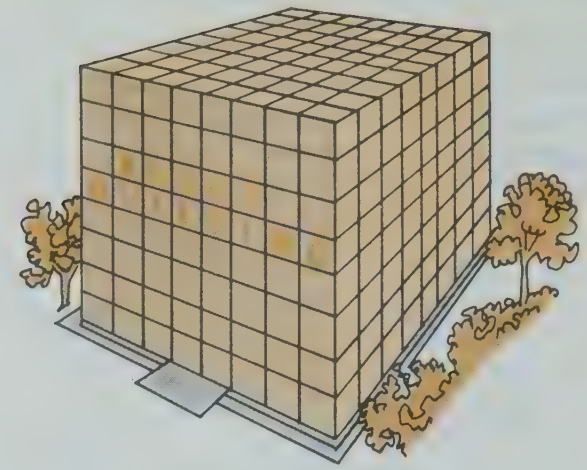
think

Find n and p. Then explain the pattern shown.

1. $3 \cdot 5 = (n \cdot n) - 1 = p$
2. $4 \cdot 6 = (n \cdot n) - 1 = p$
3. $5 \cdot 7 = (n \cdot n) - 1 = p$
4. $9 \cdot 11 = (n \cdot n) - 1 = p$
5. $19 \cdot 21 = (n \cdot n) - 1 = p$
6. $99 \cdot 101 = (n \cdot n) - 1 = p$

Investigating the Ideas

Each small cube represents a room in the Block Building.



? How many rooms are in the Block Building?

Discussing the Ideas

1. A How many rooms are in each row?
   B How many rooms are on each floor (or story) of the building?
   C How many floors (stories) are in the building?
   D Explain how you found the number of rooms in the building.

2. When the same number is used as a **factor** several times, you can use an **exponent** to show how many times the factor **(base)** is used.
   A How many times is 10 used as a factor in 10^5?

   Exponent → $10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100\,000$ ← Base
   B You can read 10^5 as "ten to the fifth power." How would you read 6^4? 5^8? 9^7?

3. Express the number of rooms in the Block Building using an exponent.

4. The process of finding the area of a square and the volume of a cube suggests a reason why 4^2 is called "four squared" and 4^3 is called "four cubed." Give this reason.

5. You can discover a rule for writing the decimal numeral for any **power of ten**. Explain this rule.

Power of ten		Numeral
10^1	=	10
10^2	=	100
10^3	=	1000
10^4	=	10 000
⋮		⋮

1. Write each product using an exponent.

A $7 \cdot 7$ **D** $10 \cdot 10 \cdot 10 \cdot 10$ **G** $5 \cdot 5$

B $3 \cdot 3 \cdot 3$ **E** $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$ **H** $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$

C $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ **F** $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ **I** $4 \cdot 4 \cdot 4$

2. Tell how many times ten is used as a factor in each part.

A 10^3 **B** 10^2 **C** 10^6 **D** 10^7 **E** 10^{19} **F** 10^{50} **G** 10^{1000}

3. Give the usual numeral. EXAMPLE: $10^2 = 100$

A 10^3 **B** 10^5 **C** 10^6 **D** 10^8 **E** 10^9 **F** 10^{12} **G** 10^1

4. Write the product as a power of ten.

EXAMPLE: $10^2 \cdot 10^1 = (10 \cdot 10) \cdot 10 = 10^3$

A $10^3 \cdot 10^1$ **D** $10^2 \cdot 10^2$ **G** $10^6 \cdot 10^2$

B $10^2 \cdot 10^3$ **E** $10^3 \cdot 10^4$ **H** $10^4 \cdot 10^5$

C $10^1 \cdot 10^4$ **F** $10^5 \cdot 10^1$ **I** $10^5 \cdot 10^8$

think

The number 10^{100} (1 followed by 100 zeros) is called a **googol**. Which is larger, a googol or 2^{1000}?

5. Write the quotient as a power of ten.

EXAMPLE: $10^4 \div 10^2 = \frac{10\,000}{100} = 100 = 10^2$

A $10^5 \div 10^3$ **B** $10^6 \div 10^3$ **C** $10^3 \div 10^2$ **D** $10^7 \div 10^4$

6. Give the number for each ▥.

A $10^4 \div 10^2 = 10^{4-▥}$ **B** $10^4 \div 10^3 = 10^{4-▥}$ **C** $10^4 \div 10^4 = 10^{4-▥}$

7. Since $10^4 \div 10^4 = \frac{10\,000}{10\,000} = 1$ and $10^4 \div 10^4 = 10^{4-4} = 10^0$,

we define $10^0 = 1$. Similarly, $7^0 = 1$ and $3^0 = 1$.

In general, if n is a non-zero number, then $n^0 = 1$.

The symbol 0^0 is undefined.

Which part is **not** a symbol for 1?

A 8^0 **B** $10^3 \div 10^3$ **C** 1^2 **D** $2^1 \cdot 2^0$

8. Base-ten numerals can be written in **expanded notation**.

EXAMPLE: $3764 = (3 \cdot 10^3) + (7 \cdot 10^2) + (6 \cdot 10^1) + (4 \cdot 10^0)$

Write each base-ten numeral in expanded notation.

A 27 **B** 473 **C** 507 **D** 9625 **E** 87 034 **F** 4 632 470

More practice, page S-2, Set 1

Investigating the Ideas

The middle number of each set of three numbers is always between the other two.

A

280
278
270

C

47 000
46 083
46 000

E

2 500 000
439 637
2 400 000

B

3200
3192
3100

D

760 000
755 298
750 000

F

54 000 000
53 644 139
53 000 000

Can you find which of the other two numbers is closer to the middle number in each set?

Discussing the Ideas

1. Because 278 is nearer to 280 than to 270, you can say "278 **rounded to the nearest ten** is 280."
   To round 278 to the nearest **hundred**, you must decide whether 278 is nearer to 200 or 300.
   What is 278 rounded to the nearest hundred?

2. In part **B** above, what is 3192 when rounded to the nearest hundred?

3. Describe the rounding for each of the parts **C** through **F** above.

4. Why do you think it is helpful to be able to round numbers?

5. The number 2500 is halfway between 2000 and 3000.
   In cases such as this, we agree to round "up."
   What is 2500 rounded to the nearest thousand?

6. **A** Estimate the number of students that attend your school.
   B Find the exact number of students in your school and round it to the nearest hundred.
   C How close was your estimate to the rounded number for part **B**?

1. **A** Is 6287 nearer to 6000 or 7000?

B What is 6287 when rounded to the nearest thousand?

2. Choose the number that best compietes the sentence.

A 2 857 456 rounded to the nearest ten is $\begin{cases} 2\,857\,450. \\ 2\,857\,460. \end{cases}$

B 2 857 456 rounded to the nearest hundred is $\begin{cases} 2\,857\,400. \\ 2\,857\,500. \end{cases}$

C 2 857 456 rounded to the nearest thousand is $\begin{cases} 2\,857\,000. \\ 2\,858\,000. \end{cases}$

D 2 857 456 rounded to the nearest ten thousand is $\begin{cases} 2\,850\,000. \\ 2\,860\,000. \end{cases}$

E 2 857 456 rounded to the nearest hundred thousand is $\begin{cases} 2\,800\,000. \\ 2\,900\,000. \end{cases}$

3. Round each number in the table to the nearest thousand.

	PROVINCE	SCHOOL ENROLLMENT
A	Alberta	437 150
B	Ontario	2 083 951
C	Quebec	1 639 335
D	P.E.I.	30 648

4. The area of Canada is 9 976 185 square kilometres.

A Round this number to the nearest thousand.

B Round this number to the nearest million.

5. The highest active volcano is Guallatiri in Chile. It is 6060 metres high.

A Round the height to the nearest hundred metres.

B Round the height to the nearest thousand metres.

6. The population of Mexico City in 1970 was 8 541 070.

A To the nearest hundred thousand, what was the population?

B To the nearest million, what was the population?

think

The ones' digit of a 3-digit numeral is three times as large as the tens' digit. The hundreds' digit is twice as large as the tens' digit. What is the number if it is greater than 500?

More practice, page S-2, Set 2

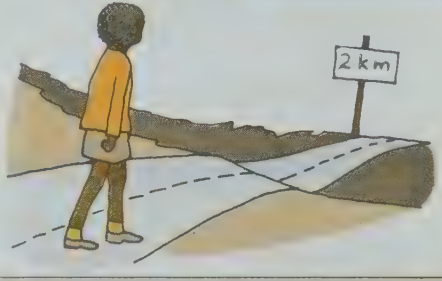
Investigating the Ideas

One of the more useful skills in mathematics is the ability to make a **reasonable estimate** of the answer to a problem before actually computing the exact answer to the problem.

Can you make estimates to one or more of these problems?

A What is the total of the ages of all the students in your math class?

B How many days have you lived?



C About how many steps would you take in walking a distance of two kilometres?

Discussing the Ideas

1. In Bob's eighth grade math class there are 29 students. Bob is 13 years old.
   - **A** Is 300 too low an estimate for the total age of all the students?
   - **B** Is 600 too high an estimate?
   - **C** How would you make an estimate of the total age of all the students in Bob's class?

2. Explain your estimate for Investigation question **B**.

3. Explain how you made your estimate for Investigation question **C**.

4. To make estimates, you can round the numbers to the nearest multiple of 10, 100, or 1000. Explain how to round the numbers and estimate the answers to each of these problems.
   - **A** $219 + 398 + 134 + 372$ **B** $63 \cdot 589$ **C** $39\overline{)1611}$

1. Use multiples of 10, 100, or 1000 for each ▒.

- **A** To estimate 297 + 609, you can find the sum 300 + ▒.
- **B** To estimate 684 · 9, you can find the product ▒ · 9.
- **C** To estimate 63 · 78, you can find the product 60 · ▒.
- **D** To estimate 2139 ÷ 69, you can find the quotient 2100 ÷ ▒.
- **E** To estimate 692 − 176, you can find the difference ▒ − 200.
- **F** To estimate 6 · 4195, you can find the product 6 · ▒.
- **G** To estimate 524 + 395, you can find the sum ▒ + 400.
- **H** To estimate 63 · 295, you can find the product 60 · ▒.
- **I** To estimate 78 + 63 + 91, you can find the sum 80 + 60 + ▒.
- **J** To estimate 8199 + 2411, you can find the sum ▒ + 2000.
- **K** To estimate 381 · 714, you can find ▒ · 700.

2. Give an estimate for the answer for each part in Exercise 1.

3. Give an estimate for each of the following problems.

A 721 + 695	**D** 32 · 77	**G** 1984 + 6911	**J** 625 + 379
B 843 + 275	**E** 778 ÷ 4	**H** 78 + 24 + 98	**K** 7684 + 3211
C 629 − 179	**F** 71 · 688	**I** 213 · 498	**L** 566 · 32

In Exercises 4 through 6, give an estimate for the answer.

4. The area of Alaska is 1 519 809 square kilometres (km^2). The population in 1970 was 302 173. About how many square kilometres per person are there?

5. In 1971, the average salary of 2 061 000 public school teachers was $10 100. About how much total salary was earned by these teachers in 1971?

6. The USSR has an area of 22 402 200 square kilometres and a population of 242 612 000. Canada has an area of 9 976 185 square kilometres and a population of 21 830 000.

- **A** About how many Russians per square kilometre?
- **B** About how many Canadians per square kilometre?
- **C** Which country is more densely populated?

More practice, page S-3, Set 3

Investigating the Ideas

Can you compute the answers to these problems and then check your answers by breaking the code?

1. $7296 + 874 + 4287 + 1099$

2. $319\,801 - 174\,966$

3. 478×39

4. $79\overline{)29\,743}$

Coded answers: 1. GIAAB 2. GJJDIA 3. GDBJH 4. ICB,rIE

Discussing the Ideas

1. What are some shortcuts you can use in addition problems?

2. Explain why the subtraction problem is incorrect even though the screens cover some of the digits.

$$\begin{array}{r} 6\ 2\blacksquare \\ -3\ 8\blacksquare \\ \hline 3\ \blacksquare\blacksquare \end{array}$$

3. How can you check your work for a subtraction problem?

4. Estimate the product for the multiplication problem $274 \cdot 308$.

5. Explain why this student did not get the correct answer.

$$\begin{array}{r} 274 \\ \times 308 \\ \hline 2192 \\ 822 \\ \hline 10412 \end{array}$$

6. Which is the best estimate of the quotient $64\overline{)38\,954}$?

   A 60 **B** 600 **C** 6000

7. Find the quotient and remainder for Exercise 6.

8. How can you check your work in a division problem?

1. A **palindrome** is a numeral which represents the same number if the digits of the numeral are written in reverse order. Examples of palindromes are 171, 2662, and 35653. The example at the right shows how to get a palindrome by reversing the digits of a number and adding. Find the palindrome for each number using the addition method.

A 312 **C** 692 **E** 3587
B 813 **D** 1962 **F** 57919

EXAMPLE:

number →	178
reverse →	+871
sum →	1049
reverse →	+9401
sum →	10450
reverse →	+05401
palindrome →	15851

2. Study and complete part **A**. Then perform the same operations with the numbers given in the other parts.

A

8 7 6	← number
−6 7 8	← reverse
1 9 8	← difference
+8 9 1	← reverse
▒▒▒▒	← sum

B 624 **C** 902 **D** 753

3. Find the products. They may be interesting to compare.

A 96 × 46 **B** 69 × 64 **C** 322 × 669 **D** 223 × 966

4. Find the quotients and remainders.

A $52\overline{)2704}$ **B** $123\overline{)456\,789}$ **C** $1001\overline{)839\,839}$

5. Copy each problem, replacing the screens by digits so that the answers are correct.

A

▒ 8 ▒ 7	
+ 9 6 7 ▒	
▒ 5 ▒ 1 5	

B

4 ▒ ▒ 7	
− ▒ 9 8 ▒	
1 0 1 8	

C

▒▒▒	
× 9 9 9	
▒▒▒▒	
▒▒▒	
▒▒▒▒	
4 7 ▒ ▒ 4	

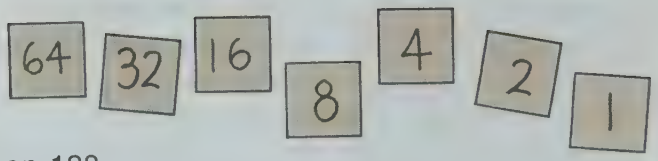
D

▒▒	
76) ▒▒▒	
▒▒	
▒▒	
▒ 6	
0	

More practice, pages S-3, 4, and 5, Sets 4, 5, 6, and 7

Investigating the Ideas

Try this game with a classmate.



RULES: **A** Make seven cards like the ones shown.

B Name any number less than 128.

C Your opponent gets one minute (or less) to choose the cards whose numbers give your number for their sum.

$79 = 64 + 8 + 4 + 2 + 1$

D Your opponent scores one point if he gets the number.

E Your opponent then challenges you with a number.

Can you play the game to 10 points with a classmate?

Discussing the Ideas

1. Do you think there are some numbers from 1 through 127 that cannot be formed from the seven cards?

2. What is special about the set of cards for these numbers?

   A 63 **B** 31 **C** 15 **D** 7 **E** 3

3. Give the missing exponent for each power.

   A $64 = 2^{\square}$ **C** $16 = 2^{\square}$ **E** $4 = 2^{\square}$ **G** $1 = 2^{\square}$

   B $32 = 2^{\square}$ **D** $8 = 2^{\square}$ **F** $2 = 2^{\square}$

4. Mary Binary, who was very good at the game, made a table like the one below to help her find the correct cards quickly. She tallied a "1" if a card was needed and a "0" if it was not needed.

   What are the numbers for **A**, **B**, and **C** in the table?

NUMBER	$64 = 2^6$	$32 = 2^5$	$16 = 2^4$	$8 = 2^3$	$4 = 2^2$	$2 = 2^1$	$1 = 2^0$
75	1	0	0	1	0	1	1
46		1	0	1	1	1	0
A	1	1	0	0	1	0	0
B				1	1	1	1
C		1	1	0	1	0	1

1. Complete the "tally sheet" for each number.

	NUMBER	64	32	16	8	4	2	1
	26			1	1	0	1	0
A	70							
B	111							
C	49							
D	83							
E	122							
F	98							

2. What is the number indicated by the tallies?

NUMBER	2^6	2^5	2^4	2^3	2^2	2^1	2^0
A	1	1	0	0	1	1	0
B		1	0	0	1	1	1
C	1	0	1	0	1	0	1
D		1	1	1	0	1	1
E		1	0	1	1	0	0
F	1	1	1	1	1	1	0

3. The symbols below represent tallies from the charts above.
Give the number that each tally represents.

A

1	0	0	0	0	0	1

B

	1	1	0	1	0	1

C

1	0	1	1	1	1	1

D

1	0	0	1	1	0	1

E

	1	1	1	0	0	0

F

1	1	1	0	1	0	1

4. Make a tally for each of these base-ten numbers.

A	7	**C**	26	**E**	42	**G**	61	**I**	73	**K**	100
B	13	**D**	30	**F**	50	**H**	67	**J**	80	**L**	125

Investigating the Ideas

Study the base-two and base-ten tallies that show the number of dots in set **A**.

A

B

BASE-TWO						
2^6	2^5	2^4	2^3	2^2	2^1	2^0

BASE-TEN		
10^2	10^1	10^0

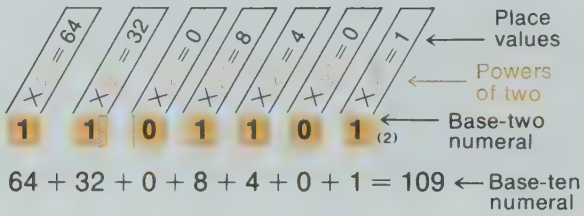
BASE-TWO						
2^6	2^5	2^4	2^3	2^2	2^1	2^0

BASE-TEN		
10^2	10^1	10^0

Can you show base-two and base-ten tallies for the number of dots in set **B**?

Discussing the Ideas

In the decimal system of numeration, we group by tens in order to write numerals for numbers. In the **binary system** (base-two), we group by twos. The diagram below should help you understand the binary system.



$64 + 32 + 0 + 8 + 4 + 0 + 1 = 109$ ← Base-ten numeral

1. In the decimal numeration system, the place-value names are powers of ten. What powers are the place-value names in the binary system?

2. In the base-ten, or decimal, system of numeration, we use the digits 0 through 9 to write numerals for any whole number. In the base-two, or binary, system, what are the only digits needed?

3. We write $101_{(2)}$ to show $1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$.
The symbol $101_{(2)}$ is read as "One zero one, base two."
This means one group of four, zero groups of two, and one.
Read and give the meaning of each of these base-two numerals.

A $11_{(2)}$ **B** $100_{(2)}$ **C** $111_{(2)}$ **D** $1011_{(2)}$ **E** $10101_{(2)}$

1. Complete each sentence concerning the digit printed in color.

EXAMPLE: $1101_{(2)}$ → The 1 in the fours' place represents 1×2^2

A $101_{(2)}$ → The ? place represents ?.

B $10101_{(2)}$ → The 1 in the ? place represents ?.

C $1000001_{(2)}$ → The 1 in the ? place represents ?.

2. Write the base-ten numeral for each base-two numeral.

A $1_{(2)}$	**C** $111_{(2)}$	**E** $101_{(2)}$	**G** $11_{(2)}$	**I** $1001_{(2)}$	**K** $1011_{(2)}$
B $10_{(2)}$	**D** $100_{(2)}$	**F** $110_{(2)}$	**H** $1100_{(2)}$	**J** $1010_{(2)}$	**L** $1111_{(2)}$

3. Copy and complete the table for powers of two. You will need the results in later exercises.

Powers of two	
$2^0 =$	$2^7 =$
$2^1 =$	$2^8 =$
$2^2 =$	$2^9 =$
$2^3 =$	$2^{10} =$
$2^4 =$	$2^{11} =$
$2^5 =$	$2^{12} =$
$2^6 =$	$2^{13} =$

4. Expand each base-two numeral as shown in the example. Then use the table of powers of two and calculate the base-ten numeral that represents the same number.

EXAMPLE: $1101_{(2)} = (1 \cdot 2^3) + (1 \cdot 2^2) + (0 \cdot 2^1) + 1$
$= 8 + 4 + 0 + 1 = 13$

A $1110_{(2)}$	**C** $11100_{(2)}$	**E** $11111_{(2)}$	**G** $1110111_{(2)}$
B $1000_{(2)}$	**D** $10000_{(2)}$	**F** $100000_{(2)}$	**H** $10101010_{(2)}$

5. Write a simple base-two numeral for each part.

A $(1 \cdot 2^1) + (1 \cdot 2^0)$

B $(1 \cdot 2^2) + (0 \cdot 2^1) + (1 \cdot 2^0)$

C $(1 \cdot 2^4) + (1 \cdot 2^3) + (0 \cdot 2^2) + (0 \cdot 2^1) + (0 \cdot 2^0)$

D $(1 \cdot 2^5) + (0 \cdot 2^4) + (1 \cdot 2^3) + (1 \cdot 2^2) + (1 \cdot 2^1) + (0 \cdot 2^0)$

6. Convert these base-ten numerals to base-two numerals. Refer to the powers of two table in Exercise 2.

A 156	**C** 539	**E** 900	**G** 1279
B 372	**D** 684	**F** 1023	**H** 1584

think

Examples **A** and **B** show a shortcut for converting a binary numeral to a base-ten numeral. Can you discover how to use the shortcut and use it to give the base-ten numerals for these numerals?

A $1\,0\,0\,0\,1\,1\,1_{(2)} = 71_{(ten)}$

B $1\,0\,1\,1\,0\,0\,0\,1_{(2)} = 177_{(ten)}$

A $10111_{(2)}$ **B** $10000101_{(2)}$ **C** $11100001_{(2)}$ **D** $111111111_{(2)}$ **E** $110110001_{(2)}$

Investigating the Ideas

Each digit of a binary numeral is called a **bit**. Suppose the letters of the alphabet were coded in order with five-bit binary numerals as shown in the table. For example, the binary code for MATH would be:

01101 → M
00001 → A
10100 → T
01000 → H

A Five-bit Binary Code	
A	0 0 0 0 1
B	0 0 0 1 0
C	0 0 0 1 1
D	0 0 1 0 0
⋮	⋮ ⋮ ⋮ ⋮ ⋮
M	0 1 1 0 1
⋮	⋮ ⋮ ⋮ ⋮ ⋮
Z	1 1 0 1 0

Can you complete the five-bit binary code table and write a word or message using the binary code?

Discussing the Ideas

1. Since E is the fifth letter of the alphabet, what five-bit binary numeral would be the code for E?

2. The code for M is 01101. What base-ten numeral gives its position in the alphabet?

3. Each group of five-bit binary numerals is a coded name. Decode each name.

A

0	0	1	0	0
0	0	0	0	1
1	0	1	1	0
0	0	1	0	1

B

0	1	1	0	1
0	0	0	0	1
1	0	0	1	0
1	1	0	0	1

C

1	0	0	0	0
0	0	0	0	1
1	0	1	0	1
0	1	1	0	0
0	0	0	0	1

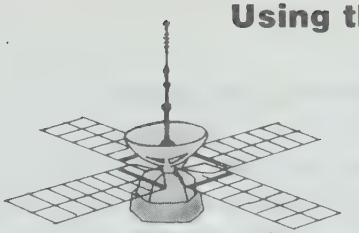
4. Explain how this palindromic message could be written in five-bit binary code:

N E V E R O D D O R E V E N

5. What is the largest number of symbols that could be coded using five-bit binary numerals?

6. How many symbols could be coded with six-bit numerals?

The Mariner spacecraft (1965–1971) took pictures of the planet Mars which were radioed back to earth and produced with the help of a computer and binary numerals. The computer received the "pictures" in the form of binary numerals consisting of six bits. Each group of six bits determined the shade of one dot in the final picture. The shading of the dots varied in 64 different shades from completely white to completely black as shown in the table at the right. All binary numerals representing numbers between 0 and 63 represent various shades of gray between white and black.



Mariner Spacecraft

Shade of dot	Six-bit binary numeral	Base-ten numeral
White	000000	0
Almost white	000001	1
⋮	⋮	⋮
Almost black	111110	62
Black	111111	63

1. **A** Which number from 0 through 63 is represented by 101001?
   B Does this number represent a shade of gray closer to white or to black?

2. In the picture transmission just described, what binary numeral would represent the darkest shade of gray that is not black?

3. Which binary numeral would represent the lighest shade of gray?

4. Give two binary numerals that would represent shades of gray midway between white and black.

5. Suppose that an imaginary spacecraft, Sparrow I, was sent on a close flight past an imaginary planet Nova. After taking pictures, radio signals were sent back and amplified to give this array of two-bit binary numerals. Use this array, and the code below to convert the numbers into a picture on a 7-by-7 grid as shown.

	1	2	3	4	5	6	7
a	01	11	11	11	11	11	01
b	01	11	00	11	10	11	01
c	11	11	11	11	10	11	01
d	01	11	11	11	10	11	01
e	01	01	11	11	11	11	01
f	01	11	11	11	11	11	01
g	01	01	01	10	10	01	01

Code

Base-ten numeral	Binary numeral	
0 →	00 →	
1 →	01 →	
2 →	10 →	
3 →	11 →	

Grid

	1	2	3	4	5	6	7
a							
b							
c							
d							
e							
f							
g							

Investigating the Ideas

In order to compute efficiently using base-ten numerals, you must know a large number of basic addition and multiplication "facts."

To compute using the base-two numerals, you need to know only the eight facts for the base-two tables.

BASE TEN	
(+) *facts*	(×) *facts*
$0 + 0 = 0$	$0 \cdot 0 = 0$
$0 + 1 = 1$	$1 \cdot 1 = 1$
$1 + 1 = 2$	$1 \cdot 2 = 2$
$1 + 2 = 3$	$2 \cdot 2 = 4$
⋮	⋮
$9 + 9 = 18$	$9 \cdot 9 = 81$

Can you copy and complete the base-two addition and multiplication tables?

BASE TWO						
+	0	1		×	0	1
0				0		
1				1		

Discussing the Ideas

1. The only "hard" addition fact is $1_{(2)} + 1_{(2)} = 10_{(2)}$. Are any of the multiplication facts "hard"?

2. Here is the way a student showed an addition problem using base-two numerals. Can you explain how the sum was found?

$$\begin{array}{r} {\scriptstyle 1\ 1\ \ \ } \\ 101_{(2)} \\ +\,11_{(2)} \\ \hline 1000_{(2)} \end{array}$$

3. What is $10_{(2)} - 1_{(2)}$?

4. How would you complete this subtraction problem?

$$\begin{array}{r} 1\ 1\ \cancel{1}^{0}\ \cancel{0}^{10}{}_{(2)} \\ -\,1\ 1\ 1_{(2)} \\ \hline 1_{(2)} \end{array}$$

5. Explain the steps in this multiplication problem.

$$\begin{array}{r} 1\ 1\ 1_{(2)} \\ \times\,1\ 1_{(2)} \\ \hline 1\ 1\ 1 \\ 1\ 1\ 1\ \ \\ \hline 1\ 0\ 1\ 0\ 1_{(2)} \end{array}$$

6. Try this division problem. $10_{(2)}\overline{)10110_{(2)}}$ Check your work by using multiplication.

All numerals in the exercises are base-two numerals.

1. Find the sums.

A $\begin{array}{r} 10 \\ +\ 1 \\ \hline \end{array}$ **B** $\begin{array}{r} 11 \\ +10 \\ \hline \end{array}$ **C** $\begin{array}{r} 11 \\ +11 \\ \hline \end{array}$ **D** $\begin{array}{r} 101 \\ 110 \\ +111 \\ \hline \end{array}$ **E** $\begin{array}{r} 100 \\ 111 \\ 110 \\ +101 \\ \hline \end{array}$ **F** $\begin{array}{r} 111 \\ 11 \\ +\ 1 \\ \hline \end{array}$

2. Find the differences.

A $\begin{array}{r} 11 \\ -\ 1 \\ \hline \end{array}$ **B** $\begin{array}{r} 10 \\ -\ 1 \\ \hline \end{array}$ **C** $\begin{array}{r} 100 \\ -\ 11 \\ \hline \end{array}$ **D** $\begin{array}{r} 1101 \\ -1011 \\ \hline \end{array}$ **E** $\begin{array}{r} 110110 \\ -\ 11011 \\ \hline \end{array}$ **F** $\begin{array}{r} 1000 \\ -\ \ 1 \\ \hline \end{array}$

3. Find the products.

A $\begin{array}{r} 11 \\ \times 1 \\ \hline \end{array}$ **C** $\begin{array}{r} 10 \\ \times 10 \\ \hline \end{array}$ **E** $\begin{array}{r} 100 \\ \times 100 \\ \hline \end{array}$ **G** $\begin{array}{r} 101 \\ \times 11 \\ \hline \end{array}$ **I** $\begin{array}{r} 100 \\ \times 110 \\ \hline \end{array}$ **K** $\begin{array}{r} 101 \\ \times 111 \\ \hline \end{array}$

B $\begin{array}{r} 10 \\ \times 0 \\ \hline \end{array}$ **D** $\begin{array}{r} 100 \\ \times 10 \\ \hline \end{array}$ **F** $\begin{array}{r} 11 \\ \times 11 \\ \hline \end{array}$ **H** $\begin{array}{r} 111 \\ \times 111 \\ \hline \end{array}$ **J** $\begin{array}{r} 111 \\ \times 11 \\ \hline \end{array}$ **L** $\begin{array}{r} 101 \\ \times 110 \\ \hline \end{array}$

4. Find the quotients.

A $1\overline{)101}$ **B** $11\overline{)110}$ **C** $10\overline{)10110}$ **D** $110\overline{)110000}$

5. Write a base-two numeral for the number n in each equation.

A $11 + n = 101$ **D** $101 \cdot n = 1111$ **G** $11 \cdot n = 1100$

B $111 - n = 110$ **E** $110 \div 10 = n$ **H** $n + 100 = 1000$

C $10 \cdot n = 100$ **F** $n - 11 = 1011$ **I** $1000 - n = 111$

think

The **duodecimal system** (base twelve) requires two extra digits. Suppose we use the digits and names shown in the diagram.

0	1	2	3	4	5	6	7	8	9	D	ϵ
zero	one	two	three	four	five	six	seven	eight	nine	dek	el
										(from **decade**)	(from **eleven**)

1. Count from $1_{(12)}$ to $100_{(12)}$. For example, for the number after $10_{(12)}$, you would say "one one." For the number after $99_{(12)}$, you would say "nine dek."
2. Complete two duodecimal charts (one for multiplication and one for addition) similar to the ones for base two. You will need to use twelve digits $\{0, 1, 3, \ldots, 9, \text{D}, \epsilon\}$ across the top and down the side of each chart.

REVIEWING THE IDEAS

1. Write a Hindu-Arabic numeral for each numeral.

A

B

C MCDLXIV

2. Write a base-ten numeral for each number.

A 3 billion **B** 748 million

3. Which digit in the numeral 634 396 863 represents 30 000 000?

4. Write a subtraction equation that can be used to find the missing addend n in the addition equation $27 + n = 53$.

5. Write a division equation that can be used to find the missing factor y in the multiplication equation $9 \cdot y = 234$.

6. Which principle is illustrated in each of the following equations?

A $7 + 9 = 9 + 7$

B $3462 \cdot 1 = 3462$

C $(2 \cdot 3) \cdot 7 = 7 \cdot (2 \cdot 3)$

D $(9 + 16) + 4 = 9 + (16 + 4)$

E $42 + 0 = 42$

7. Write each product using an exponent.

A $6 \cdot 6 \cdot 6$ **C** $10 \cdot 10$

B $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ **D** $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

8. What is the missing exponent? $10^{\square} = 1$

9. Solve the equations.

A $10^2 \cdot 10^3 = n$ **D** $8^2 \cdot 8^1 = n$

B $2^3 \cdot 2^4 = n$ **E** $n \div 7^2 = 7^3$

C $10^4 \cdot 10^0 = n$ **F** $10^4 \div 10^3 = n$

10. Round 7 463 487 to the nearest

A hundred **B** thousand **C** million

11. Which number is the best estimate for $179 \cdot 73$?

A 16 000 **B** 14 000 **C** 12 000

12. Estimate the quotient for $58\overline{)29\,437}$ to the nearest hundred.

13. Compute.

A $\begin{array}{r} 9685 \\ +\,8796 \\ \hline \end{array}$ **B** $\begin{array}{r} 8007 \\ -\,2369 \\ \hline \end{array}$ **C** $805 \cdot 496$

D $87\overline{)21\,503}$

14. Write a base-two numeral that tells how many dots there are in this set.

15. Write each base-ten numeral as a base-two numeral.

A 5 **C** 19 **E** 32

B 12 **D** 29 **F** 66

16. Give the base-ten numeral for each base-two numeral.

A $110_{(2)}$ **C** $1000_{(2)}$ **E** $1111_{(2)}$

B $1011_{(2)}$ **D** $11011_{(2)}$ **F** $101011_{(2)}$

17. Compute in base-two.

A $\begin{array}{r} 111_{(2)} \\ +\ 11_{(2)} \\ \hline \end{array}$ **B** $\begin{array}{r} 10000_{(2)} \\ -\ 111_{(2)} \\ \hline \end{array}$ **C** $\begin{array}{r} 111_{(2)} \\ \times 101_{(2)} \\ \hline \end{array}$

D $10_{(2)}\overline{)10010_{(2)}}$

TEST YOURSELF

1. What Hindu-Arabic numeral shows the same number as MCMLXXVI?

2. In the decimal numeral 537 426, the digit 3 stands for 3
   - A thousands.
   - B ten thousands.
   - C hundred thousands.

3. Write the product $6 \cdot 6 \cdot 6 \cdot 6$ using an exponent.

4. Round 347 896 to the nearest ten thousand.

5. Which equation illustrates the distributive principle?
   - A $2 \cdot (7 \cdot 4) = (2 \cdot 7) \cdot 4$
   - B $2 \cdot (7 + 4) = 2 \cdot 7 + 2 \cdot 4$
   - C $(2 \cdot 7) \cdot 4 = 4 \cdot (2 \cdot 7)$

6. Estimate the product $385 \cdot 83$.

7. Compute.

   A
   $$\begin{array}{r} 596 \\ 8129 \\ +\,2748 \\ \hline \end{array}$$

   B
   $$\begin{array}{r} 33\,091 \\ -\,28\,747 \\ \hline \end{array}$$

   C $389 \cdot 76$

   D $83\overline{)2988}$

8. Write a base-two numeral for each base-ten numeral.
   - A 13
   - B 37

9. Write each base-two numeral as a base-ten numeral.
   - A $11010_{(2)}$
   - B $101111_{(2)}$

10. Compute and write the answer using base-two numerals.

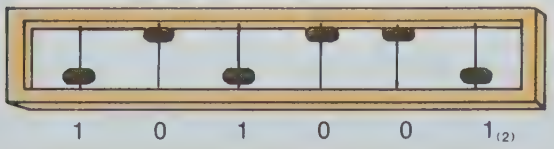
    A
    $$\begin{array}{r} 1011_{(2)} \\ 111_{(2)} \\ +\,1101_{(2)} \\ \hline \end{array}$$

    B
    $$\begin{array}{r} 1101_{(2)} \\ \times\ 11_{(2)} \\ \hline \end{array}$$

RESEARCH PROJECTS

A A method of computing called "finger reckoning" was practiced in the 14th century in Europe. Find out how to use your fingers to find products. (See *From Fingers to Computers* by Margaret F. Willerding; Pasadena: Franklin Publications, 1968; Grolier.)

B A binary abacus has only one counter in each place.



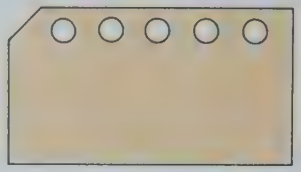
Make a model of a binary abacus. Learn how to add and subtract using binary numerals with the abacus.

C The **duodecimal system** (base twelve) requires two extra digits (one for ten and one for eleven). Using ᴅ for ten and ᴇ for eleven (see page E-25), make up some addition and multiplication problems in base twelve. (See *Understanding Arithmetic* by Robert L. Swain; New York: Rinehart and Company, 1957; Holt, Rinehart and Winston.)

D The binary system is utilized by modern electronic computers. Find out about them and make a poster. (See *Mathematics* by David Bergamini and the Editors of *Life,* Life Science Library; New York: Time, Inc., 1963; General Learning Corporation.)

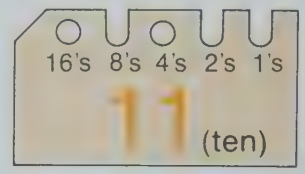
A BASE-TWO CARD SORTER

You will need 32 index cards, a paper punch, and some scissors for this project.

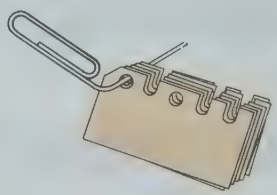


Cut off one corner and punch five holes through an index card as shown. Use this card as a pattern to cut and punch the other 31 cards.

Cut slots to the appropriate holes so that each card represents one of the base-two numerals from $0_{(ten)}$ through $31_{(ten)}$. Number the cards. You can select any number from 0 through $31_{(ten)}$ in five sortings performed as follows:



Suppose that you wish to select card $19_{(ten)}$. $19_{(ten)} = 10011$



Use a paper clip. Put one end through all the holes in the 16's place. Retain the cards that do not stay on the clip. These have a slot in the 16's place. Discard those on the clip. Put the paper clip through the holes in the 8's place. Keep the cards that stay on the clip, discard the others. Put the clip through the hole in the 4's place, retain those cards that stay on the clip. Put the clip through the 2's place. Keep the cards that do not stay on the clip. Put the clip through the 1's place. The card that comes off has a slot in the 16's place, the 2's place, and the ones' place; therefore, it must be $19_{(ten)}$.

Try to locate other cards by this method.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

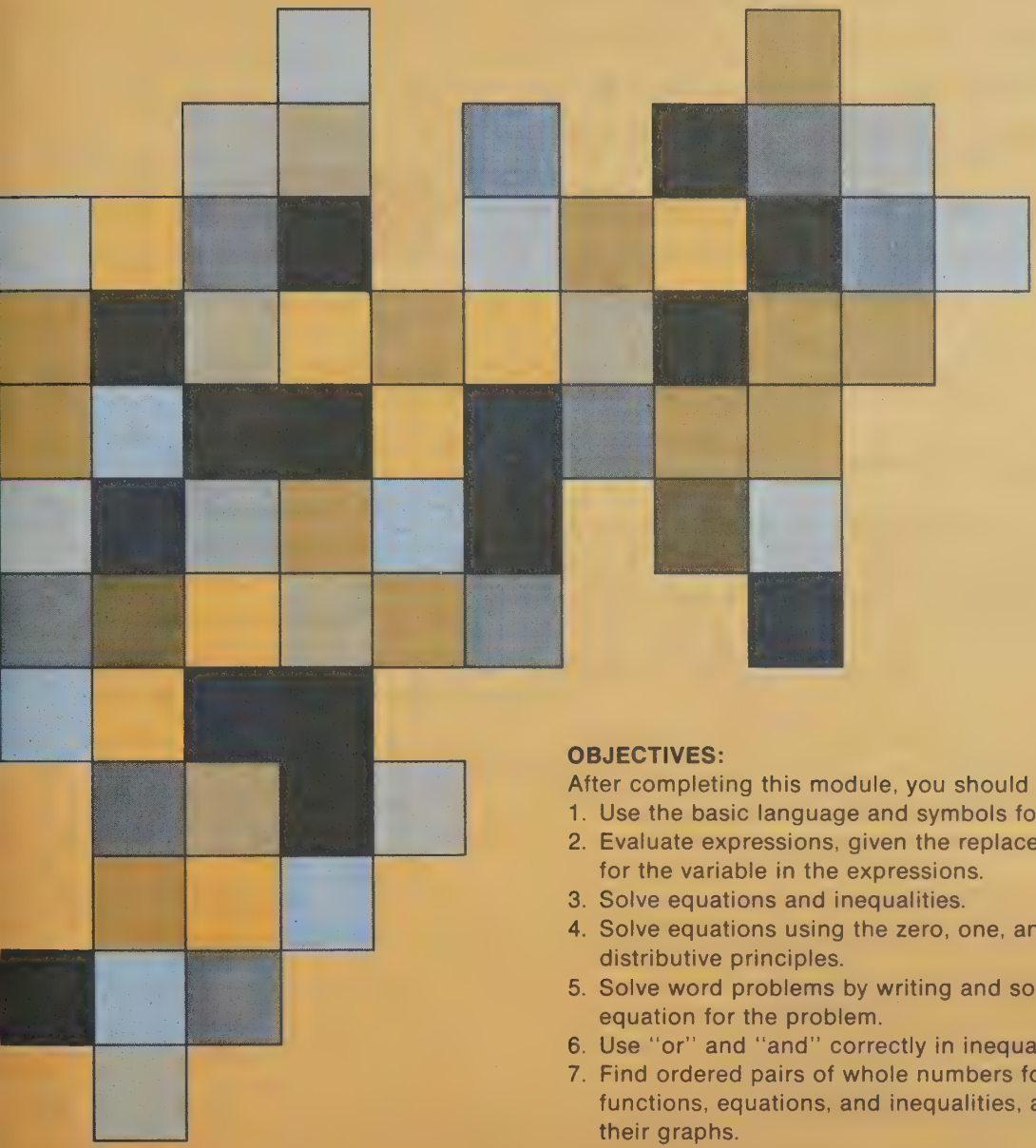
1. 1976 **2. B** **3.** 6^4 **4.** 350 000 **5. B** **6.** 32 000

7. A 11 473; **B** 4 344; **C** 29 564; **D** 36 **8. A** $111_{(2)}$; **B** $10010_{(2)}$

9. A 26; **B** 47 **10. A** $11111_{(2)}$; **B** $100111_{(2)}$

UNIT E: *Integers*

MODULE 2: *Sets, Equations, and Inequalities*



OBJECTIVES:

After completing this module, you should be able to:

1. Use the basic language and symbols for sets.
2. Evaluate expressions, given the replacement set for the variable in the expressions.
3. Solve equations and inequalities.
4. Solve equations using the zero, one, and distributive principles.
5. Solve word problems by writing and solving an equation for the problem.
6. Use "or" and "and" correctly in inequalities.
7. Find ordered pairs of whole numbers for functions, equations, and inequalities, and draw their graphs.

Sets – Language and Symbols

Investigating the Ideas

The language of sets can be found throughout all of mathematics. Some examples are given below.

A The **set** of whole numbers: $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$

B The **set** of points on a line:

C The **set** of even whole numbers less than ten: $\{0, 2, 4, 6, 8\}$

Can you show or list the sets described below?

1 The set of odd whole numbers.

2 The set of points inside a circle.

3 The set of numbers that are the squares of the whole numbers.

Discussing the Ideas

1. The three dots in set A indicate that the numbers in the set continue forever in this pattern. The set of whole numbers is an **infinite set**. If a set is not infinite, then it is **finite**. What kind of a set is each of the three sets that you found for the Investigation question?

2. In set C above, we say that 2 is an element of the set and write this as $2 \in C$. The number 3 is not an element of set C and we write $3 \notin C$.
   A Is $8 \in C$? B Is $7 \in C$? C Is $12 \in C$?

3. If $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{2, 4, 6\}$, then Y is a **subset** of X because every element in Y is an element in X. We write: $Y \subset X$.
   Which of these sets are subsets of X?
   $A = \{0, 1, 2, 3\}$ $B = \{4, 5\}$ $C = \{3, 5, 7\}$ $D = \{1, 2, 3, 4, 5, 6\}$

4. The set which has no elements is called the **empty set** and is denoted by $\{\ \}$ or ϕ. The set of whole numbers greater than 4 and less than 5 is the empty set. Describe the empty set in other ways.

5. The empty set is a subset of every set. Does the empty set have any subsets?

1. Tell whether each set is **infinite** or **finite**.

- **A** The set of multiples of 3.
- **B** The set of points on the sides of a square.
- **C** The set of whole numbers that are less than or equal to one trillion.
- **D** The set of men who have walked on the moon.

2. Suppose $R = \{0, 1, 2, 3, 4, 5, 6\}$, $S = \{2, 4, 6\}$, and $T = \{0, 1, 2, 3, 4\}$. Which statements are true and which are false?

A $4 \in R$ **C** $5 \in T$ **E** $1 \in T$ **G** $S \subset R$ **I** $T \subset R$

B $5 \in S$ **D** $0 \in S$ **F** $R \subset S$ **H** $S \subset T$ **J** $T \subset \phi$

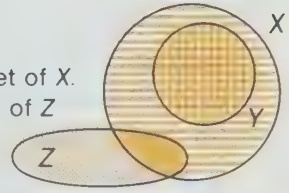
3. List all of the subsets of S in Exercise 2.

4. How many subsets does a set of four elements have?

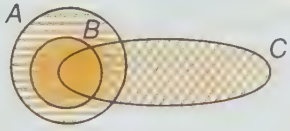
5. A **Venn diagram** is used to show relationships between two or more sets. Think of the ovals or circles as enclosing the elements of the sets.

$Y \subset X$

Z is not a subset of X. Some elements of Z are also in X.



Describe three relationships between sets A, B, and C shown at the right.



6. What are the relationships between sets M, N, and Q shown by the Venn diagram at the right?

7. Suppose $J \subset K$ and $K \subset L$. What other relationship must be true?

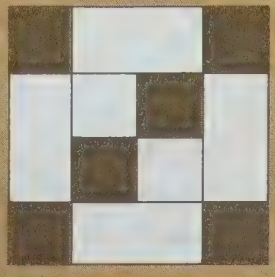
8. If $A \subset B$ and $B \subset C$, what can you conclude?

9. The set $W = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$, the whole numbers, is an infinite set.

- **A** Give a subset of W which is also an infinite set.
- **B** Give a subset of the subset for Part **a** which is also an infinite set.

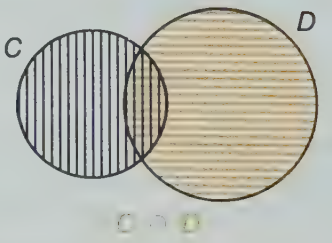
think

How many different squares can you find in the figure?

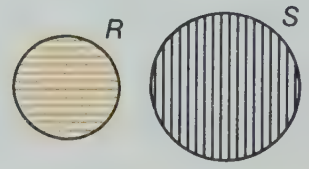


Discussing the Ideas

1. The union ( ) of set A and set B is the set containing all the elements in A or in B or in both A and B. We write $A \cup B$ for the union of A and B.
   - **A** In the Venn diagram, what parts of circular regions A and B represent the union of set A and set B?
   - **B** If $A = \{5, 6, 7, 8, 9, 10\}$ and $B = \{7, 8, 9, 10, 11, 12\}$, which elements are in set $A \cup B$?

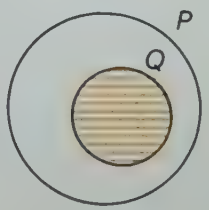


2. The intersection ( ) of set C and set D is the set containing all the elements that are common to both C and D. We write $C \cap D$ for the intersection of C and D.
   - **A** In the Venn diagram, what parts of circular regions C and D represent the intersection of set C and set D?
   - **B** If $C = \{5, 6, 7, 8, 9, 10\}$ and $D = \{7, 8, 9, 10, 11, 12\}$ which elements are in $C \cap D$?



$R \cap S = \phi$

3. If set R and set S have no elements in common, their intersection is the empty set. R and S are **disjoint sets**. Suppose $X = \{1\,2, 3, 4, 5\}$, $Y = \{2, 4, 6, 8\}$, and $Z = \{6, 7, 8, 9\}$. Which pair of sets are disjoint?



4. Suppose sets P and Q are such that $Q \subset P$.
   - **A** Describe $P \cup Q$.
   - **B** Describe $P \cap Q$.

1. If $A = \{3, 6, 9, 12\}$ and $B = \{1, 3, 5, 7, 9\}$, give the elements in each of the following sets:

A $A \cup B$ **B** $A \cap B$

2. If $R = \{0, 1, 2, 3, 4, 5\}$, $S = \{3, 4, 5, 6, 7\}$, and $T = \{7, 8, 9\}$, list the elements in each of the following sets.

A $R \cup S$ **B** $R \cup T$ **C** $S \cup T$ **D** $R \cap S$ **E** $R \cap T$ **F** $S \cap T$

3. Use the sets R, S, and T of Exercise 2 to form the following sets.

A $(R \cup S) \cap T$
B $(R \cap S) \cup T$
C $(R \cap S) \cap T$
D $(R \cup S) \cap (R \cup T)$
E $(R \cap S) \cup (R \cap T)$
F $R \cup (S \cap T)$
G $(R \cap T) \cup (S \cap T)$
H $(R \cup T) \cap (S \cup T)$
I $S \cup (R \cap T)$
J $R \cap (S \cup T)$
K $(R \cap S) \cup (T \cap S)$
L $(R \cup T) \cap S$

4. Draw a Venn diagram that will show how the three sets of Exercise 2 are related.

5. **A** Let S be the subset of the whole numbers which are exactly divisible by 4. List several numbers in S.

B Let T be the subset of the whole numbers which are exactly divisible by 6. List several elements in this set.

C List several numbers in $S \cap T$.

D What special properties do the numbers in $S \cap T$ have?

think

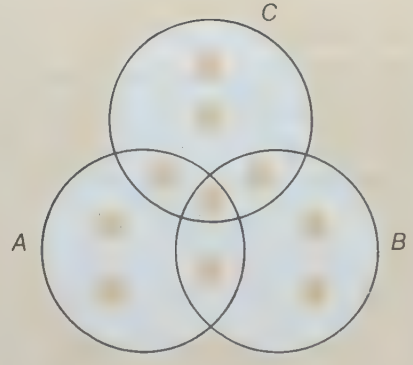
1. Use the numbers in set

$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

to replace the ▒ in sets A, B, and C so that

A each number in X is used exactly once and

B the sum of the numbers in each of the sets A, B, and C is 27.



2. Repeat Exercise 1 making the sum in each set 22.

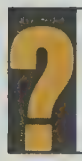
Investigating the Ideas

In algebraic problems, you must often evaluate expressions like these. →

A	$x + 6$
B	$2 \cdot x + 3$
C	$x^2 + 5$

To evaluate the expression, you replace the letter with a number. Suppose that you replaced **x** in expression **A** with each of the numbers in the set {0, 1, 2, 3, 4, 5,}, you would get →

$x = 0 \rightarrow 0 + 6 = 6$
$x = 1 \rightarrow 1 + 6 = 7$
$x = 2 \rightarrow 2 + 6 = 8$
$x = 3 \rightarrow 3 + 6 = 9$
$x = 4 \rightarrow 4 + 6 = 10$
$x = 5 \rightarrow 5 + 6 = 11$



Can you evaluate expressions **B** and **C** using the numbers in the set {0, 1, 2, 3, 4, 5, 6}?

Discussing the Ideas

1. Explain how you evaluated expression **B**.

2. What did you do to evaluate expression **C**?

3. The set of numbers {0, 1, 2, 3, 4, 5} that you used is called a **replacement set**. Each number in the set is a replacement for the **variable x**. Any letter can be used as a variable in an expression. What are the variables in each of these expressions?

   A $y + 1$ **B** $3 \cdot n + 7$ **C** $r^2 - 1$ **D** $z \div 4$

4. Evaluate each expression for the replacement given.

   A If $a = 7$, then $2 \cdot a =$ ▒▒.
   B If $r = 3$, then $11 - r =$ ▒▒.
   C If $s = 10$, then $3 \cdot s + 13 =$ ▒▒.
   D If $y = 6$, then $y^2 - 4 =$ ▒▒.
   E If $t = 8$, then $13 - t =$ ▒▒.
   F If $q = 35$, then $q \div 7 =$ ▒▒.
   G If $n = 5$, then $n^3 - 5 =$ ▒▒.
   H If $w = 15$, then $8 \cdot w =$ ▒▒.

1. Evaluate the expression $x + 9$ for each number in the set $\{1, 2, 3\}$.

2. Evaluate $12 - y$ if the replacement set is $\{5, 6, 7, 8\}$.

3. Evaluate $3 \cdot x - 2$. The replacement set is $\{0, 2, 4, 6\}$.

4. Find the missing number for each part.
   - **A** If $s = 2$, then $8 + s =$ ▒.
   - **B** If $w = 9$, then $6 \cdot w =$ ▒.
   - **C** If $z = 12$, then $z \div 4 =$ ▒.
   - **D** If $x = 7$, then $x \cdot x =$ ▒.
   - **E** If $n = 6$, then $(5 \cdot n) - 1 =$ ▒.
   - **F** If $t = 2$, then $\frac{12 - t}{5} =$ ▒.
   - **G** If $a = 24$, then $a \div 3 =$ ▒.
   - **H** If $d = 7$, then $(6 \cdot d) \div 7 =$ ▒.

5. Suppose the replacement set for $4 \cdot n - 13$ is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. What is the replacement for n that will make the value of the expression 19?

6. In the expression $x^2 - x$, each x must be replaced with the same number from the replacement set.
   If $x = 3$, then $x^2 - x = 3^2 - 3 = 9 - 3 = 6$.
   Evaluate $n^2 + n + 1$ for each replacement in the set $\{0, 1, 2, 3, 4\}$.

7. **A** Evaluate $y^2 + 2 \cdot y + 1$ using the numbers in set $\{1, 2, 3\}$ as the replacement set.
   B Evaluate $(y + 1)^2$ using the numbers in set $\{1, 2, 3\}$ as the replacement set.

think

Can you find a pattern for the last digit of the numbers that are of the form n^n where n is a whole number?

n	1	2	3	4	5	6	. . .
n^n	$1^1 = 1$	$2^2 = 4$	$3^3 = 27$	$4^4 = 256$	$5^5 = 3125$	$6^6 = ?$	. . .
Last digit	1	4	7	6	5	6	. . .

More practice, page S-5, Set 8

Investigating the Ideas

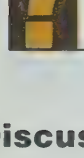
In algebra, equations and inequalities such as those at the right are often called **open sentences**. These sentences can be either **true** or **false** depending upon the replacements for **x**, **y**, and **z**. For example, if the replacement set for **A** is {0, 1, 2, 3}, we think:

A $(3 \cdot \quad) + 1 = 7$

B $(5 \cdot \quad) - 4 = 11$

C $2 \cdot \quad > 4$

If $x = 0$, then $(3 \cdot x) + 1 = 7$ becomes $(3 \cdot 0) + 1 = 7$; which is **false**.
If $x = 1$, then $(3 \cdot x) + 1 = 7$ becomes $(3 \cdot 1) + 1 = 7$; which is **false**.
If $x = 2$, then $(3 \cdot x) + 1 = 7$ becomes $(3 \cdot 2) + 1 = 7$; which is **true**.
If $x = 3$, then $(3 \cdot x) + 1 = 7$ becomes $(3 \cdot 3) + 1 = 7$; which is **false**.



Can you use the replacement set {0, 1, 2, 3} for sentences **B** and **C** and decide when they become true or false?

Discussing the Ideas

1. The numbers which replace the variable in an open sentence to form true statements are **solutions** of the sentence. The **solution set** contains all the numbers of the replacement set that are solutions to the sentence. The solution set for sentence **A** is {2}.
   - **A** What is the solution set for sentence **B**?
   - **B** What is the solution set for sentence **C**?

2. If the replacement set for open sentence $x < 6$ is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, what is the solution set?

3. If the replacement set for $x + 5 = 0$ is {0, 1, 2, 3, 4, 5}, are there any solutions?

4. If every replacement of the variable in an open sentence gives a false statement, **the solution set is the empty set**. Which open sentences below have empty solution sets if {0, 1, 2, 3, 4, 5} is the replacement set?

   A $3 \cdot x - 1 = 11$ **B** $x > 7$ **C** $x + 2 = 7 + x$

1. Solve each equation. The replacement set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

A $x + 3 = 11$
B $n \cdot 6 = 30$
C $21 \div y = 7$
D $8 \cdot r = 8$
E $t + 2 = 2 + t$
F $x \div 5 = 2$
G $(2 \cdot y) - 3 = 11$
H $3 \cdot n = 4 \cdot n$
I $x + x + x = 3 \cdot x$

2. Find the solution set for each inequality using $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as the replacement set.

A $x < 4$
B $s + 7 > 12$
C $n + n < 11$
D $x \cdot 3 > 14$
E $y - 2 > 7$
F $10 - r < 5$
G $t^2 < 40$
H $z^3 + 3 < 100$
I $n^2 - n > 20$

3. If every element in the replacement set is a solution for an inequality or an equation, the equation or inequality is said to be a **generalization** with respect to the replacement set.

EXAMPLE: Replacement set: $\{0, 1, 2, 3, 4, 5\}$

$2 + n = n + 2$ is a generalization.

$x - 5 = 5 - x$ is not a generalization.

Determine which equations or inequalities are generalizations. Use $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ as the replacement set.

A $x + 4 > x + 3$
B $x + x = x^2$
C $1 \cdot x = x$
D $(2 \cdot x) \cdot 5 = x \cdot 10$
E $x + 9 < 15$
F $(4 \cdot x) + (3 \cdot x) = 7 \cdot x$
G $0 + x = x$
H $8 \cdot x = (7 \cdot x) + x$
I $x \cdot x = x$
J $x \div x = 1$
K $x \cdot (x + 1) = x^2 + x$
L $(6 \cdot x) \div (2 \cdot x) = 3$

4. Which generalizations in Exercise 3 would also be generalizations if the replacement set were the set of all whole numbers?

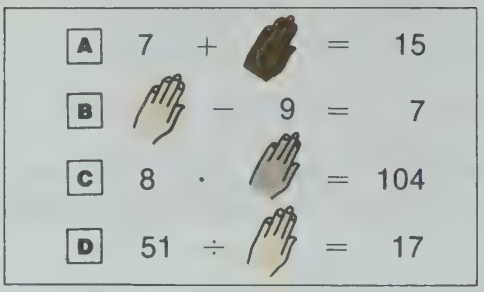
think

Suppose you were paid \$5 for each equation you can solve correctly and fined \$3 for each one incorrectly solved. If you "break even" on 16 equations, how many did you solve correctly?

More practice, page S-6, Set 9

Investigating the Ideas

When the replacement set for an open sentence is the set of all whole numbers, then it is impossible to try every number as a possible solution. Instead, you can often find the solutions by thinking about missing addends, sums, factors, or products.



Suppose the hand in each equation covers the number that makes each sentence a true statement.

Can you find the number that is covered by the hand in each equation?

Discussing the Ideas

1. **A** In equation A above, did the hand cover a missing addend or a sum?
   B What was the number covered by the hand?
   C In what way is equation A like the equation $7 + x = 15$?

2. **A** In equation B, did the hand cover a missing addend or a sum?
   B What was the number covered by the hand?

3. **A** In equation C, does the hand cover a factor or a product?
   B In equation D, does the hand cover a factor or a product?
   C What are the numbers covered in equations C and D?

4. We often omit the multiplication symbol between a numeral and a variable or between two variables.
   Instead of $6 \cdot x$, we write $6x$.
   Instead of $r \cdot s$, we write rs.
   Instead of $5 \cdot (a + 4)$, we write $5(a + 4)$.
   How would you write each of the following without using the multiplication symbol?
   A $8 \cdot y$ **B** $\ell \cdot w$ **C** $(z + 2) \cdot 7$ **D** 6 times r

5. Why would you not write a product like $3 \cdot 7$ without using the multiplication symbol?

The replacement set for the equations on this page is the set of whole numbers.

1. Give the number for each ▥. Then give the number for n.

A $5 + n = 12$
$n = 12 - ▥$
$n = ?$

B $3n = 21$
$n = ▥ \div 3$
$n = ?$

C $n - 8 = 3$
$n = 3 + ▥$
$n = ?$

D $n \div 6 = 3$
$n = 3 \cdot ▥$
$n = ?$

E $9n = 45$
$n = 45 \div ▥$
$n = ?$

F $13 - n = 6$
$6 + n = ▥$
$n = ?$

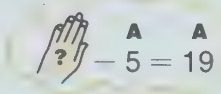
2. Solve the equations.

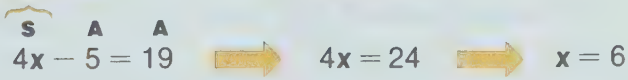
A $x + 5 = 9$
B $7 + a = 13$
C $20 = m + 1$
D $s + 9 = 10$
E $7d = 56$
F $8 + b = 34$
G $56 = 18 + g$
H $10w = 70$
I $10 + m = 100$
J $10n = 100$
K $6a = 72$
L $49 = 7 + c$
M $r \div 3 = 8$
N $9 = 28 - a$
O $5 = 15 \div s$
P $q - 8 = 4$

3. Study the example.

EQUATION: $4x - 5 = 19$

SOLUTION:





Give the number for the ▥. Then give the number for x.

A $2x + 3 = 11$
$2x = ▥$
$x = ?$

C $2x + 4 = 24$
$2x = ▥$
$x = ?$

E $2x + 3 = 17$
$2x = ▥$
$x = ?$

B $4x - 3 = 17$
$4x = ▥$
$x = ?$

D $3x - 5 = 13$
$3x = ▥$
$x = ?$

F $25 = 5 + 5x$
$▥ = 5x$
$? = x$

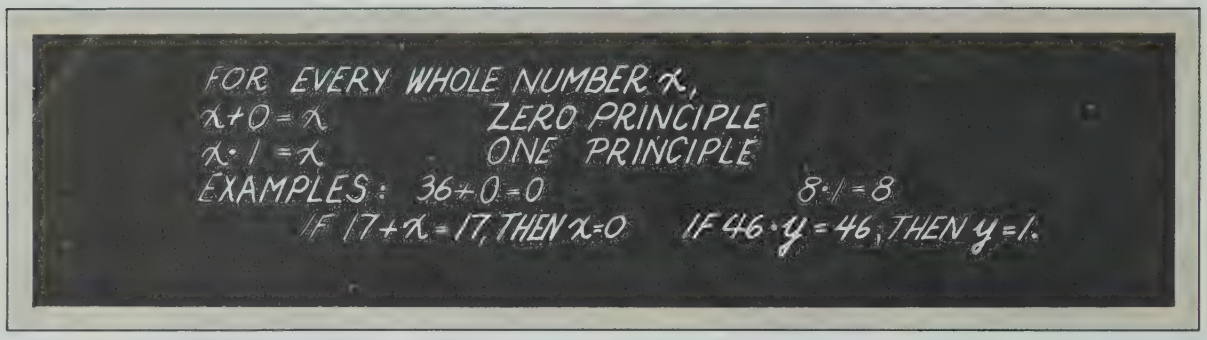
4. Solve the equations.

A $4x + 1 = 21$
B $3x + 7 = 37$
C $5x - 1 = 14$
D $8x + 3 = 3$
E $2x + 9 = 11$
F $6x - 5 = 31$

More practice, page S-6, Set 10

Investigating the Ideas

The zero principle and the one principle are examples of generalizations because they are true statements for every whole number.



How many of the these equations are generalizations?

A $x - x = 0$ **C** $x + x = 2x$ **E** $x \div 1 = x$

B $x \cdot 0 = 0$ **D** $x - 0 = x$ **F** $x \div x = 1$

Discussing the Ideas

1. Are there any equations in **A** through **F** above that are not generalizations? Explain.

2. **A** In equation **F**, if **x** is replaced by 0, you have $0 \div 0 = 1$. Is this a true statement?
   B Is equation **F** true if **x** is replaced by any whole number **except** 0?

3. Explain how to find the number for each screen. Think about the zero principle.

   If $21 + (x - 5) = 21$,
   then $x - 5 =$ ▒▒
   and then $x =$ ▒▒

4. Explain how to find the number for each screen. Think about the one principle.

   If $8 (x - 3) = 8$,
   then $x - 3 =$ ▒▒
   and then $x =$ ▒▒

1. Give the number for each ▒. Then find the number for x.

A $6 + (x - 1) = 6$
$x - 1 =$ ▒
$x =$?

B $7(3 - x) = 7$
$3 - x =$ ▒
$x =$?

C $4(x - 2) = 0$
$x - 2 =$ ▒
$x =$?

D $9(5 - x) = 9$
$5 - x =$ ▒
$x =$?

E $13 + (x - 11) = 13$
$x - 11 =$ ▒
$x =$?

F $7 - (x + 3) = 0$
$x + 3 =$ ▒
$x =$?

G $8 \div (x + 2) = 1$
$x + 2 =$ ▒
$x =$?

H $6x + 18 = 18$
$6x =$ ▒
$x =$?

I $(x - 8) - 3 = 0$
$x - 8 =$ ▒
$x =$?

J $(x + 3) \div 10 = 1$
$x + 3 =$ ▒
$x =$?

K $(x + 3) + 10 = 25$
$x + 3 =$ ▒
$x =$?

L $9(12 \div x) = 54$
$12 \div x = 6$
$x =$?

2. Copy and solve the equations.

A $17 - (8 - x) = 17$

B $6(x - 4) = 6$

C $3(x - 5) = 0$

D $7(3x - 6) = 0$

E $(x + 3) \div 1 = 10$

F $3(x - 5) = 3$

G $5(2x - 10) = 0$

H $9 + (2x - 12) = 9$

I $8 - (5 - x) = 7$

think

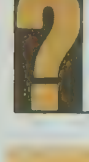
Try this Addition Mystery Trick on a friend. Ask your friend to write a four-digit number. You look at the number and then write the Mystery Sum on another piece of paper. Ask your friend to write two more four-digit numbers. Then you quickly write the last two numbers to give the Mystery Sum.

4728	Friend
3524	Friend
6953	Friend
6475	You
3046	You
24726	Mystery Sum

1. Study the example and explain the trick.

2. What would the Mystery Sum be if the first number were 2369?

Investigating the Ideas



Can you evaluate $2x + 3x$ for each of the numbers in the set $\{0, 1, 2, 3, 4, 5\}$?

$x = 0 \rightarrow 2x + 3x =$ ▮ $x = 2 \rightarrow 2x + 3x =$ ▮ $x = 4 \rightarrow 2x + 3x =$ ▮

$x = 1 \rightarrow 2x + 3x =$ ▮ $x = 3 \rightarrow 2x + 3x =$ ▮ $x = 5 \rightarrow 2x + 3x =$ ▮

Discussing the Ideas

1. Describe any pattern that you see in the set of numbers you found when you evaluated $2x + 3x$.

2. **A** Do you think $2x + 3x = 5x$ is a generalization for the set $\{0, 1, 2, 3, 4, 5\}$?
   B Do you think $2x + 3x = 5x$ is a generalization for all whole numbers?

3. The **distributive principle** can help you understand why $2x + 3x = 5x$ is a generalization. Use this principle to give the number for each screen.

   The Distributive Principle: $ba + ca = (b + c)a$

   $2x + 3x = (▮ + ▮)x = ▮x$

4. Use the distributive principle to give the number for ▮.
   A $6x + 2x = ▮x$ **C** $4x + 5x = ▮x$ **E** $1x + 4x = ▮x$
   B $3x + 7x = ▮x$ **D** $7x + 6x = ▮x$ **F** $8x + 3x = ▮x$

5. The distributive principle can be extended to more than two addends. Thus, $ax + bx + cx = (a + b + c)x$. Use this *extended form of the distributive principle* to find the sums.
   A $2x + 5x + 6x$ **B** $3x + 4x + 10x$ **C** $9x + 9x + 9x$

6. Multiplication also distributes over subtraction:
   $$ba - ca = (b - c)a$$
   A What is $5x - 3x$? **B** What is $10x - 4x$?

1. Find the sum or difference.

A $5x + 7x$
B $2x + 9x$
C $8x - 2x$
D $4x + 8x$
E $10x - 3x$
F $7x + 1x$
G $6x - 5x$
H $2x + 4x + 3x$
I $6x + 6x + 6x$
J $7x + 10x + 4x$
K $8x + 3x + 12x$
L $(7x - 2x) + 3x$

2. Since $x \cdot 1 = 1 \cdot x = x$ for all x, we often omit the numeral 1 before the variable. Find each sum or difference.

A $x + 2x$
B $x + x + x$
C $3x - x$
D $5x + x$
E $10x - x$
F $x + x + x + x$
G $x + x + 6x$
H $(x + 9x) - 2x$
I $(7x - 3x) - x$
J $x + (8x - 5x)$
K $x + 2x + 3x + 4x$
L $9x - (10x - x)$

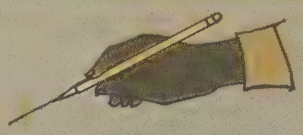
3. Give the expression for each ▥.

A $8x = 3x +$ ▥
ANSWER: $5x$
B $9x =$ ▥ $+ 5x$
C $2x +$ ▥ $= 10x$
D ▥ $+ 6x = 13x$
E $4x = 6x -$ ▥
F $9x -$ ▥ $= x$
G $x +$ ▥ $= 2x$
H $3x -$ ▥ $= 2x$
I $6x +$ ▥ $+ 3x = 15x$
J ▥ $+ x + 2x = 4x$
K $(5x -$ ▥$) + 2x = 6x$

4. Evaluate each expression.

A $7x + 2x$ when $x = 5$
B $4x + 6x$ when $x = 9$
C $x + 8x$ when $x = 7$
D $11x - 5x$ when $x = 8$
E $23x + 7x$ when $x = 4$
F $15x - x$ when $x = 10$
G $10x - 3x$ when $x = 2$
H $x + 5x$ when $x = 10$
I $9x - x$ when $x = 0$
J $10x - 10x$ when $x = 3$
K $5x - 4x$ when $x = 5$
L $(3x - 2x) + x$ when $x = 1$

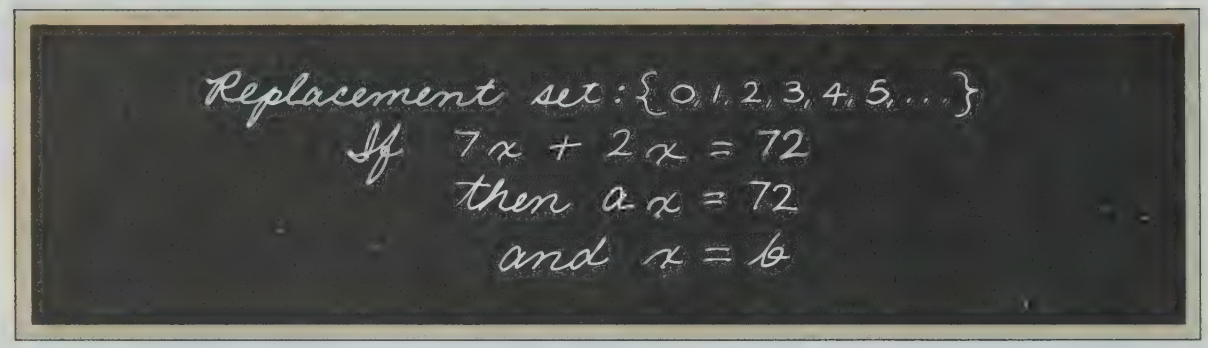
think



This strange "word" may seem not to have any meaning, but you can have **some math fun** by crossing out **five letters**. Can you do it?

Investigating the Ideas

Study the equations below.



Can you find the numbers for ***a*** and ***b***?

Discussing the Ideas

1. A How did you find the number for ***a***?
   B How did you find the number for ***b***?

2. Complete each step in the solution of this equation: $5x + 3x = 56$

$$8x = \blacksquare$$
$$x = \blacksquare$$

3. Explain each step in the solution of this equation:

$$9x - 4x = 40$$
$$5x = 40$$
$$x = 8$$

CHECK: $9 \cdot 8 - 4 \cdot 8 = 72 - 32 = 40$

4. Explain how the check of the solution was made.

5. Check the solution you found for the equation in the Investigation.

6. Explain or show how you would solve and check this equation. $(5x + x) + 1 = 19$

1. Copy the equations, giving the correct number for each ▒.

A $2x + 4x = 18$
$6x = 18$
$x =$ ▒

B $2x + 3x = 30$
$5x = 30$
$x =$ ▒

C $10x - 4x = 42$
$6x = 42$
$x =$ ▒

D $4x + 3x = 7$
▒$x = 7$
$x =$ ▒

E $5x - 2x = 12$
▒$x = 12$
$x =$ ▒

F $(2x + 3x) + 4x = 27$
$9x =$ ▒
$x =$ ▒

G $2x + 3x + 2 = 22$
$5x =$ ▒
$x =$ ▒

H $4x + x - 5 = 30$
$5x =$ ▒
$x =$ ▒

I $(11x - 4x) - 4 = 10$
$7x =$ ▒
$x =$ ▒

2. Copy and solve the equations.

A $3x + 4x = 14$

B $5x + 3x = 32$

C $x + 5x = 24$

D $7x - 6x = 5$

E $(8x - 3x) - 2x = 15$

F $11x - 4x = 14$

G $9x - 2x = 21$

H $x + (3x - 3) = 17$

I $5x + (2 + 3x) = 18$

3. Copy and solve the equations.

A $(2x + 3x) + 2 = 17$

B $4x - x = 9$

C $(3x + 4x) + 1 = 29$

D $(x + 5x) + 3 = 39$

E $(8x - 3x) - 2x = 15$

F $(10x - 5x) + 3 = 18$

G $6x = 15 + 3x$

H $8x = 2x + 30$

I $26 + 3x = 2 + 7x$

Think

The number 5 can be partitioned as a sum of whole numbers in the seven ways shown at the right. Do not count the different orders of addends due to commutativity, but include the number itself as one partition.

How many partitions does 6 have?
How many partitions does 7 have?

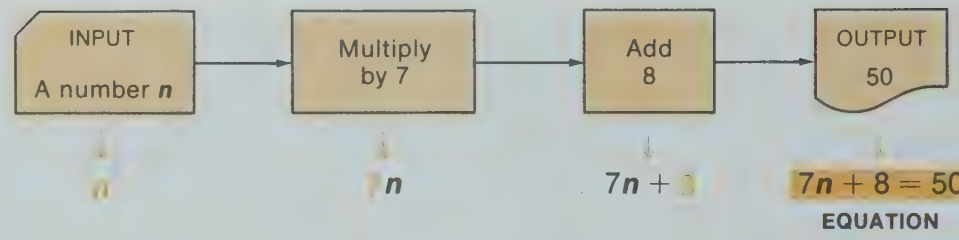
Partitions of 5

$1 + 1 + 1 + 1 + 1$
$1 + 1 + 1 + 2$
$1 + 1 + 3$
$1 + 2 + 2$
$1 + 4$
$3 + 2$
5

More practice, page S-7, Set 11

Investigating the Ideas

The flow chart below suggests an equation.

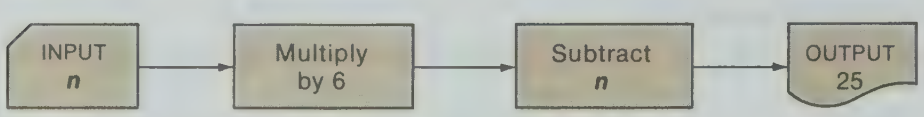


Can you write an equation for each of these flow charts?

A

B

C



Discussing the Ideas

1. The equation for the first flow chart above could be stated in words as, "If a certain number n is multiplied by 7 and then 8 is added to the product, the result is 50." How would you state the equation for flow charts A, B, and C?

2. A Make a flow chart suggested by the word problem below.

   If 3 is subtracted from a certain number and the difference multiplied by 7, the product will be 28.

   B Which equation is correct for the word problem?

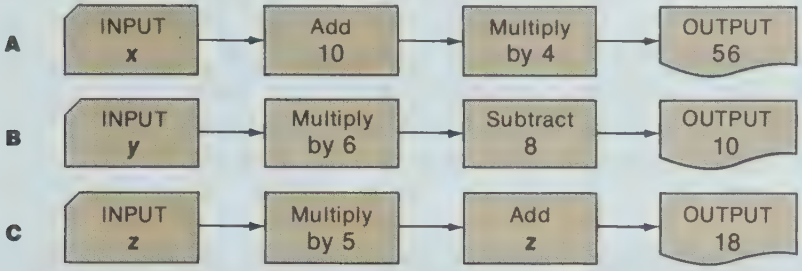
   A Let n = the number
   $(3 - n)\,7 = 28$

   B Let x = the number
   $(x - 3)\,7 = 28$

   C Let t = the number
   $t - 3 \cdot 7 = 28$

3. What are the solutions for the equations that you wrote for the flow charts in the Investigation?

1. Write the equation suggested by each flow chart.



2. Solve each equation for Exercise 1.

In Exercises 3 through 7, design a flow chart to illustrate each problem. Then write and solve an equation for the flow chart.

3. Five times a certain number plus 6 is 46. What is the number?

4. If 6 is subtracted from a certain number and the difference is multiplied by 10, the product is 70. What is the number?

5. Margaret said, "If you multiply my age by 3 and add 1, you will get my mother's age." If Margaret's mother is 40 years old, how old is Margaret?

6. Rex had 87 cents. After paying for five candy bars he had 12 cents left. How much did each bar cost?

7. Diane multiplied a certain number by 5. Karl multiplied the same number by 3. Then they added the two products together and got a sum of 56. What was the number?

8. Beth is 5 years older than Tina. The sum of their ages 10 years from now will be 53.

- **A** What is the sum of their ages now?
- **B** How old is each girl?

9. Dave is 3 years older than Chuck. When Chuck is as old as Dave is now, the sum of their ages will be 39. How old is each one now?

think

Mark four points on a piece of paper with no three of the points on the same line. Now connect each point to the other three so that six segments are formed. Show how you can arrange the points so that none of the segments cross each other.

More practice, page S-7, Set 12

Discussing the Ideas

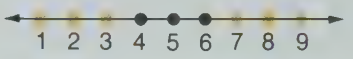
The examples in the display below will help you understand the logical use of the words "and" and "or" in algebra and how these two words are related to the idea of union and intersection of sets. The replacement set for these examples is {1, 2, 3, 4, 5, 6, 7, 8, 9}.

Inequality	Solution set
$x < 7$	$A = \{1, 2, 3, 4, 5, 6\}$
$x > 4$	$B = \{5, 6, 7, 8, 9\}$
$x > 4$ and $x < 7$	$A \cap B = \{5, 6\}$

Inequality	Solution set
$x < 4$	$C = \{1, 2, 3\}$
$x > 6$	$D = \{7, 8, 9\}$
$x < 4$ or $x > 6$	$C \cup D = \{1, 2, 3, 7, 8, 9\}$

1. The inequality $x > 4$ and $x < 7$ is often written as $4 < x < 7$. You can think of this as meaning "all numbers x from the replacement set that are between 4 and 7."
   What does each of these inequalities mean?
   A $1 < x < 5$ **B** $3 < x < 9$ **C** $7 > x > 2$

2. Does the set of points shown in color on the number line represent the solution set to inequality A or B?



   A $x < 4$ and $x > 6$ **B** $x < 4$ or $x > 6$

3. An inequality such as $x < 3$ or $x = 3$ is often denoted by the special symbol $\leq$ and written $x \leq 3$. It is read "x is less than or equal to 3." The symbol $\geq$ means "is greater than or equal to."
   A Using {1, 2, 3, 4, 5, 6, 7, 8, 9} as the replacement set, what is the solution set for $x \leq 3$?
   B What is the solution set for $x \geq 3$?

4. How could you write each inequality in a shorter way?
   A $x < 7$ or $x = 7$ **C** $x > 5$ or $x = 5$ **E** $x < 5$ and $x > 3$
   B $x < 8$ and $x > 6$ **D** $x > 1$ and $x < 7$ **F** $x = 6$ *or* $x > 6$

1. The replacement set for this exercise is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Give the solution set for each inequality.

A	$x < 8$ $A = \{ ? \}$	$x > 5$ $B = \{ ? \}$	$x < 8$ and $x > 5$ $A \cap B = \{ ? \}$
B	$x > 2$ $C = \{ ? \}$	$x < 9$ $D = \{ ? \}$	$x > 2$ and $x < 9$ $C \cap D = \{ ? \}$
C	$x < 3$ $E = \{ ? \}$	$x > 7$ $F = \{ ? \}$	$x < 3$ or $x > 7$ $E \cup F = \{ ? \}$
D	$x > 5$ $G = \{ ? \}$	$x = 2$ $H = \{ ? \}$	$x > 5$ or $x = 2$ $G \cup H = \{ ? \}$

2. Write the shortened version for each inequality.

A	$x > 2$ and $x < 10$	**C**	$x < 11$ and $x > 0$	**E**	$x = 9$ or $x < 9$
B	$x < 19$ and $x > 12$	**D**	$x > a$ and $x < b$	**F**	$x > 3$ or $x = 3$

3. Give the solution set for each inequality.
The replacement set is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

A	$5 < x < 10$	**D**	$x \geq 7$	**G**	$1 \leq x \leq 7$
B	$2 < x < 7$	**E**	$x \leq 4$	**H**	$5 \leq x \leq 7$
C	$x < 3$ or $x > 7$	**F**	$x \geq 6$	**I**	$8 \leq x \leq 9$

4. Give the solution set for each inequality. If the solution set is large, give only a description of it. Replacement set: All whole numbers.

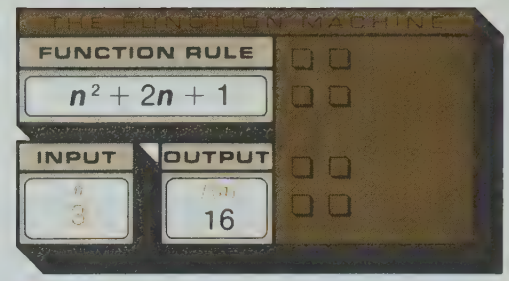
A	$23 < x < 31$	**D**	$10^0 \leq x \leq 3$	**G**	$3^5 < x < 7^3$
B	$4 \leq x \leq 19$	**E**	$10^2 < x < 10^3$	**H**	$24^2 < x < 26^2$
C	$4997 < x < 5002$	**F**	$2^3 < x < 3^2$	**I**	$3^3 \leq x \leq 3^4$

5. Find the solution set for each inequality.
Replacement set: All whole numbers.

A	$6 < x - 2 < 10$	**D**	$10 < x - 2 < 12$	**G**	$3 < x - 4 < 9$
B	$3 < x + 4 < 12$	**E**	$8 \leq 3x \leq 27$	**H**	$10 < 4 - x < 17$
C	$0 < 2x < 10$	**F**	$169 < x^2 < 225$	**I**	$0 < x - (x - 1) < 10$

Investigating the Ideas

For each **input number** n from a set of numbers, the function machine applied a **function rule** to produce an output number. The **output number** is denoted by $f(n)$. This symbol is read: "f of n." The function machine shows that when the input number is 3, the output number is $f(3) = 3^2 + 2 \cdot 3 + 1 = 9 + 6 + 1 = 16$.



Can you use this function rule to find the output numbers below?

A $f(1)$ **B** $f(2)$ **C** $f(10)$ **D** $f(15)$ **E** $f(21)$ **F** $f(42)$ **G** $f(96)$ **H** $f(100)$

Discussing the Ideas

1. You can show the input-output pairs for a function by a **function table**.

How would you complete this function table for the other input numbers in the Investigation?

Function Rule: $n^2 + 2n + 1$	
n	$f(n)$
1	4
2	9
3	16
4	
10	

2. What pattern can you see concerning the output numbers?

3. The function rule has odd numbers as inputs. Find the missing numbers in the table. What do you notice about the output numbers?

Function Rule: $(n^2 - 1) \div 8$	
n	$f(n)$
1	0
3	1
5	
7	
9	
11	

4. Invent a function rule of your own. Make a function table for your function.

In Exercises 1 through 6, copy and complete the function tables.

1.

Function Rule: $(n-1)^2$	
n	$f(n)$
7	36
3	4
A 10	
B 4	
C 1	

2.

Function Rule: $7n-n$	
n	$f(n)$
1	6
4	24
A 7	
B 12	
C 36	

3.

Function Rule: $3(5n-2)$	
n	$f(n)$
1	9
2	24
A 3	
B 4	
C 5	

4.

Function Rule: n^3-n^2	
n	$f(n)$
2	4
A 3	
B 5	
C 8	
D 10	

5.

Function Rule: $\frac{n^2-n}{n}$	
n	$f(n)$
3	2
A 4	
B 5	
C 10	
D 100	

★ **6.**

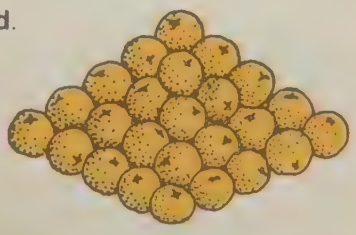
Function Rule: 2^n+1	
n	$f(n)$
1	3
2	5
A 3	
B 4	
C 5	
D 6	

★ **7.** What is the function rule for this table?

Function Rule: [illegible]	
n	$f(n)$
1	9
2	19
3	29
4	39
5	49

think

Dan piled some oranges in the shape of a **triangular pyramid**. The bottom layer was a triangle with five oranges on each side.



How many oranges were in the pile?

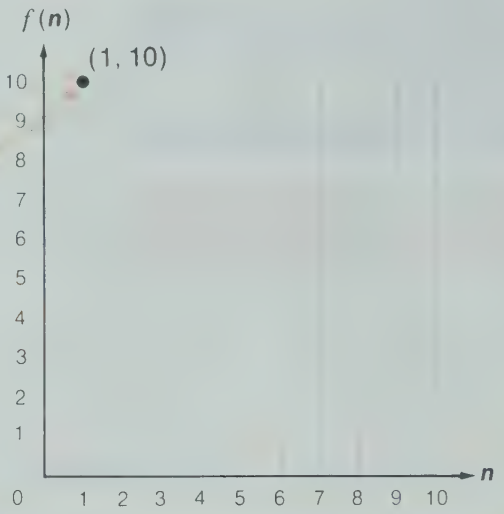
Investigating the Ideas

The input-output numbers for a function can be used as the **co-ordinates** of points in the plane. When you graph the points for the co-ordinates, you have **graphed** the function.

Function Rule

$12 - (2n)$

n	$f(n)$	Co-ordinates
1	10	→ (1, 10)
2	8	→ (2, 8)
3	[illegible]	→ (3, [illegible])
4	[illegible]	→ (4, [illegible])
5	[illegible]	→ (5, [illegible])
6	[illegible]	→ (6, [illegible])



Can you complete the function table and then graph the points for the function rule above?

Discussing the Ideas

1. Do the points for the input-output pairs of the function fall in a pattern on the graph? Describe the pattern.

2. A Find the missing numbers in the function table.

Function rule: $f(n) = 12 \div n$

n	1	2	3	4	6
$f(n)$	12	6	[illegible]	[illegible]	[illegible]

B Draw the graph for the pairs of this function.

C Do the points in the graph for the function fall on a line?

3. Invent a function rule of your own. Find some input-output pairs and then draw a graph of the function.

In Exercises 1, 2, and 3, complete the function tables and then draw the graph of each function.

1. Function Rule: $n^2 - n + 1$

	n	$f(n)$
	0	1
	1	1
A	2	
B	3	
C	4	

2. Function Rule: $(n^2 + 17) - 8n$

	n	$f(n)$
	1	10
A	2	
B	3	
C	4	
D	5	

3. Function Rule: $6n - n^2$

	n	$f(n)$
A	0	
B	1	
C	2	
D	3	
E	4	

4. The function given by the rule $f(x) = x$ is called an **identity function**. Draw a graph of this function using the whole numbers 0 through 10 as input numbers.

5. A **constant function** is a function whose output is just one number. Draw a graph of the constant function given by the rule $f(n) = 7$ using the whole numbers 1 through 8 as input numbers.

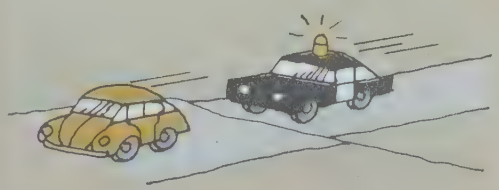
6. Air temperature can be thought of as a function of the time of day. Make a graph of the temperature function given in the table below.

Function Rule: Temperature

n (time)	$f(n)$ (temperature C)
11:00 A.M.	20
12:00 Noon	22
1:00 P.M.	25
2:00 P.M.	28
3:00 P.M.	29
4:00 P.M.	27

think

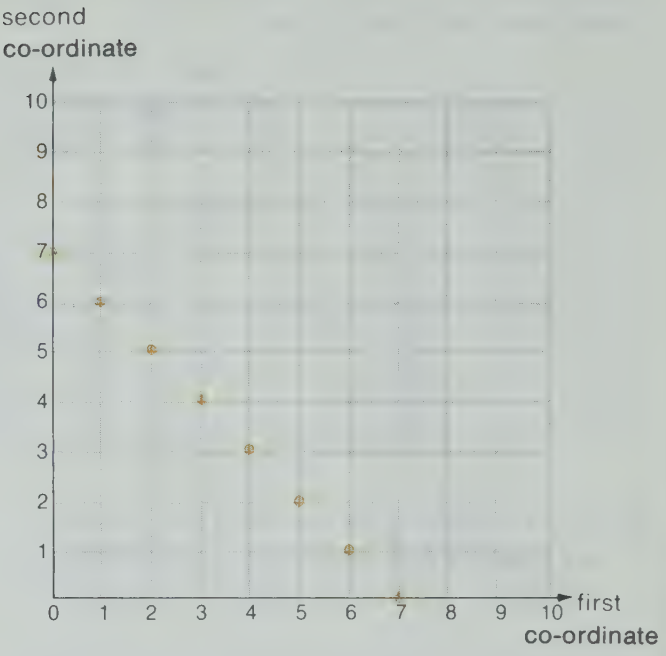
Suppose that in Speed City, four policemen can write four tickets for speeding in four minutes. If the entire police force can write eighty tickets in one hour and 20 minutes, how many persons are on the police force in Speed City?



Investigating the Ideas

The co-ordinates for the points shown graphed at the right are related in a special way.

Which one of the statements below best describes the relation between the numbers for the points?



A The first co-ordinate is greater than the second.

B The difference of the two co-ordinates in each pair is 1.

C The sum of the two co-ordinates in each pair is 7.

D The product of the two co-ordinates in each pair is 7.

Discussing the Ideas

1. Suppose x designates the first co-ordinate and y designates the second co-ordinate, then the points graphed in the Investigation consist of those ordered pairs of whole numbers (x, y) such that $x + y = 7$.

   For $(7, 0)$, we have $7 + 0 = 7$.
   For $(6, 1)$, we have $6 + 1 = 7$.
   For $(5, 2)$, we have $5 + 2 = 7$.

   What are some other replacements for x and y in the equation $x + y = 7$?

2. To draw the graph of the equation $2x - y = 10$, you can make a table of the pairs for x and y and then graph the points for the pairs.

   A Find the missing numbers in the table.

   B List the pairs (x, y) in the table.

   C Graph the pairs (x, y) in part **B**.

x	5	6	7	8	9	10
$2x$	10	12				
y	0	2				

1. Complete the table of numbers for the equation $x - y = 2$.

x	10	9	8	7	6	5	4	3	2
y	8	7							

2. Make a graph of the equation $x - y = 2$ using the pairs (x, y) in the table for Exercise 1.

3. Complete the table for the equation $x + 2y = 15$.

x	1	3	5	7	9
$2y$	14	12			
y	7	6			

4. Graph the equation $x + 2y = 15$ using the pairs (x, y) for Exercise 3.

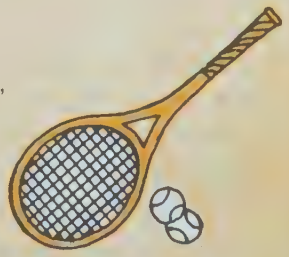
5. Complete the table for the equation $2x - 3 = y$.

x	2	3	4	5	6
$2x$	4	6	8		
y	1	3			

6. Graph the equation $2x - 3 = y$ using the pairs (x, y) for Exercise 5.

7. Make a table of pairs (x, y) for each equation. Then graph the equation.
   - A $x - y = 4$
   - B $x + y = 9$
   - C $x = y$
   - D $x + 2 = y$
   - E $x - y = 0$
   - F $2x - y = 0$

Think

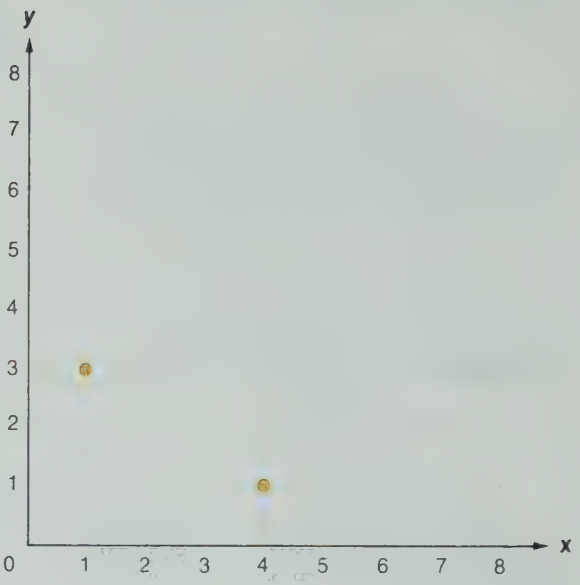


After a set of tennis, Chris said to Juanita, "If I had won three more games than I did, I would have beaten you by the same score that you beat me." "That's right," said Juanita, "but, of course, I won twice as many games as you did." How many games did each person win?

More practice, page S-8, Set 13

Investigating the Ideas

How many ordered pairs of whole numbers can you find which have a sum that is less than or equal to 5? Two such pairs are (1, 3) and (4, 1)



Can you mark the points for the ordered pairs on a co-ordinate grid?

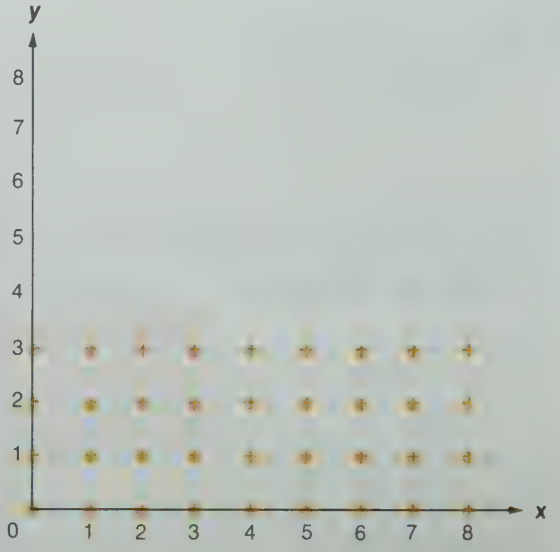
Discussing the Ideas

1. The set of all ordered pairs of whole numbers whose sum is less than or equal to 5 is the **solution set** for the inequality $x + y \leq 5$
   The pair (2, 1) is a **solution** because $2 + 1 \leq 5$ is **true**.
   The pair (5, 1) is not a solution because $5 + 1 \leq 5$ is **false**.
   How many ordered pairs are in the solution set for the inequality $x + y \leq 5$?

2. The graph at the right shows the set of all ordered pairs of whole numbers (x, y) such that $y < 4$. Any ordered pair in which the second number is less than 4 is a solution for this inequality.

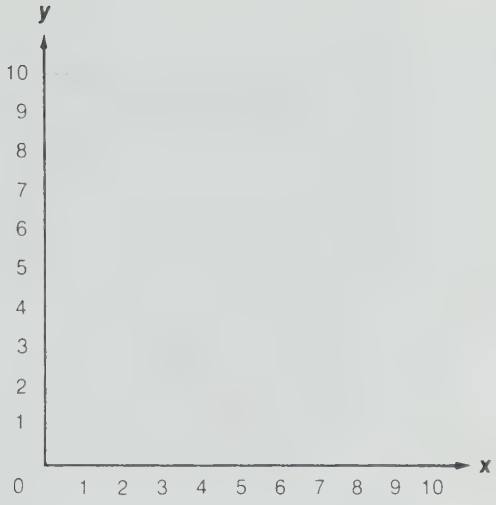
   A What does the graph of $x < 4$ look like?

   B What pairs (x, y) are in the solution set for $x < 4$ **and** $y < 4$?



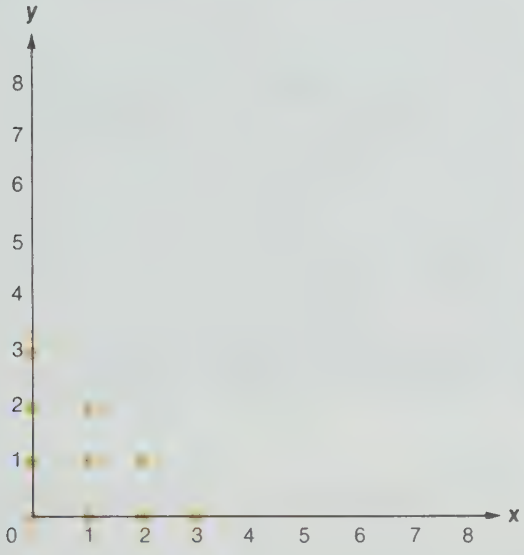
1. Draw graphs of these inequalities. The replacement set for x and y is the set of whole numbers.

- **A** $x + y \leq 7$
- **B** $x + 2y < 6$
- **C** $x < 5$
- **D** $y < 5$
- **E** $x < 5$ and $y < 5$
- **F** $x - y < 2$
- **G** $x < 4$ and $x + y < 7$
- **H** $x + y < 4$ and $x + y < 8$

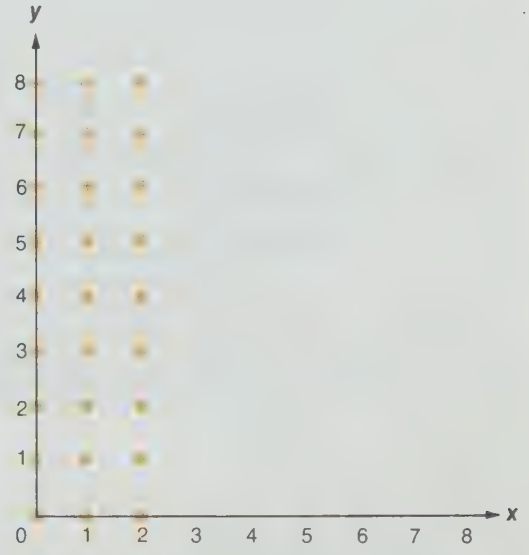


2. Write an inequality for the set of points shown on each graph.

A



B



☆ **3.** Draw a graph of all ordered pairs of whole numbers x and y such that $2 < x < 6$ and $3 < y < 7$.

☆ **4.** Draw a graph of all ordered pairs of whole numbers x and y such that $x = 5$ and $4 \leq y \leq 9$.

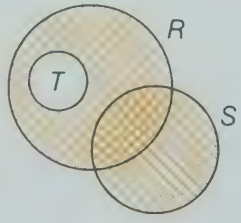
REVIEWING THE IDEAS

1. $X = \{0, 2, 4, 6\}$ and $Y = \{0, 1, 2, 3, 4, 5, 6\}$

A Is $2 \in X$? **C** Is $5 \in X$?

B Is $6 \in Y$? **D** Is $X \subset Y$?

2. Study the Venn diagram, then decide whether each statement is true or false.



A $T \subset R$

B $T \subset S$

C $T \cup R = R$

D $S \cap T = \phi$

E $R \cap T = R$

3. $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, and $C = \{1, 3\}$. Find the elements in each of the following sets.

A $A \cup B$ **C** $A \cup C$ **E** $B \cup C$

B $A \cap B$ **D** $A \cap C$ **F** $B \cap C$

4. Draw a Venn diagram which shows the relationships between the three sets in Exercise 3.

5. Evaluate $(n + 7) \div 4$ when $n = 17$.

6. Evaluate $5(x^2 + 1)$ when $x = 3$.

7. Find the solution set for each open sentence. Use $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ as the replacement set.

A $x + 17 = 26$ **C** $x < 4$

B $6n = 48$ **D** $y + 2 > 9$

8. Is $x \cdot 1 = x$ a generalization for the set of whole numbers?

9. Solve the equations.

A $10 + (n - 7) = 10$

B $5(4 - x) = 5$

10. Give the correct number for each ▒.

A $3r + 8r = ▒r$ **B** $17m - 8m = ▒m$

11. Give the correct number for each ▒ in the equations.

A $5x + 6x = 44$

$▒x = 44$

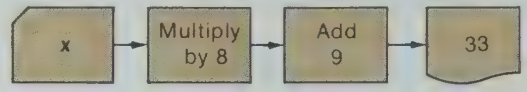
$x = ▒$

B $6x - 2x = 20$

$▒x = 20$

$x = ▒$

12. Write an equation for this flow chart.



13. Solve the equation for Exercise 12.

14. Write and solve an equation for this problem:

A certain number is added to 5 and the sum is multiplied by 4. The product is 48. What is the number?

15. Using $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ as the replacement set, find the solution set for each part.

A $x < 8$ and $x > 5$ **C** $x < 5$ or $x > 5$

B $x < 2$ or $x > 9$ **D** $3 \leq x \leq 8$

16. **A** Complete the function table.

B Draw a graph of the function.

	Function Rule $n^2 + 3$	
	n	$f(n)$
A	0	▒
B	1	▒
C	2	▒
D	3	▒

17. Draw a graph of the equation $2x - y = 1$.

18. Graph the ordered pairs of whole number (x, y) such that $x \leq 4$ and $y \leq 3$.

TEST YOURSELF

1. $C = \{0, 1, 2, 3\}$ and $D = \{1, 3, 5, 7\}$

A List the elements in $C \cup D$.

B List the elements in $C \cap D$.

2. Give the number for the ▒.

A If $x = 4$, then $x + 7 =$ ▒.

B If $y = 7$, then $2y - 5 =$ ▒.

3. What is the solution set for the open sentence $x < 5$? The replacement set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

4. Solve the equations using the set of whole numbers as the replacement set.

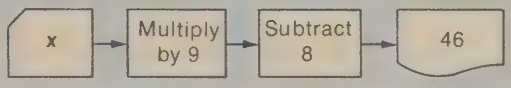
A $4(x - 5) = 4$

B $9 + (x - 3) = 9$

5. Solve the equations. The replacement set is the set of whole numbers.

A $5x + 2x = 35$ **B** $9x - x = 56$

6. Write an equation for this flow chart:



7. Write and solve an equation for this problem:

If 3 is subtracted from a certain number and the difference is multiplied by 8, the product is 16. What is the number?

8. Suppose the replacement set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, what is the solution set for each part?

A $x < 3$ or $x > 8$ **C** $x \leq 6$

B $x > 4$ and $x < 9$ **D** $x > 10$

9. Complete the function rule table.

	Function Rule $4n^2 - 3$
	▒
	2
A	3
B	4
C	5

10. Give the missing numbers in the table for the equation $3x + y = 12$.

x	0	1	2	3	4
y	12	9	**A**	**B**	**C**

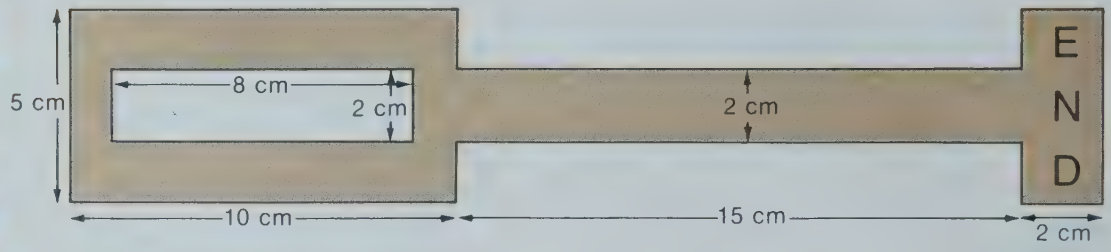
RESEARCH PROJECTS

A The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, . . ., has many interesting properties. Find some of them and make a report to the class. (See *MATHEMATICS, A Human Endeavor* by Harold R. Jacobs; San Francisco: W. H. Freeman and Company, 1970.)

B One application of set theory and logic involves switching networks. Read about these, and make and interpret some of your own. (See *Mathematics and the Modern World* by Mario F. Triola; Menlo Park, California: Cummings Publishing Company, 1973; Addison-Wesley.)

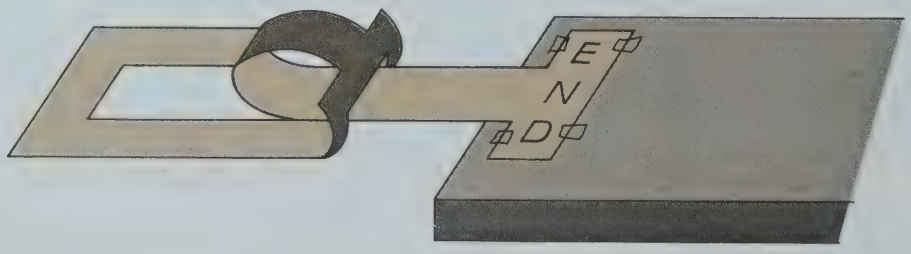
A TOPOLOGICAL PUZZLE

Topology is a kind of geometry which concerns those properties of figures which are not changed when the figures are twisted, stretched, or deformed in any way except by cutting or tearing. The figure below can be used for an interesting topological puzzle.



Cut out a shape like this from a strip of paper. (The dimensions do not have to be exactly the same as shown.)

It is easy to put the END through the hole in the strip to form the shape below.



Here is the puzzle:

Try to make the strip look like this when the END is first taped to the edge of a table.

TEST YOURSELF Answers

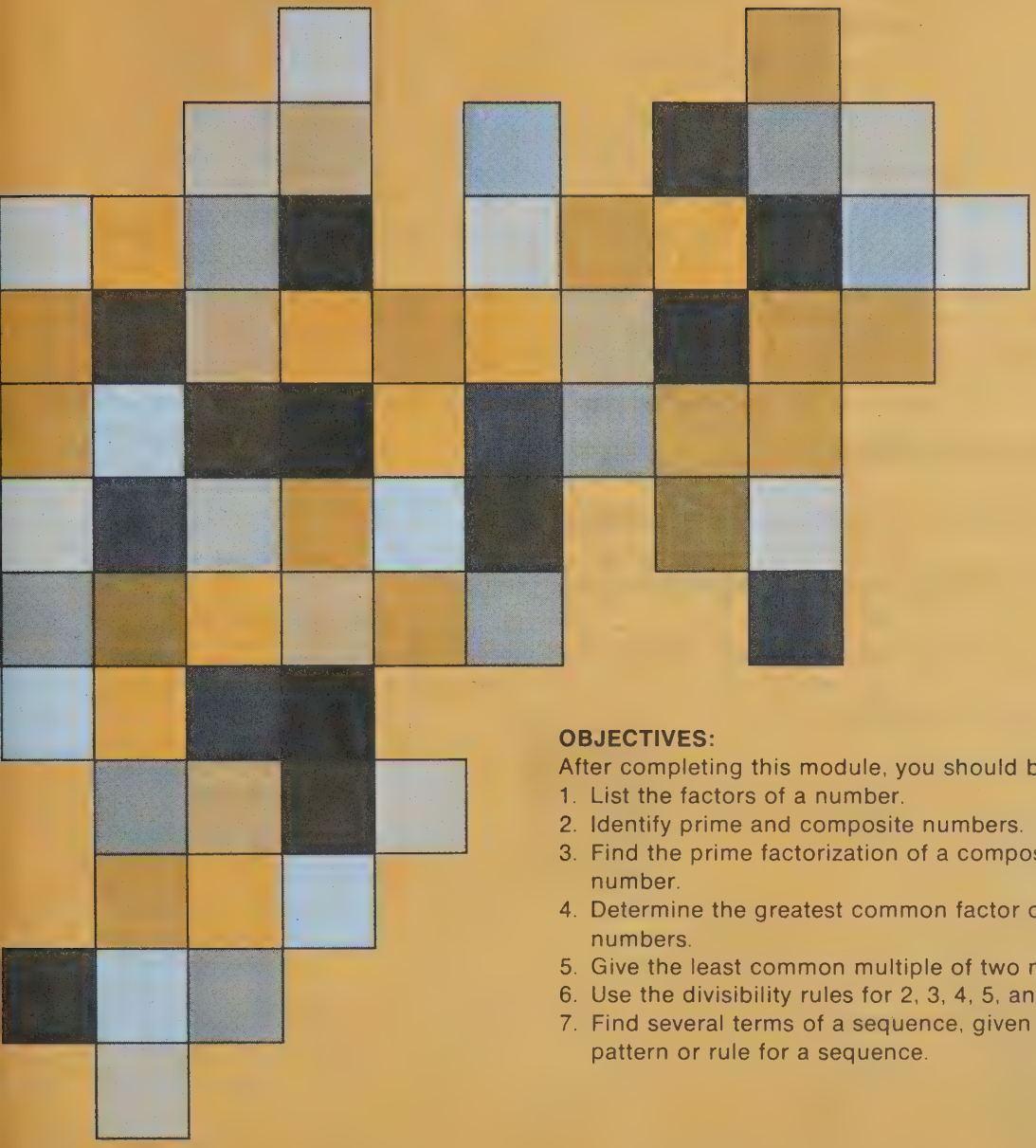
1. A $\{0, 1, 2, 3, 5, 7\}$; B $\{1, 3\}$ **2.** A 11; B 9 **3.** $\{0, 1, 2, 3, 4\}$

4. A $x = 6$; B $x = 3$ **5.** A $x = 5$; B $x = 7$ **6.** $9x - 8 = 46$

7. $8(x - 3) = 16$; $x = 5$ **8.** A $\{0, 1, 2, 9, 10\}$; B $\{5, 6, 7, 8\}$; C $\{0, 1, 2, 3, 4, 5, 6\}$; D ϕ **9.** A 33; B 61; C 97

10 A 6; B 3; C 0

UNIT E: *Integers*
MODULE 3: *Number Theory*



OBJECTIVES:
After completing this module, you should be able to:

1. List the factors of a number.
2. Identify prime and composite numbers.
3. Find the prime factorization of a composite number.
4. Determine the greatest common factor of two numbers.
5. Give the least common multiple of two numbers.
6. Use the divisibility rules for 2, 3, 4, 5, and 9.
7. Find several terms of a sequence, given the pattern or rule for a sequence.

Investigating the Ideas

If a, b, and c are whole numbers and $a \cdot b = c$, then a and b are **factors** of c.

2 and 7 are factors of 14 because $2 \cdot 7 = 14$

1 and 14 are factors of 14 because $1 \cdot 14 = 14$

No other whole numbers are factors of 14.
The set of all factors of 14 is $\{1, 2, 7, 14\}$

Number	Factors
1	{1}
2	{1, 2}
3	{1, 3}
4	{1, 2, 4}
⋮	⋮
20	⋮

Can you find the set of all factors of each whole number from 1 through 20?

Discussing the Ideas

1. What number is a factor of every number?

2. Which numbers from 1 through 20 had six factors?

3. Which numbers from 1 through 20 had only two factors?

4. **A** Is every whole number a factor of zero?
   B Is zero a factor of any whole number? Which one?

$0 \cdot 0 = 0$
$0 \cdot 1 = 0$
$0 \cdot 2 = 0$
$0 \cdot 3 = 0$
⋮

5. **A** Is 43 a factor of 731? Why?
   B Is 17 a factor of 918? Explain.
   C Is 17 a factor of 497? Explain.

$$17\overline{)731}\quad 43,\ 68,\ 51,\ 51,\ 0$$

6. The division shows that 29 is not a factor of 843. Can you decide if 29 is a factor of 841 without dividing? Explain.

$$29\overline{)843}\quad 29,\ 58,\ 263,\ 261,\ 2$$

1. Solve each equation.

A $1 \cdot x = 84$ **C** $3 \cdot z = 84$ **E** $6 \cdot s = 84$

B $2 \cdot y = 84$ **D** $4 \cdot r = 84$ **F** $7 \cdot t = 84$

2. Use the results of Exercise 1 to list all the factors of 84.

3. The whole number 9 has exactly three factors: 1, 3, and 9. Find a number smaller than 9 that also has exactly three factors.

4. Find the smallest number with exactly four factors.

5. Give the set of all factors of each number.

A 70 **B** 66 **C** 48 **D** 150 **E** 420

6. Use division, if necessary, to answer each question **yes** or **no**.

A Is 7 a factor of 91?

B Is 12 a factor of 148?

C Is 37 a factor of 111?

D Is 89 a factor of 3115?

E Is 123 a factor of 1371?

F Is 143 a factor of 1001?

7. The number 6 is called a **perfect number** because the sum of its factors other than itself is 6: $1 + 2 + 3 = 6$.

A Another perfect number is found between 20 and 30. What is this number?

B Another perfect number is found between 495 and 500. What is this number?

8. A number is known to have 2, 3, and 5 among its factors. Give five more factors of this number.

9. **A** The number $1365 = 3 \cdot 5 \cdot 7 \cdot 13$ has 16 factors. What are these factors?

B The number 2310 has 32 factors. What are these factors?

think

John has a total of ten pockets in his pants, shirt, and jacket.

Can he place 44 pennies in his pockets so that each pocket has a different number of pennies?

Investigating the Ideas

The set of whole numbers can be separated into different subsets according to the number of factors each number has. The table at the right shows that 0 has every whole number as a factor. The number 1 has only one factor. Numbers such as 2, 3, and 5 have exactly two factors. Other numbers have more than two factors.

Whole number	Set of all factors
0	{0, 1, 2, 3, 4, 5, 6, . . .}
1	{1}
2	{1, 2}
3	{1, 3}
4	{1, 2, 4}
5	{1, 5}
6	{1, 2, 3, 6}
⋮	⋮

How many numbers less than 80 can you find that have exactly two factors?

Discussing the Ideas

1. The whole numbers which have exactly two factors are **prime numbers**. How many prime numbers are there that are less than 80?

2. Is 57 a prime number? Why or why not?

3. A whole number, other than 0, is **composite** if it has more than two factors.
   - **A** Which numbers in the table are composite?
   - **B** Which are prime?

	Whole number	Set of all factors
A	51	{1, 3, 17, 51}
B	109	{1, 109}
C	187	{1, 11, 17, 187}
D	1001	{1, 7, 11, 13, 77, 91, 143, 1001}
E	1009	{1, 1009}

4. **A** Is 0 either a prime or a composite number?
   B Is 1 either a prime or a composite number?

5. Are there any even prime numbers?

6. Which numbers in the set below are not prime numbers?
   {7, 17, 27, 37, 47, 57, 67, 77, 87, 97}

Eratosthenes, a Greek mathematician who lived over 2000 years ago, invented a "sieve" for finding prime numbers. The exercises below illustrate the **Sieve of Eratosthenes**.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

1. Make a table of whole numbers 1 through 100 in six columns.

2. The first number, 1, is not prime. Mark it out. The number 2 is the smallest prime. Circle it. Then strike out all other numbers that have 2 as a factor. What is the first number greater than 2 to eliminate from the table?

1	2	3	4	5	6
7	8	9	10	11	12

3. **A** Circle the first number after 2 that has not been eliminated before. What prime number is it?

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	2		
25	26	27			

B Now strike out all the numbers that have 3 as a factor. Some of these numbers will have been eliminated before. What is the first number to be eliminated from the table twice? What are the prime factors of this number?

C What is the first "new" number to be eliminated from the table?

D What is the next number larger than 3 that has not been eliminated from the table?

4. **A** Circle the number 5 in the table, for it is prime, and then strike out each of the numbers that has 5 as a factor.

B What is the first new number eliminated from the table this time?

5. Continue the process above until you reach a point where no new number is eliminated from the table. The numbers remaining are the primes less than 100.

★ **6.** Extend the sieve to find all primes less than 500.

Investigating the Ideas

The table below gives the first 199 prime numbers. Using this table, you can find that the 43rd prime is 191, for it is found in the row headed "4" and the column headed "3."

Can you use the table of prime numbers to help you complete the chart below?

A short table of prime numbers

	0	1	2	3	4	5	6	7	8	9
0		2	3	5	7	11	13	17	19	23
1	29	31	37	41	43	47	53	59	61	67
2	71	73	79	83	89	97	101	103	107	109
3	113	127	131	137	139	149	151	157	163	167
4	173	179	181	191	193	197	199	211	223	227
5	229	233	239	241	251	257	263	269	271	277
6	281	283	293	307	311	313	317	331	337	347
7	349	353	359	367	373	379	383	389	397	401
8	409	419	421	431	433	439	443	449	457	461
9	463	467	479	487	491	499	503	509	521	523
10	541	547	557	563	569	571	577	587	593	599
11	601	607	613	617	619	631	641	643	647	653
12	659	661	673	677	683	691	701	709	719	727
13	733	739	743	751	757	761	769	773	787	797
14	809	811	821	823	827	829	839	853	857	859
15	863	877	881	883	887	907	911	919	929	937
16	941	947	953	967	971	977	983	991	997	1009
17	1013	1019	1021	1031	1033	1039	1049	1051	1061	1063
18	1069	1087	1091	1093	1097	1103	1109	1117	1123	1129
19	1151	1153	1163	1171	1181	1187	1193	1201	1213	1217

Numbers between	How many primes?
0 and 100	
100 and 200	
200 and 300	
300 and 400	
400 and 500	
500 and 600	
600 and 700	
700 and 800	
800 and 900	
900 and 1000	
1000 and 1100	
1100 and 1200	

Discussing the Ideas

1. Between which pair of numbers in the chart did the greatest number of primes occur? How many?

2. What was the least number of primes occurring between any pair of the numbers in the chart?

3. Using the table of primes, find the following:
   - **A** 125th prime
   - **B** 11th prime
   - **C** 147th prime
   - **D** 53rd prime
   - **E** 80th prime
   - **F** 182nd prime
   - **G** 127th prime
   - **H** 155th prime

4. Using the table, determine the relative position of each of the following primes. EXAMPLE: 89 is the 24th prime.
   A 431 **B** 719 **C** 101 **D** 607 **E** 1181 **F** 523 **G** 157

1. Use the table of primes to find the following:

A 22nd prime **B** 41st prime **C** 75th prime **D** 92nd prime

2. Is 279 a prime? If not, list all of its factors.

3. Using the table, give the relative position of each prime.

A 311 **B** 461 **C** 11 **D** 659 **E** 1213 **F** 991

4. Which numbers in the set {1, 13, 113, 1113} are prime?

5. Which numbers in this set are prime?
{101, 201, 301, 401, 501, 601, 701, 801, 901, 1001, 1101, 1201}

6. Pairs of primes, such as 11 and 13 or 29 and 31, are called **twin primes** because they have a difference of 2.

A List all the pairs of twin primes that are less than 100.

B How many pairs of twin primes are there between 1000 and 1100?

C Show that the prime 1217 does not have a twin.

Think

List all the whole numbers that are less than 60 and have an odd number of factors. What property does the set of numbers have?

7. There are three consecutive odd numbers which are prime. What are these numbers?

8. The three primes 11, 13, and 17 are three consecutive primes. They have the form p, $p + 2$, and $p + 6$ where p stands for a prime. Find another set of primes with this form.

9. For each input number n, the function machine multiplies the input number by 1 less than the input number and then adds 11 to the product.

A Copy and complete the function table.

B Are all of the output numbers $f(n)$ prime numbers?

Function Rule

$n\ (n - 1) + 11$

n	$f(n)$	n	$f(n)$
1	11	7	
2	13	8	
3		9	
4		10	
5		11	
6		12	

10. Using the function rule $n\ (n - 1) + 17$ and the input numbers 1 through 20, find the numbers for $f(n)$. In each case, do you get a prime number for $f(n)$?

Investigating the Ideas

The **factor tree** at the right shows that 48 can be expressed as the product $2 \cdot 3 \cdot 2 \cdot 2 \cdot 2$.

Can you complete these other factor trees for 48?

A

B

C

Discussing the Ideas

1. A Are all of the factors in the top row of each factor tree for 48 prime numbers?
   B Except for the order in which the factors occur, are the same prime factors found in the top row of each tree?

2. The factor trees illustrate an important theorem in mathematics called the **Fundamental Theorem of Arithmetic**. It is also known as the **Unique Factorization Theorem**.

   > Each composite number can be expressed as a product of primes in exactly one way, except for the order of the factors.

   Make a factor tree for 36. What are the prime factors of 36?

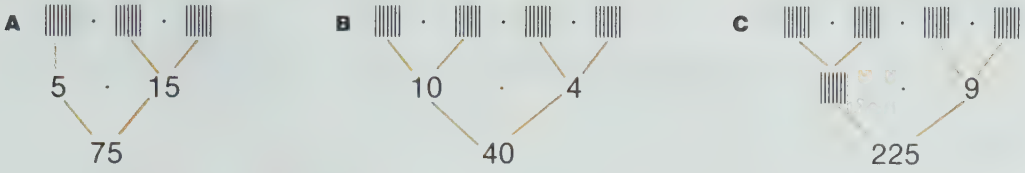
3. When a number is written as a product of prime numbers, we say that we have given a **prime factorization** of the number, or that we have **factored the number completely**. Give the missing number for each ▒ so that a prime factorization of each number is found.

   A $14 = 2 \cdot$ ▒
   B $20 = 5 \cdot 2 \cdot$ ▒
   C $26 = 2 \cdot$ ▒
   D $8 = 2 \cdot 2 \cdot$ ▒
   E $18 = 3 \cdot 2 \cdot$ ▒
   F $30 = 2 \cdot 3 \cdot$ ▒
   G $42 = 2 \cdot 3 \cdot$ ▒
   H $27 = 3 \cdot 3 \cdot$ ▒
   I $44 = 2 \cdot 2 \cdot$ ▒

4. Factor each number completely. Use a factor tree if necessary.

   A 60 B 42 C 140 D 300 E 625

1. Complete each factor tree.



2. Factor each composite number completely.

A 15	**C** 22	**E** 50	**G** 45	**I** 81	**K** 66
B 12	**D** 39	**F** 70	**H** 16	**J** 60	**L** 100

3. Each composite number is written as the product of two factors. Use these factors to help you give prime factorizations of the numbers. EXAMPLE: $140 = 4 \cdot 35$

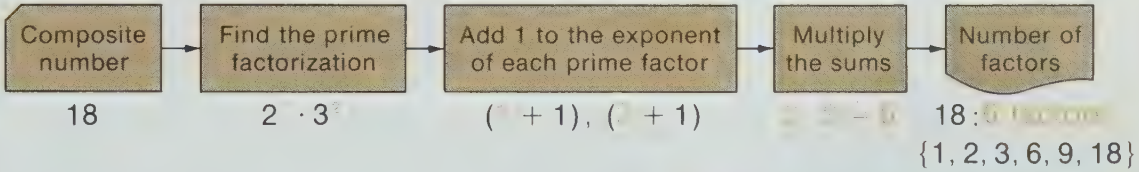
$140 = 2 \cdot 2 \cdot 5 \cdot 7$

A $98 = 2 \cdot 49$	**E** $196 = 14 \cdot 14$	**I** $240 = 15 \cdot 16$
B $63 = 9 \cdot 7$	**F** $150 = 10 \cdot 15$	**J** $189 = 9 \cdot 21$
C $60 = 6 \cdot 10$	**G** $132 = 11 \cdot 12$	**K** $144 = 16 \cdot 9$
D $156 = 12 \cdot 13$	**H** $72 = 9 \cdot 8$	**L** $490 = 10 \cdot 49$

4. Use exponents to give the prime factorizations of the numbers. For example, $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$.

A 24	**D** 81	**G** 120	**J** 484	**M** 441
B 72	**E** 216	**H** 48	**K** 288	**N** 612
C 250	**F** 1000	**I** 108	**L** 225	**O** 1089

★ **5.** The flow chart below can help you find the number of factors of any composite number.



Give the number of factors of each number.

A $24 = 2^3 \cdot 3^1$	**D** $72 = 2^3 \cdot 3^2$
B $30 = 2^1 \cdot 3^1 \cdot 5^1$	**E** $1000 = 2^3 \cdot 5^3$
C $100 = 2^2 \cdot 5^2$	**F** $1001 = 7^1 \cdot 11^1 \cdot 13^1$

★ **6.** List the set of all factors of the numbers given in Exercise 5.

think

1. Divide the figure into three pieces that are exactly alike.
2. Divide the figure into four pieces that are exactly alike.

More practice, page S-8, Set 14

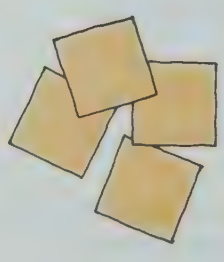
Investigating the Ideas

Jody's desk top is a rectangle 84 centimetres long by 56 centimetres wide. She wants to cover the desk top exactly with squares. Of course, 1 by 1-centimetre squares could be used or 2 by 2-centimetre squares would work also.

What is the largest size square that could be used to completely cover the desk top?

Discussing the Ideas

1. Peggy said, "The squares that fit must have sides whose lengths are factors of both 56 and 84."
   - **A** List the factors of 56.
   - **B** List the factors of 84.
   - **C** What are the common factors of 56 and 84?
   - **D** What is the largest factor common to 56 and 84?



2. The **greatest common factor (GCF)** of two numbers is the largest number that is a common factor of both numbers.

   Study the information at the right. What is the GCF of 18 and 24?

The set of factors of 18:	$A = \{ \_, \_, \_, \_, 9, 18\}$
The set of factors of 24:	$B = \{ \_, 2, 3, 4, \_, 8, 12, 24\}$
The set of common factors of 18 and 24:	$A \cap B = \{ \_, 2, 3, 6\}$
The GCF of 18 and 24 is ▒.	

3. To find the lowest-terms fraction for a given fraction, divide the numerator and the denominator by their GCF.

   EXAMPLE: $\frac{18}{24}$ The GCF of 18 and 24 is ▒. $\frac{18 \div \_}{24 \div \_} = \frac{3}{4}$ ← **Lowest-terms fraction**

   What is the GCF of the numerator and denominator of each fraction?

   A $\frac{12}{18}$ **B** $\frac{9}{12}$ **C** $\frac{50}{60}$ **D** $\frac{15}{25}$ **E** $\frac{12}{14}$

1. **A** List the set of factors of 12.
B List the set of factors of 18.
C Find the intersection of the sets in Parts **A** and **B**.
D What is the GCF of 12 and 18?

2. **A** List the set of factors of 36.
B List the set of factors of 54.
C Find the intersection of the sets in Parts **A** and **B**.
D What is the GCF of 36 and 54?

3. Find the GCF of each pair.

A 12, 20	**C** 21, 56	**E** 12, 16	**G** 14, 63	**I** 34, 51
B 8, 18	**D** 30, 45	**F** 20, 30	**H** 24, 72	**J** 180, 250

4. What is the GCF of the numerator and the denominator of each fraction?

A $\frac{10}{25}$ **B** $\frac{16}{24}$ **C** $\frac{27}{30}$ **D** $\frac{24}{72}$ **E** $\frac{56}{98}$ **F** $\frac{48}{120}$

5. If the GCF of two whole numbers is 1, we say that the numbers are **relatively prime**. Using this definition, decide which pairs of numbers are relatively prime. If the numbers are not relatively prime, give the GCF.

A 4, 15	**C** 8, 27	**E** 20, 33	**G** 26, 51	**I** 249, 303
B 15, 18	**D** 14, 15	**F** 33, 144	**H** 87, 74	**J** 221, 247

6. The examples show how prime factorizations can be used to find the GCF of a pair of numbers.

EXAMPLES:

A
$20 = \quad \cdot 5$
$28 = \quad \cdot 7$
$\text{GCF} = 2 \cdot 2 = 4$

B
$60 = 2 \cdot \quad \cdot 5$
$126 = \quad \cdot 3 \cdot 7$
$\text{GCF} = \quad = \quad$

Give the GCF of each pair.

A $30 = 2 \cdot 3 \cdot 5$, $105 = 3 \cdot 5 \cdot 7$
B $44 = 2 \cdot 2 \cdot 11$, $42 = 2 \cdot 3 \cdot 7$
C $14 = 2 \cdot 7$, $84 = 2 \cdot 2 \cdot 3 \cdot 7$
D $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, $858 = 2 \cdot 3 \cdot 11 \cdot 13$
E $540 = 2^2 \cdot 3^3 \cdot 5$, $90 = 2 \cdot 3^2 \cdot 5$
F $55\,125 = 3^2 \cdot 5^3 \cdot 7^2$, $514\,080 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 17$

Think

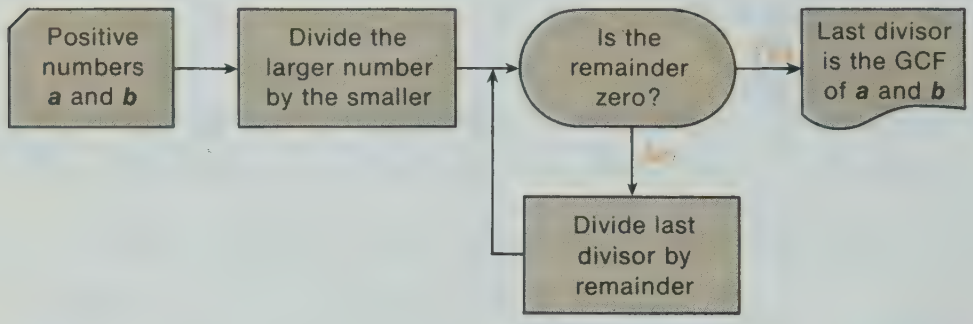
Use the digits 2, 3, and 7 to write all the 3-digit numerals you can with none of the digits repeated in the same numeral. Then find the GCF of these numbers.

More practice, page S-9, Set 15

Investigating the Ideas

If you wish to find the GCF of two rather large numbers, you may find the problem difficult. The flow chart below shows the steps of the **Euclidean Algorithm**, a method which simplifies the problem of finding the GCF of two numbers.

EUCLIDEAN ALGORITHM FLOW CHART


Positive numbers *a* and *b*
Divide the larger number by the smaller
Is the remainder zero?
Last divisor is the GCF of *a* and *b*
Divide last divisor by remainder

Can you use the Euclidean Algorithm Flow Chart to find the GCF of 1104 and 2507?

Discussing the Ideas

1. Explain how to use the flow chart when $a = 12$ and $b = 8$.

2. What is the GCF of 12 and 8? Did you get this number as the output number for the flow chart?

3. The repeated division at the right shows a shortcut way of arranging the division problems in order to find the GCF of 84 and 308.

   A Explain what was done in each step.

   B What is the GCF of 84 and 308?

A
$$84\overline{)308}\quad 3$$
$$252$$
$$56$$

B
$$56\overline{)84}\quad 1$$
$$56$$
$$28$$

C
$$28\overline{)56}\quad 2$$
$$56$$
$$0$$

4. Find the GCF of 141 and 376 using the Euclidean Algorithm.

1. Find the GCF of each pair by using the Euclidean Algorithm.

A 14, 210	**D** 135, 351	**G** 368, 80	**J** 475, 1501
B 184, 69	**E** 70, 525	**H** 259, 888	**K** 7560, 1232
C 115, 391	**F** 64, 128	**I** 2599, 2825	**L** 38 127; 31 737

2. Prove that 340 and 819 are relatively prime; that is, show that their GCF is 1 using the Euclidean Algorithm.

3. Any two consecutive whole numbers such as 8 and 9 or 342 and 343 are relatively prime.

A What is the first remainder using the Euclidean Algorithm?

B What is the second remainder?

4. Are 1001 and 100 001 relatively prime? What is their GCF?

5. There are 256 seventh grade students at Edishtron Junior High School. There are 224 eighth grade students in the school also. The principal of the school arranged the homerooms so that both grades have the same number of students in each homeroom. Furthermore, the number of students in each homeroom is as large as possible.

A How many students are in each homeroom?

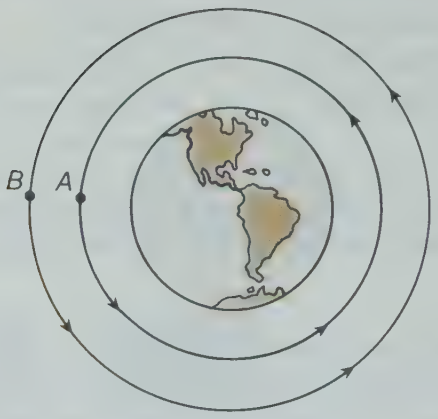
B How many homerooms are needed?

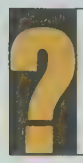
think

1. A student chooses a whole number between 1 and 1000. You are to discover the chosen number by asking ten or fewer questions of the student. The questions must be stated so that they can be answered by Yes or No. Can you find the number?
2. How many questions would you need to find any whole number less than a million by asking Yes or No questions?

Investigating the Ideas

Suppose two satellites, Alpha (*A*) and Beta (*B*), are orbiting the earth. Satellite Alpha makes one complete orbit in 90 minutes and Beta makes one complete orbit in 120 minutes. The illustration shows the present position of Alpha and Beta.





Can you find how many minutes it will take for Alpha and Beta to get back together in the same position?

Discussing the Ideas

1. Explain how you found your answer to the Investigation question.

2. Marty said, "I counted multiples of 90 until I reached a multiple of 120." What number did Marty find?

Multiples of 90: {90, 180, 270, [redacted], [redacted], . . .}
Multiples of 120: {120, 240, [redacted], [redacted], [redacted], . . .}

3. Marty found the least common multiple (LCM) of 90 and 120. It is the smallest nonzero common multiple of two numbers. Study the example below. Give the number for the [redacted].

Set of nonzero multiples of 8: $A = \{8, 16, 24, 32, 40, [redacted], 56, 64, \ldots\}$
Set of nonzero multiples of 12: $B = \{12, 24, 36, [redacted], 60, \ldots\}$
Common nonzero multiples of 8 and 12: $A \cap B = \{[redacted], [redacted], \ldots\}$
The smallest nonzero common multiple of 8 and 12 is [redacted].
The LCM of 8 and 12 is [redacted]

4. Study the example and then find the LCM for the pairs of numbers below.

EXAMPLE: Find the LCM of 8 and 10.
Nonzero multiples of 8 → {8, 16, 24, 32, [redacted], 48, . . .}
↑
40 is the first nonzero multiple of 8 that is also a multiple of 10.
The LCM of 8 and 10 is 40

A 6 and 15 **B** 12 and 20 **C** 9 and 12

1. **A** List the nonzero multiples of 6 that are less than 72.

B List the nonzero multiples of 20 that are less than 80.

C List the set of nonzero common multiples of 6 and 20 that are less than 72.

D What is the LCM of 6 and 20?

2. Find the LCM of each pair of numbers by thinking about multiples of the larger number until you reach a nonzero multiple of the smaller number.

EXAMPLE: Find the LCM of 6 and 8.
Nonzero multiples of 8 → {8, 16, 24, . . .}
The LCM of 6 and 8 is 24

A 4, 6	**C** 9, 12	**E** 8, 7	**G** 8, 10	**I** 6, 21
B 6, 15	**D** 10, 15	**F** 8, 20	**H** 10, 16	**J** 15, 24

3. To find the LCM of 2, 3, and 5, first find the LCM of 3 and 5 (which is 15). Now find the LCM of 2 and 15 (which is 30). The LCM of 2, 3, and 5 is 30. Find the LCM of each group.

A 4, 6, 10	**D** 6, 10, 12	**G** 8, 10, 12	**J** 10, 15, 20
B 2, 3, 4	**E** 4, 6, 8	**H** 9, 12, 15	**K** 6, 12, 18
C 4, 6, 9	**F** 3, 4, 6	**I** 2, 4, 6	**L** 30, 40, 60

4. If the prime factors of two numbers are known, it is easy to choose the LCM. Study the example.

EXAMPLE: Find the LCM of 18 and 15.

$18 = 2 \cdot 3 \cdot 3$
$15 = 3 \cdot 5$

The LCM of 18 and 15 must contain all the factors of 18 as well as those factors of 15 which are not factors of 18.

The LCM of 18 and 15 is $2 \cdot 3 \cdot 3 \cdot 5 = 90$

Find the LCM of each pair.

A $4 = 2 \cdot 2$, $14 = 2 \cdot 7$

B $20 = 2 \cdot 2 \cdot 5$, $25 = 5 \cdot 5$

C $42 = 2 \cdot 3 \cdot 7$, $70 = 2 \cdot 5 \cdot 7$

D $8 = 2 \cdot 2 \cdot 2$, $9 = 3 \cdot 3$

E $33 = 3 \cdot 11$, $15 = 3 \cdot 5$

F $18 = 2 \cdot 3^2$, $60 = 2^2 \cdot 3 \cdot 5$

G $27 = 3^3$, $36 = 2^2 \cdot 3^2$

H $48 = 2^4 \cdot 3$, $54 = 2 \cdot 3^3$

think

Two cars, *A* and *B*, travel in the same direction. Car *A* travels toward *B* at a rate of **60 kilometres per hour. Car *B*** travels at a rate of 50 km/h and is now 200 km ahead of *A*.

1. How many hours will it take *A* to catch *B*?
2. How many km must *A* travel in order to catch *B*?

More practice, page S-9, Set 16

Investigating the Ideas

It is useful to know when one number is divisible by another number.

36 is **divisible** by 4 because $4 \cdot 9 = 36$.

42 is **divisible** by 3 because $3 \cdot 14 = 42$.

However, 15 is **not divisible** by 4 because

$$4\overline{)15}\quad 3,\ 12,\ 3$$

Also, 27 is **not divisible** by 6 because

$$6\overline{)27}\quad 4,\ 24,\ 3$$

Can you decide which of these numbers are divisible by 9?

A 126 **B** 2483 **C** 7696 **D** 828 135

Discussing the Ideas

1. Using the equation $7 \cdot 23 = 161$, you can say that "161 **is divisible by** 7," or "7 **is a divisor of** 161," or "7 **is a factor of** 161."
   What similar statements can you make regarding the numbers 23 and 161?

2. There are some simple divisibility rules that you may already know.
   - **A** How can you tell whether or not a number is divisible by 2?
   - **B** How can you tell whether or not a number is divisible by 5?
   - **C** How can you tell whether or not a number is divisible by 10?
   - **D** How can you tell whether or not a number is divisible by 25?
   - **E** How can you tell whether or not a number is divisible by 100?

3. Study the table at the right.
   Then complete this statement:
   If a number is divisible by 3, then the sum of the digits of the number is ? .

$3 \cdot 4 =$ [illegible] $= 3$
$3 \cdot 8 =$ [illegible] $= 6$
$3 \cdot 102 =$ [illegible] $= 9$
$3 \cdot 727 =$ [illegible] $= 12$
$3 \cdot 1254 =$ [illegible] $= 18$

4. Which of these numbers are divisible by 3?
   A 421 **B** 7114 **C** 705 **D** 121 212

1. In the set {12, 15, 17, 20, 24, 27, 30, 35, 48, 51, 70, 87}, which numbers are

A divisible by 2? **B** divisible by 5? **C** divisible by 10?

2. Tell whether each statement is true or false.

A Every number which divisible by 10 is also divisible by 2.

B Every number which is divisible by 2 is also divisible by 10.

C Every number which is divisible by 10 is also divisible by 5.

D Every number which is divisible by 10 is also divisible by both 2 and 5.

3. Solve the equation, then answer the question.

A $1 + 2 + 3 = n$: is 123 divisible by 3?

B $2 + 1 + 1 = n$: is 211 divisible by 3?

C $7 + 0 + 5 = n$: is 705 divisible by 3?

D $7 + 5 + 6 = n$: is 756 divisible by 3?

E $2 + 0 + 0 + 1 = n$: is 2001 divisible by 3?

F $5 + 5 + 4 + 2 = n$: is 5542 divisible by 3?

G $7 + 8 + 9 + 2 + 1 = n$: is 78 921 divisible by 3?

4. Without actually dividing, tell which numbers are divisible by 3.

A 231	**C** 519	**E** 1968	**G** 20 001	**I** 1 010 100
B 426	**D** 733	**F** 4440	**H** 72 641	**J** 23 456

5. A number is divisible by 6 if and only if it is even **and** also divisible by 3. Which of the numbers are divisible by 6?

A 132	**C** 765	**E** 842	**G** 1260	**I** 8088	**K** 3234
B 321	**D** 576	**F** 702	**H** 888	**J** 8808	**L** 1968

6. Divisibility rules can help you find the prime factorization of a number. Study the example, then give the prime factorization of each number.

EXAMPLE: Give the prime factorization of 402.

SOLUTION: Since $4 + 0 + 2 = 6$, 3 is a factor of 402.
Also, 402 is an even number; therefore, 2 is a factor.
$402 = 2 \cdot 3 \cdot 67$

A 111 **B** 114 **C** 267 **D** 195 **E** 11 010

Other Divisibility Rules

Investigating the Ideas

Study the examples below to discover a rule for divisibility by 11.

$253 \rightarrow 5 - 5 = 0 \rightarrow$ 253 is divisible by 11. $\rightarrow 253 = 11 \cdot 23$
$924 \rightarrow 13 - 2 = 11 \rightarrow$ 924 is divisible by 11. $\rightarrow 924 = 11 \cdot 84$
$942 \rightarrow 11 - 4 = 7 \rightarrow$ 942 is **not** divisible by 11.
$4928 \rightarrow 17 - 6 = 11 \rightarrow$ 4928 is divisible by 11. $\rightarrow 4928 = 11 \cdot 448$
$25\,674 \rightarrow 12 - 12 = 0 \rightarrow$ 25 674 is divisible by 11. $\rightarrow 25\,674 = 11 \cdot 2334$
$342\,715 \rightarrow 16 - 6 = 10 \rightarrow$ 342 715 is **not** divisible by 11.

Can you decide which of these numbers are divisible by 11?
A 748 **B** 407 **C** 4721 **D** 98 626 **E** 804 716

Discussing the Ideas

1. What is the divisibility rule for 11 illustrated by the examples in the Investigation?

2. If a number is less than 100, you can quickly decide by using direct division whether or not this number is divisible by 4.
   100 is divisible by 4 because $4 \cdot 25 = 100$.
   Every multiple of 100 is also divisible by 4.
   Study these examples: $104 = 100 + 4 \rightarrow$ 104 is divisible by 4.
   $114 = 100 + 14 \rightarrow$ 114 is not divisible by 4.
   $224 = 200 + 24 \rightarrow$ 224 is divisible by 4.
   $39[illegible] = 3900 + 36 \rightarrow$ 3936 is divisible by 4.
   A What divisibility rule for 4 do the examples suggest?
   B Is 728 divisible by 4? Is 2650 divisible by 4?

3. The rule for divisibility by 9 is much the same as the rule for divisibility by 3. Study the table of information.

$2 \cdot 9 =$ [illegible] $= 9$	$111 \cdot 9 =$ [illegible] $= 27$
$5 \cdot 9 =$ [illegible] $= 9$	$217 \cdot 9 =$ [illegible] $= 18$
$11 \cdot 9 =$ [illegible] $= 18$	$834 \cdot 9 =$ [illegible] $= 18$
$26 \cdot 9 =$ [illegible] $= 9$	$2541 \cdot 9 =$ [illegible]

   A State a rule for divisibility by 9.
   B Is 279 divisible by 9? Is 1663 divisible by 9?

1. Which of the following numbers are divisible by 11? Verify by actual division.

A	9394	**C**	8251	**E**	8250	**G**	71 577	**I**	437 404
B	8371	**D**	8252	**F**	3465	**H**	395 824	**J**	437 415

2. Which of the numbers are divisible by 4?

A	204	**C**	626	**E**	996	**G**	332	**I**	344	**K**	7112
B	740	**D**	1148	**F**	116	**H**	443	**J**	3008	**L**	6152

3. **A** If a number is divisible by 12, is it also divisible by 4?
B If a number is divisible by 12, is it also divisible by 3?
C If a number is divisible by 3 and also divisible by 4, is it necessarily divisible by 12?

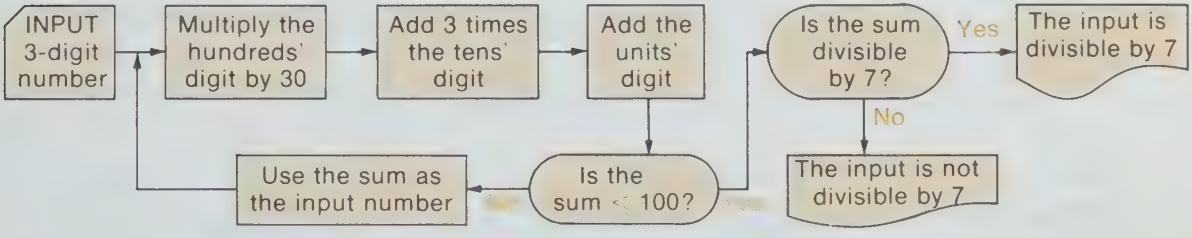
4. Which numbers are divisible by 12?

A	324	**B**	216	**C**	448	**D**	720	**E**	468	**F**	678

5. Which numbers are divisible by 9?

A	414	**C**	7001	**E**	16 164	**G**	441	**I**	3 244 617
B	3153	**D**	8001	**F**	809	**H**	123 456	**J**	38 496

★ **6.** Most divisibility rules for 7 are complicated. The flow chart shows the steps for one such rule for 3-digit numbers only. Study the flow chart and the example.


INPUT 3-digit number
Multiply the hundreds' digit by 30
Add 3 times the tens' digit
Add the units' digit
Is the sum divisible by 7?
Yes
The input is divisible by 7
No
The input is not divisible by 7
Is the sum < 100?
Use the sum as the input number

EXAMPLE: 126 → 6 = 6
3 · 2 = 6
30 · 1 = 30

Since 42 is divisible by 7, the number 126 is divisible by 7.

Which of these numbers are divisible by 7?

A	246	**D**	647	**G**	973
B	222	**E**	595	**H**	861
C	378	**F**	196	**I**	241

think

What is the smallest number, larger than zero, which is divisible by all whole numbers 1 through 9?

Investigating the Ideas

Study the sequences at the right. Try to discover a pattern for each of them.

Can you find the next three numbers for any of these sequences?

A 1, 4, 7, 10, 13, . . .

B 2, 5, 9, 14, 20, . . .

C 2, 5, 10, 17, 26, . . .

D 1, 3, 9, 27, 81, . . .

E 23, 161, 299, 437, 575, . . .

Discussing the Ideas

1. What pattern did you find for sequence **A**?

2. Explain the patterns for sequences **B** through **E**.

3. The terms of the sequence are the numbers of the sequence. We speak of the **first** term, the **second** term, the **third** term, and, in general, the n^{th} term of the sequence.

Term:	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}		n^{th}	
	↓	↓	↓	↓	↓		↓	
Sequence:	1 ,	3 ,	5 ,	7 ,	9 ,	. . . ,	$2n-1$,	. . .

In the sequence above, the rule for the n^{th} term is $2n - 1$.
Thus, if $n =$, then $2 \cdot - 1 = 7$. (The 4^{th} term is 7.)
if $n = 6$, then $2 \cdot - 1 = 11$. (The 6^{th} term is 11.)

A What is 7^{th} term of the sequence?
B What is the 10^{th} term of the sequence?
C What is the 11^{th} term of the sequence?

4. The rule for the n^{th} term of a sequence is $5n - 4$. Use the rule to give three more terms of the sequence.

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	
↓	↓	↓	↓	↓	↓	
$5 \cdot - 4$,	$5 \cdot - 4$,	$5 \cdot - 4$,	,	,	,	. . .
↓	↓	↓				
1 ,	6 ,	11 ,	,	,	,	. . .

1. Give the next three numbers for each sequence.

A 1, 3, 5, 7, 9, . . .

B 1, 10, 100, 1000, . . .

C 1, 2, 4, 8, 16, . . .

D 3, 7, 11, 15, 19, . . .

2. Give the 4th, 5th, and 6th terms of each sequence.
Use the rule for the n^{th} ***term.***

	TERM:							
SEQUENCES	**A**	4	7	10	▒	▒	▒	$3n + 1$
	B	1	3	6	▒	▒	▒	$\frac{n(n+1)}{2}$
	C	1	4	9	▒	▒	▒	n^2
	D	2	3	6	▒	▒	▒	$(n-1)^2 + 2$

3. Write the first five terms of the sequence whose n^{th} term is $3n - 1$.

4. Write the first five terms of the sequence whose n^{th} term is 0 if n is even and 1 if n is odd.

☆**5.** What is the rule for the n^{th} term of each sequence?

A 3, 6, 9, 12, . . .

B 2, 3, 4, 5, . . .

C $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, \ldots$

D 1, 4, 9, 16, 25, . . .

☆**6.** The **Fibonacci Sequence**, 1, 1, 2, 3, 5, 8, 13, 21, . . . , is a famous sequence that has many interesting properties.

A Give the next four numbers of the sequence.

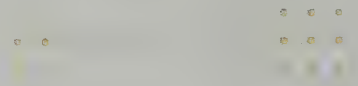
B Can you explain the rule for the n^{th} term of the sequence?

think

The n^{th} term of a sequence is given by the rule $\frac{n^3 - 5n + 6}{6}$
Find the first six terms of the sequence.

SQUARE NUMBERS

1. The sequence 1, 4, 9, . . . , n^2, . . . , is a sequence of **square numbers**. The first three square numbers are represented by the arrays of dots. Give the next four square numbers and draw arrays of dots for them.



$1^2 = 1$ $2^2 = 4$ $3^2 = 9$

2. Study Figures **A** through **C**.

A **B** **C**

$1 = 1^2$ $1 + 3 = 2^2$ $1 + 3 + 5 = 3^2$

Draw square arrays of dots for the next three square numbers and write addition statements for each square.

3. Solve the equations. Try to avoid computation by using the pattern established in Exercise 2.

A $1 + 3 + 5 = n$

B $1 + 3 + 5 + 7 = n$

C $1 + 3 + 5 + 7 + 9 = n$

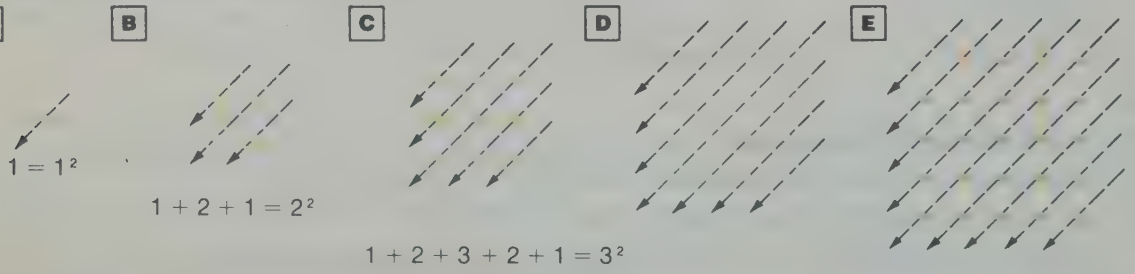
D $1 + 3 + 5 + 7 + 9 + 11 = n$

E $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = n$

F $1 + 3 + 5 + 7 + \ldots + 95 + 97 + 99 = n$

4. Figures **A** through **E** show that we may think of the square numbers as the following sums:

A **B** **C** **D** **E**



$1 = 1^2$

$1 + 2 + 1 = 2^2$

$1 + 2 + 3 + 2 + 1 = 3^2$

A Write similar addition statements for the numbers pictures in Figures **D** and **E**.

B Write 64 as a sum.

C Write 100 as a sum.

5. Find the square number for each sum.

A $1 + 2 + 3 + 4 + 5 + 6 + 5 + 4 + 3 + 2 + 1 =$ ▒.

B $1 + 2 + 3 + 4 + 5 + 6 + 7 + 6 + 5 + 4 + 3 + 2 + 1 =$ ▒.

C $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 =$ ▒.

Triangular Numbers

1. The sequence of **triangular numbers** is the sequence whose n^{th} term is $\frac{n(n+1)}{2}$. The first three triangular numbers are represented by these arrays of dots.

1 3 6

Give the next four triangular numbers and draw arrays of dots for these numbers.

2. Give the next ten triangular numbers for the set $\{1, 3, 6, 10, \ldots\}$.

3. Each triangular number greater than 1 is the sum of consecutive whole numbers beginning with 1.

1 , 3 , 6 , 10 , . . .

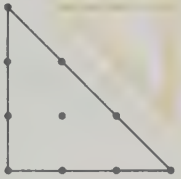
(1 + 2) (1 + 2 + 3) (1 + 2 + 3 + 4)

Represent the next four larger triangular numbers as the sum of consecutive whole numbers.

4. Every square number greater than 1 is the sum of two consecutive triangular numbers. The examples below illustrate this.

$+ 3 = 4 = 2^2$

$+ 6 = 9 = 3^2$



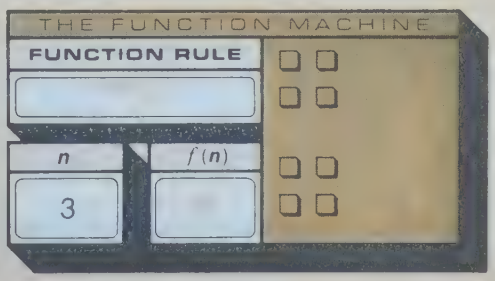
$+ 10 = 16 = 4^2$

A Express each number 25, 36, and 49 as the sum of two triangular numbers.

B Express 100 as the sum of two triangular numbers.

5. This function machine produces triangular numbers. Find $f(n)$ for each number in the table.

n	3	7	10	20	100	1000
$f(n)$	6					



Investigating the Ideas

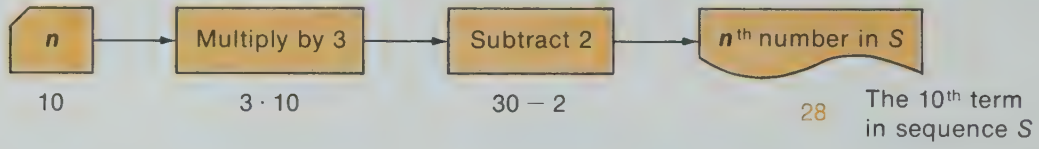
The sequence S given below has some interesting properties that you will study in this lesson.

S: 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, . . . , $3n - 2$, . . .

Can you list all the numbers in this sequence that are less than 150?

Discussing the Ideas

1. The flow chart below can help you find any term in sequence S.

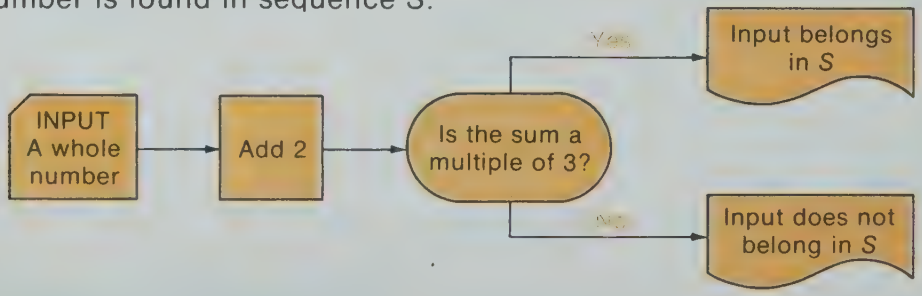


A What is the 16th term in S?

B What is the 20th term in S?

C What is the 100th term in S?

2. This flow chart can help you decide whether a given whole number is found in sequence S.



Use the flow chart to decide which of these numbers are terms in S.

A 70 **B** 116 **C** 213 **D** 997 **E** 4132

3. If you choose any two numbers in S, do you think their product is also a number in S?

1. Find the designated term in sequence *S*.

A 8^{th} term **C** 50^{th} term **E** 105^{th} term
B 26^{th} term **D** 82^{nd} term **F** 1000^{th} term

2. Is the given number a term of sequence *S*? Use the flow chart in Discussion Exercise 2 to help you decide.

A 127 **C** 250 **E** 306 **G** 888 **I** 1002
B 168 **D** 481 **F** 762 **H** 1001 **J** 1003

3. Each number below is a number from sequence *S*. List all the factors of the number which are also numbers in *S*.

EXAMPLE: **40** Factors in *S*: {1, 4, 10, 40}

A 28 **C** 10 **E** 70 **G** 25 **I** 52
B 16 **D** 34 **F** 76 **H** 100 **J** 91

4. Suppose we call a number from *S* which is greater than 1 and which has exactly two factors in *S* an **S-prime**. Then 4 is an *S*-prime because 1 and 4 are the only numbers in *S* that are factors of 4.

A Is 10 an *S*-prime? **D** Is 16 an *S*-prime?
B Is 13 an *S*-prime? **E** Is 88 an *S*-prime?
C Is 25 an *S*-prime? **F** Is 46 an *S*-prime?

5. What is the smallest number in *S*, greater than 1, that is an *S*-prime?

6. If a number greater than 1 in *S* is not an *S*-prime, then it can be expressed as a product of *S*-primes.

EXAMPLE: $40 = 4 \cdot 10$ (4 and 10 are *S*-primes.)
$208 = 4 \cdot 4 \cdot 13$ (4 and 13 are *S*-primes.)

Express each number from *S* as a product of primes.

A 15 **B** 88 **C** 70 **D** 52 **E** 49

7. Show that 100 can be expressed as a product of two different sets of *S*-primes.

REVIEWING THE IDEAS

1. Give the set of all factors of the following numbers:

EXAMPLE: 24
ANSWER: {1, 2, 3, 4, 6, 8, 12, 24}

A 18	C 22	E 16	G 75
B 30	D 50	F 28	H 128

2. What is the smallest number that has exactly three factors?

3. Only one of the numbers in the set is prime. Which is it?

{33, 27, 49, 87, 51, 91, 78, 97, 57, 39}

4. One of the numbers in the set is a composite number. Which is it?

{71, 37, 17, 67, 47, 69, 23, 31, 41}

5. Give the prime factorization of each number.

A 54	D 81	G 284
B 40	E 210	H 500
C 20	F 220	I 900

6. Which of the following prime numbers have twin primes? If the number has a twin, give both numbers.

A 17	C 43	E 31	G 97
B 23	D 89	F 59	H 71

7. What is the GCF of each pair?

A 16, 24	D 14, 35	G 6, 35
B 20, 25	E 66, 88	H 13, 52
C 9, 21	F 60, 75	I 48, 120

8. Find the GCF of each pair using the Euclidean Algorithm.

A 168, 315	C 1763, 2021
B 629, 851	D 2397, 3337

9. A List the first 10 nonzero multiples of 6.
B List the first 10 nonzero multiples of 10.
C List the first four common multiples of 6 and 10.
D What is the LCM of 6 and 10?

10. What is the LCM of each pair?

A 6, 8	E 12, 15	I $3^2 \cdot 5, 3 \cdot 5^2$
B 5, 8	F 4, 20	J $5^2 \cdot 7, 5 \cdot 7^2$
C 9, 12	G 7, 11	K $2 \cdot 3 \cdot 5, 2^2 \cdot 3$
D 14, 21	H 4, 18	L $7^2 \cdot 11^2, 7^2 \cdot 17$

11. What is the LCM in each group?

A 3, 4, 6	D 3, 5, 7	G 10, 20, 30
B 3, 6, 8	E 4, 6, 9	H 6, 9, 27
C 6, 8, 10	F 3, 4, 5	I 12, 18, 20

12. Which numbers in the set {54, 65, 66, 111, 70, 222} are divisible by

A 2? B 3? C 6? D 5?

13. For each number, tell if it is divisible by 4.

A 316 B 702 C 328 D 114

14. What is a divisibility rule for 9?

15. Which numbers are divisible by 9?

A 727 B 1512 C 6381 D 23 760

16. Give the next four numbers in each sequence.

A 1, 6, 11, 16, 21, . . . B 1, 3, 9, 27, . . .

17. Write the first four numbers of a sequence whose n^{th} term is $3n + 2$.

18. For each number 6, 7, 8, 9, and 10, tell whether it is **square**, **triangular**, or **prime**.

TEST YOURSELF

1. **A** Is 8 a factor of 144?
   B Is 6 a factor of 266?
2. List all the factors of 20.
3. Is 87 a prime number or a composite number?
4. Give the prime factorization of 140.
5. Complete the prime factorization of this number.
   $483 = 23 \cdot \blacksquare \cdot \blacksquare$
6. What is the greatest common factor of 42 and 77?
7. What is the GCF of two numbers that are relatively prime?
8. What is the LCM of 12 and 15?
9. Is 324 divisible by
   A 4? **B** 3? **C** 9? **D** 11?
10. Give the next three terms for each sequence.
    A 2, 5, 8, . . . , $3n - 1$, . . .
    B 2, 5, 10, 17, . . . , $n^2 + 1$, . . .

RESEARCH PROJECTS

A A famous unsolved problem in mathematics, called **Goldbach's Conjecture**, is concerned with whether or not every even number greater than 2 can be expressed as the sum of two primes. For example, $24 = 11 + 13$ and $44 = 31 + 13 = 37 + 7$. Can you show that each even number less than 100 can be expressed as the sum of two prime numbers?

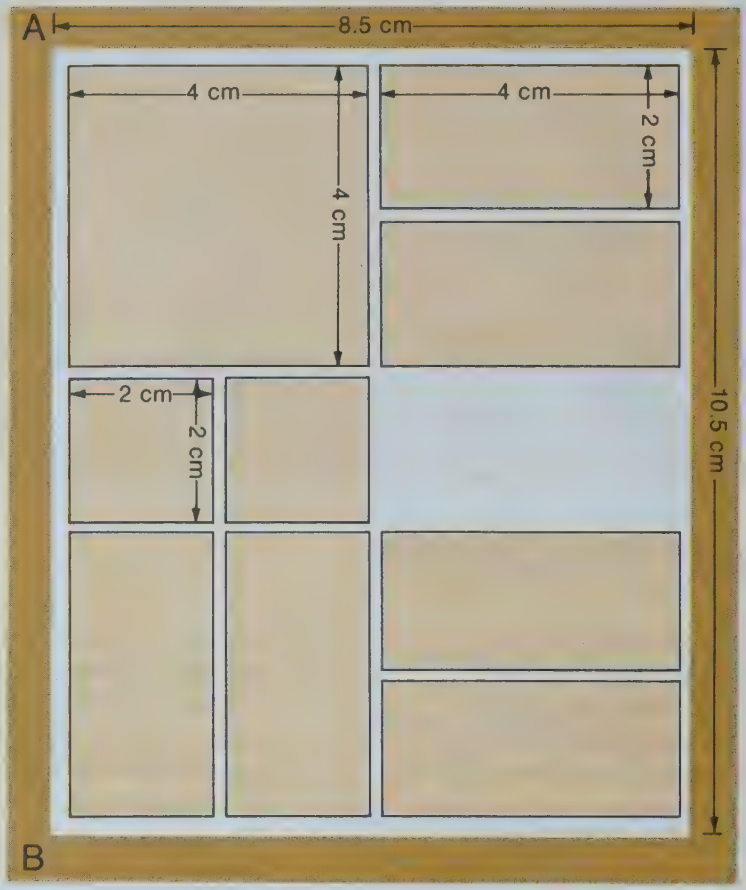
B In reply to a question about the definition of a "friend," Pythagoras said: A friend is "one who is the other I, such are 220 and 284." Mathematically 220 and 284 are examples of **amicable** numbers. The sum of the proper divisors of each number equals the other number. Read about the history of amicable numbers and see how many you can find. (See *Number—The Language of Science* by Tobias Dantzig; New York: Doubleday and Company, 1956.)

C Another famous conjecture about primes is attributed to Edward Waring (1734–1793). He asserted that every odd number greater than 1 which is not a prime can be expressed as the sum of three primes. For example, $9 = 2 + 2 + 5$ and $27 = 3 + 7 + 17$. Show that Waring's conjecture is true for all the odd non-prime numbers less than 100 but greater than 1.

D Many formulas have been developed that give a finite number of primes. Read about some of these formulas and report on them to your class. (See *One, Two, Three, . . . Infinity*, by George Gamow; New York: The Viking Press, 1961; Macmillan.)

E Make a chart of the whole numbers from zero to 100, showing how each number can be classified as odd, even, prime, composite, triangular, square, perfect, cubic, and so forth. Make your chart suitable for bulletin board display.

A TRANSLATION PUZZLE



Cut out the nine pieces from centimetre graph paper and arrange them inside a cardboard frame as shown. Now try to move the large square from corner A to corner B by sliding the pieces. You cannot jump over a piece, turn a piece, or cover a piece.

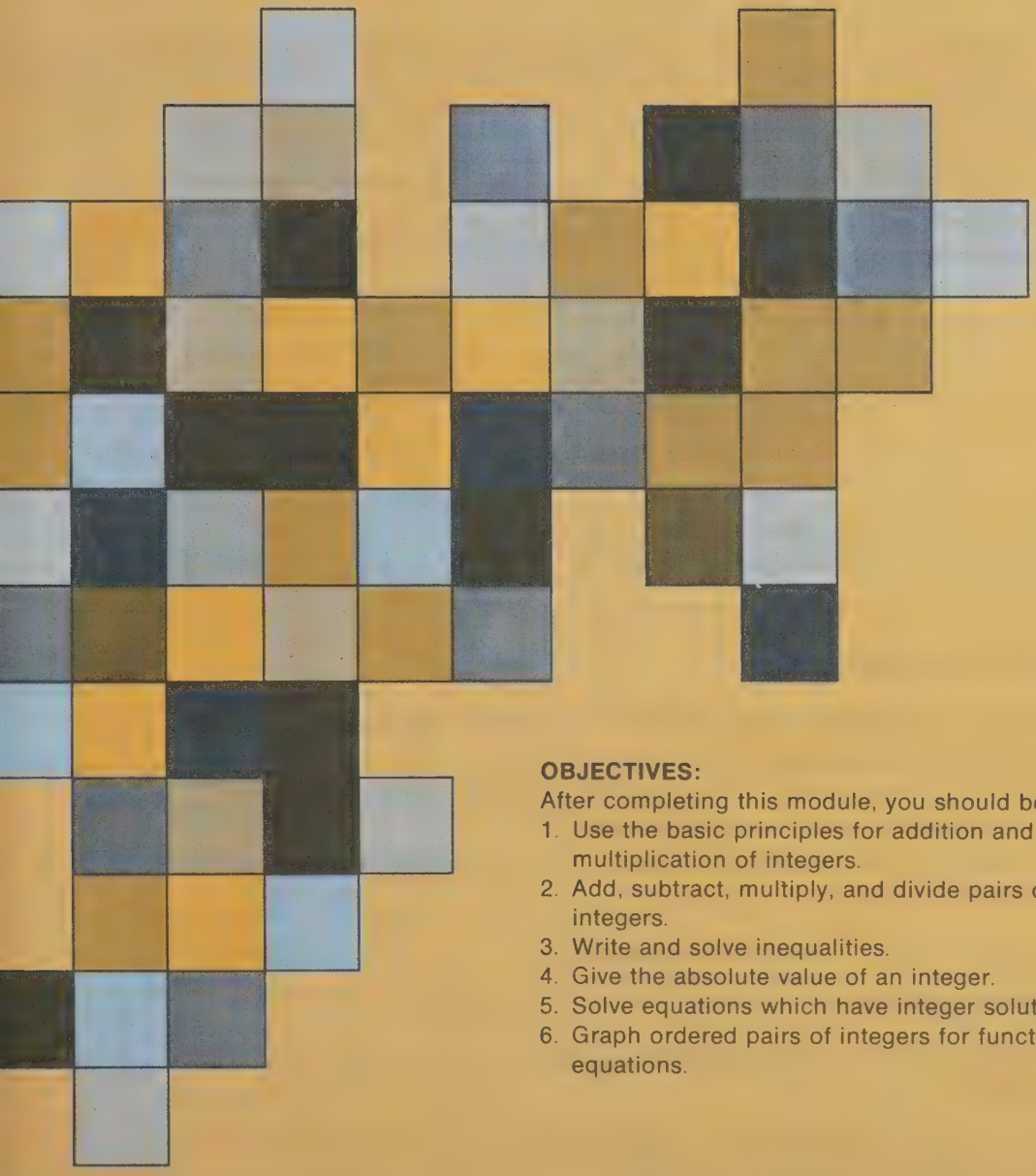
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A yes; B no **2.** {1, 2, 4, 5, 10, 20} **3.** composite **4.** $2^2 \cdot 5$
5. $483 = 23 \cdot 3 \cdot 7$ **6.** 7 **7.** 1 **8.** 60
9. A yes; B yes; C yes; D no **10.** A 11, 14, 17; B 26, 37, 50

UNIT E: *Integers*

MODULE 4: *Integers*



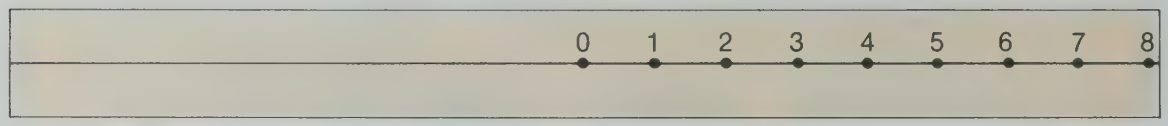
OBJECTIVES:

After completing this module, you should be able to:

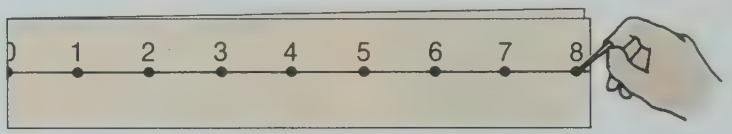
1. Use the basic principles for addition and multiplication of integers.
2. Add, subtract, multiply, and divide pairs of integers.
3. Write and solve inequalities.
4. Give the absolute value of an integer.
5. Solve equations which have integer solutions.
6. Graph ordered pairs of integers for functions and equations.

Investigating the Ideas

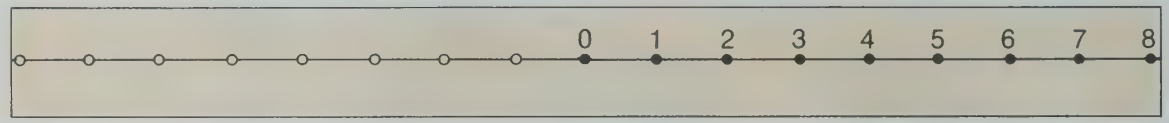
A Draw a number line on a strip of paper.
Start with 0 in the middle of the strip.



B Fold the strip at 0.
Punch pin holes through each point starting at 1.



C Open up your strip.



What numbers go with the points to the left of zero on your number line?	Label all the points you can on the number line.

Discussing the Ideas

1. The **negative integers** are paired with the points opposite each of the **positive integers** 1, 2, 3, 4, 5, ...
The symbol $^{-}1$ (negative 1) denotes the negative integer, which is opposite from 1 on the number line.
What are some other negative integers that you could name on your number line?

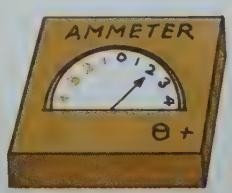
2. The integers 1 and $^{-}1$ are called **opposites**.

   "1 is the opposite of $^{-}1$" and "$^{-}1$ is the opposite of 1."

   What are some other pairs of integers that are opposites of each other?

3. Can you think of some practical uses of negative integers?
The figures may suggest some uses.

Stock	Sales in 100's	Open	High	Low	Close	Net change
Gropex	12	32	35	31	34	$^{+}2$
IS&M	57	49	53	46	47	



1. Give the opposite of each integer.

A 3	**C** 10	**E** $^{-}4$	**G** $^{-}11$	**I** $^{-}89\,376$	**K** $^{-}1\,000\,000$
B 7	**D** 100	**F** $^{-}8$	**H** $^{-}99$	**J** 10 000	**L** 10 000 001

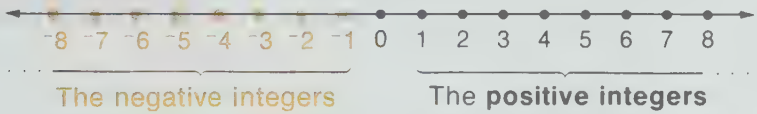
2. The temperature at the equator of the planet Mars in the daytime may be as high as 29°C. At night, the temperature may be as low as 87°C below zero. What integer represents the nighttime temperature?

Mars

3. Death Valley, California, is the lowest point in the North American continent. It is 86 metres below sea level. What integer could be used to describe the altitude of Death Valley?

4. For each number given, indicate whether it is a positive integer, a negative integer, or zero. The number line may help you.

The set of integers



A 7 **B** $^{-}6$ **C** $^{-}1$ **D** $^{-}99$ **E** 27

F The opposite of 4.

G The opposite of 10.

H The opposite of the opposite of 3.

I The opposite of the opposite of $^{-}5$.

J The only integer that is the opposite of itself.

5. Complete the sentences.

A The opposite of a positive integer is a ? integer.

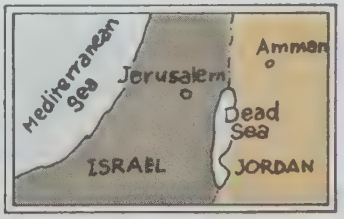
B The opposite of a negative integer is a ? integer.

C If the opposite of a number is a positive integer, the number must be a ? integer.

D If the opposite of a number is a negative integer, the number must be a ? integer.

E ? is the only integer that is neither positive nor negative.

6. The lowest land point in the world surrounds the Dead Sea. It is 306 metres lower than Death Valley (Exercise 3). What integer gives the altitude of the Dead Sea?

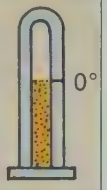


Investigating the Ideas

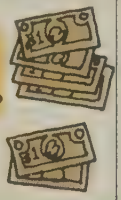
The examples below may suggest some ways for finding the sum of two integers.

A Had a score of 20 points → 20 + Went set 20 points → $^{-}20$ = Total score? → *a*

B From 0°, the temperature rose 10°. → 10 + Then it fell 4°. → $^{-}4$ = Above or below 0°? How far? → *b*



C Spend $4.00 → $^{-}4$ + Spend $2.00 more. → $^{-}2$ = Richer or poorer? How much? → *c*



D Walk 8 km west. → $^{-}8$ + Walk 3 km east. → 3 = End up to the east or west of starting point? How far? → *d*

Can you find the sum for each of the equations above?

Discussing the Ideas

1. When using integers to represent numbers in physical situations, we must agree upon the interpretation of positive and negative integers.
   - **A** What agreement must be made in keeping score in Example **A**?
   - **B** What interpretation is given to positive and negative integers in Example **B**? Example **C**? Example **D**?

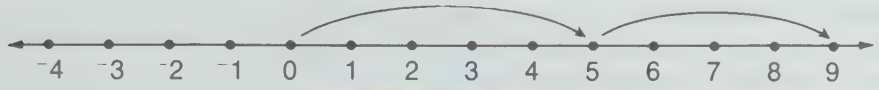
2. What addition problem does this number-line picture show?

3. What are these sums? Use a number line to help you.
   A $5 + {}^{-}4$ **B** $^{-}3 + 6$ **C** $^{-}4 + 4$ **D** $^{-}2 + {}^{-}3$

4. Write and solve an addition equation using integers for this problem. Then explain your solution.
   One morning the temperature was 9°C below zero.
   During the day, the temperature rose 14°C.
   What was the high temperature?

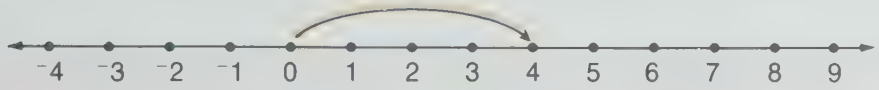
1. On the number line, jumps to the **right** show addition of positive integers. Jumps in the opposite direction (to the left) show addition of negative integers. Study the number lines and solve the equations.

A



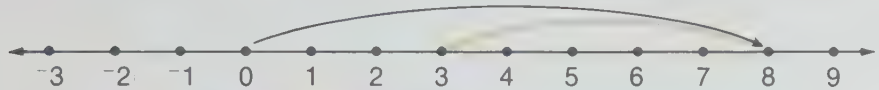
$5 + 4 = n$

B



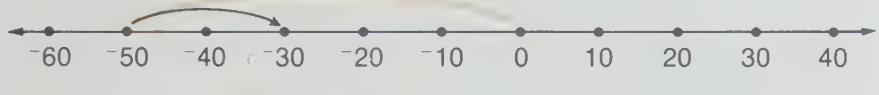
$4 + {}^{-}4 = n$

C



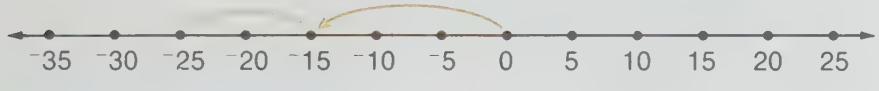
$8 + {}^{-}5 = n$

D



$^{-}50 + 20 = n$

E



$^{-}15 + {}^{-}10 = n$

2. Draw number lines and label them so that you can show the jumps for each sum. Write an equation for each number line.

A $6 + {}^{-}3$ **B** $^{-}9 + 7$ **C** $^{-}30 + {}^{-}40$ **D** $^{-}15 + 35$ **E** $^{-}800 + 200$

3. Find the sums. Think about number-line jumps.

A $^{-}4 + 1$ **C** $^{-}7 + 4$ **E** $^{-}12 + 10$ **G** $7 + {}^{-}8$ **I** $9 + {}^{-}17$

B $^{-}6 + 5$ **D** $^{-}10 + 7$ **F** $^{-}15 + 8$ **H** $6 + {}^{-}12$ **J** $^{-}21 + 18$

4. Find the sums. Use any method you choose.

A $27 + {}^{-}9$

B $^{-}18 + 11$

C $^{-}16 + {}^{-}17$

D $24 + (37 + {}^{-}24)$

E $(9 + {}^{-}14) + 9$

F $({}^{-}7 + {}^{-}11) + 20$

G $(5 + {}^{-}3) + ({}^{-}5 + 3)$

H $(4 + {}^{-}5) + (7 + {}^{-}6)$

I $(4 + {}^{-}11) + (12 + {}^{-}3)$

5. Write an **addition equation** using integers for each part. Then solve the equation.

A Bill had a score of 50 points in Canasta. The next game he lost 60 points. What was his score then?

B The temperature was 2°C below zero. Then the temperature fell 8° more. What was the new temperature?

C Kathy has $14. She spent $9 for a pair of jeans. How much money did she have then?

D Jan drove 40 km north. Then she drove back on the same road 62 km south. How far and in what direction was she from her starting point?

Basic Principles for Addition of Integers

Discussing the Ideas

Since the set of whole numbers is a subset of the integers, the basic principles for addition of integers are much the same as for whole numbers. If we accept these basic principles, we can justify the rules used to add integers.

Name of principle	Statement of principle	Examples
Opposites	The sum of each integer and its opposite is zero.	$12 + {}^{-}12 = 0$ ${}^{-}7 + 7 = 0$
Zero	For each integer a, $a + 0 = a$.	$8 + 0 = 8$ ${}^{-}9 + 0 = {}^{-}9$
Commutative	For each pair of integers a and b, $a + b = b + a$.	${}^{-}6 + 4 = 4 + {}^{-}6$ ${}^{-}9 + {}^{-}3 = {}^{-}3 + {}^{-}9$
Associative	For each three integers a, b, and c, $a + (b + c) = (a + b) + c$	$(5 + {}^{-}3) + 3 = 5 + (3 + {}^{-}3)$ ${}^{-}8 + ({}^{-}4 + 4) = ({}^{-}8 + {}^{-}4) + 4$

1. If the sum of two integers is zero, each integer is the **additive inverse** of the other.

${}^{-}7$ is the additive inverse of 7 because $7 + {}^{-}7 = 0$.
7 is also the additive inverse of ${}^{-}7$.

A What is the additive inverse of ${}^{-}11$?

B What is the additive inverse of 0?

C Does every integer have a unique additive inverse?

2. The example shows how the basic principles can be used to find the sum of 10 and ${}^{-}4$. How could you use the principles to show that

A $8 + {}^{-}3 = 5$? **B** ${}^{-}6 + 9 = 3$?

$$\begin{aligned} 10 + {}^{-}4 &= (\quad) + {}^{-}4 && \text{Addition fact} \\ &= 6 + (4 + {}^{-}4) && \text{Associative (+)} \\ &= 6 + 0 && \text{Opposites} \\ &= 6 && \text{Zero principle} \end{aligned}$$

3. Give the missing number in each step of the example. Which basic principles are used?

$$\begin{aligned} (3 + {}^{-}3) + (\ 5 + {}^{-}5) &= \blacksquare && \text{Why?} \\ (3 + \ 5) + ({}^{-}3 + {}^{-}5) &= \blacksquare && \text{Why?} \\ 8 \quad + ({}^{-}3 + {}^{-}5) &= \blacksquare && \text{Addition fact} \\ ({}^{-}3 + {}^{-}5) &= \blacksquare && \text{Why?} \end{aligned}$$

4. Give a reason for each step of the example.

$$\begin{aligned} {}^{-}7 + 5 &= ({}^{-}2 + {}^{-}5) + 5 \\ &= {}^{-}2 + ({}^{-}5 + 5) \\ &= {}^{-}2 + 0 \\ &= {}^{-}2 \end{aligned}$$

1. Find the number for ***a***. Then find the sum ***b***.

A $^{-}16 + 16 = a$ $16 + {}^{-}16 = b$

B $9 + 0 = a$ $0 + 9 = b$

C $^{-}76 + 76 = a$ $76 + {}^{-}76 = b$

D $^{-}947 + 947 = a$ $947 + {}^{-}947 = b$

2. Find the number for ***a***. Then find the sum ***b***.

A $4 + (2 + {}^{-}2) = a$ $6 + {}^{-}2 = b$

B $5 + 3 + {}^{-}3 = a$ $8 + {}^{-}3 = b$

C $^{-}4 + {}^{-}2 + 2 = a$ $^{-}6 + 2 = b$

D $^{-}5 + {}^{-}4 + 4 = a$ $^{-}9 + 4 = b$

E $^{-}2 + (^{-}3 + 3) = a$ $^{-}5 + 3 = b$

F $^{-}7 + (^{-}4 + 4) = a$ $^{-}11 + 4 = b$

G $(^{-}8 + 8) + 7 = a$ $^{-}8 + 15 = b$

H $^{-}6 + 6 + 9 = a$ $^{-}6 + 15 = b$

I $4 + {}^{-}4 + {}^{-}10 = a$ $4 + {}^{-}14 = b$

J $7 + {}^{-}7 + {}^{-}5 = a$ $7 + {}^{-}12 = b$

K $^{-}8 + {}^{-}5 + 5 = a$ $^{-}13 + 5 = b$

L $^{-}6 + {}^{-}9 + 7 = a$ $^{-}15 + 7 = b$

3. Find the sums.

A $2 + {}^{-}1$

B $4 + {}^{-}2$

C $^{-}5 + {}^{-}7$

D $^{-}4 + 1$

E $^{-}6 + 5$

F $9 + {}^{-}4$

G $8 + {}^{-}5$

H $10 + {}^{-}6$

I $^{-}9 + {}^{-}8$

J $^{-}7 + 4$

K $^{-}10 + 7$

L $^{-}12 + 10$

M $^{-}15 + 8$

N $7 + {}^{-}8$

O $12 + {}^{-}5$

P $^{-}7 + 15$

Q $20 + {}^{-}9$

R $9 + {}^{-}17$

S $12 + {}^{-}25$

T $^{-}53 + 17$

U $^{-}64 + 38$

V $^{-}99 + {}^{-}101$

W $250 + {}^{-}89$

X $^{-}138 + {}^{-}965$

4. Find the sums. (Use any method you choose.)

A $24 + (37 + {}^{-}24)$

B $(9 + {}^{-}14) + {}^{-}9$

C $(^{-}7 + {}^{-}11) + 20$

D $(5 + {}^{-}3) + (^{-}5 + 3)$

E $(120 + {}^{-}11) + (14 + {}^{-}120)$

F $(^{-}1 + {}^{-}1) + (^{-}1 + {}^{-}1) + (^{-}1 + 4)$

5. Find the sums.

A
$$\begin{array}{r} 6 \\ {}^{-}7 \\ \hline \end{array}$$

B
$$\begin{array}{r} {}^{-}8 \\ {}^{-}3 \\ \hline \end{array}$$

C
$$\begin{array}{r} {}^{-}18 \\ 9 \\ \hline \end{array}$$

D
$$\begin{array}{r} 12 \\ {}^{-}21 \\ \hline \end{array}$$

E
$$\begin{array}{r} {}^{-}3 \\ {}^{-}8 \\ {}^{-}2 \\ 9 \\ \hline \end{array}$$

F
$$\begin{array}{r} 24 \\ 12 \\ {}^{-}7 \\ {}^{-}9 \\ \hline \end{array}$$

G
$$\begin{array}{r} 6 \\ 15 \\ 30 \\ {}^{-}20 \\ \hline \end{array}$$

H
$$\begin{array}{r} {}^{-}9 \\ {}^{-}6 \\ {}^{-}1 \\ {}^{-}4 \\ \hline \end{array}$$

Think

The matches form six small squares. Remove only two of the matches in order to form four squares of the same size.

6. Solve the equations.

A $^{-}7 + n = 0$

B $5 + (3 + y) = 0$

C $z + {}^{-}4 = {}^{-}4$

D $^{-}3 + (8 + t) = {}^{-}3$

E $12 + (r + {}^{-}12) = 9$

F $(^{-}2 + {}^{-}3) + 3 = s$

G $(r + 3) + 5 = 0$

H $9 + (^{-}3 + w) = {}^{-}1$

I $^{-}6 + (^{-}4 + c) = {}^{-}4$

More practice, page S-10, Set 17

Investigating the Ideas

In each equation, the variable ***n*** represents a missing addend.

1. $n + 6 = 9$
2. $n + 6 = 2$
3. $n + 6 = ^{-}2$
4. $n + ^{-}6 = 2$
5. $n + ^{-}6 = ^{-}2$

Can you find the missing addend in these equations?

Discussing the Ideas

1. When you find a missing addend in an addition problem, you have found the difference for an equivalent subtraction problem.

   A A S
   $n + 6 = 9$

   S A A
   $9 - 6 = n$

   A What subtraction equation can you write for each equation 2 through 5 in the Investigation?

   B What are the solutions to the subtraction equations?

2. Find the differences. The addition equation in the "think cloud" may help you.

   $? + ^{-}5 = ^{-}6$ $? + ^{-}4 = ^{-}7$ $? + ^{-}3 = 0$ $? + ^{-}3 = 1$

   A S A
   $^{-}6 - ^{-}5 = A$

   B S A
   $^{-}7 - ^{-}4 = A$

   C S A
   $0 - ^{-}3 = A$

   D S A
   $1 - ^{-}3 = A$

3. The set of whole numbers is **closed** under addition because any two whole numbers can be added to give a whole number sum. The set of whole numbers is **not closed** under subtraction because if we select a first whole number and a second whole number, we cannot always subtract the second from the first to get a whole number.

   A Is the set of **integers** closed under addition; that is, is the sum of any two integers always an integer?

   B Is the set of integers closed under subtraction?

1. Find the differences by finding the missing addends.

	? + 4 = 0	? + ⁻4 = 0	? + 9 = 0	? + ⁻12 = 0
A	0 − 4 = A	F 0 − ⁻4 = A	K 0 − 9 = A	P 0 − ⁻12 = A
B	2 − ⁻3 = A	G 0 − 6 = A	L 1 − 6 = A	Q 2 − 6 = A
C	0 − 9 = A	H ⁻1 − 9 = A	M ⁻2 − 9 = A	R ⁻3 − 9 = A
D	0 − ⁻7 = A	I ⁻1 − ⁻7 = A	N ⁻2 − ⁻7 = A	S ⁻3 − ⁻7 = A
E	⁻9 − 0 = A	J ⁻9 − 1 = A	O ⁻9 − ⁻1 = A	T ⁻9 − ⁻9 = A

2. Find the integer for ***a***. Then find the integer for ***b***.

A $0 - {}^{-}5 = a$ $1 - {}^{-}5 = b$

B $0 - {}^{-}5 = a$ $2 - {}^{-}5 = b$

C $0 - {}^{-}10 = a$ $1 - {}^{-}10 = b$

D $0 - 6 = a$ $2 - 6 = b$

E $0 - 6 = a$ $3 - 6 = b$

F $0 - {}^{-}8 = a$ $3 - {}^{-}8 = b$

3. Think about what the missing addend would be if the sum were zero. Then find the correct missing addend.

A	[illegible] − ⁻8 = *n*	D [illegible] − ⁻4 = *n*	G ⁻5 − 2 = *n*	J [illegible] − ⁻3 = *n*
B	[illegible] − ⁻5 = *n*	E ⁻2 − 3 = *n*	H ⁻8 − 3 = *n*	K [illegible] − ⁻8 = *n*
C	[illegible] − ⁻3 = *n*	F ⁻4 − 6 = *n*	I [illegible] − ⁻2 = *n*	L [illegible] − ⁻4 = *n*

4. Find the differences. (Use any method you choose.)

A	18 − 3	D ⁻13 − 8	G 10 − ⁻5	J ⁻15 − ⁻2
B	3 − 18	E 10 − ⁻4	H ⁻4 − 20	K 57 − 120
C	18 − ⁻3	F ⁻4 − 10	I ⁻23 − ⁻5	L 27 − ⁻12

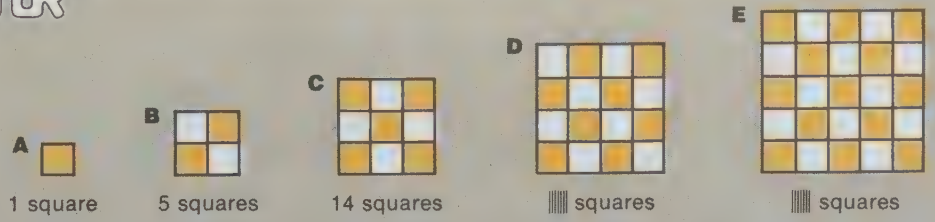
5. Give the integer for **x**.

A $(9 - {}^{-}4) + {}^{-}3 = x$

B $9 - ({}^{-}4 + {}^{-}3) = x$

C $9 + (4 + {}^{-}3) = x$

think



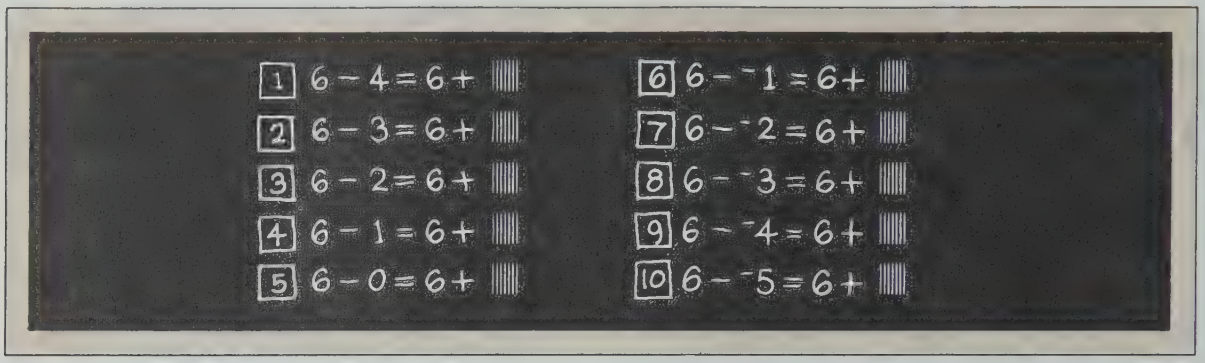
1. The **total** number of squares in each of the Figures **A** through **C** is given. What is the total number of squares in Figures **D** and **E**?

2. What is the total number of squares in an 8 by 8-square checkerboard?

More practice, page S-10, Set 18

Investigating the Ideas

Can you find the missing integer in each equation below?



Discussing the Ideas

1. Did you discover a pattern in the problems above? What is the pattern?

2. Do you agree with the statement below? Give some examples to support your answer.

   "To subtract one integer from another, just add the opposite of the integer that is to be subtracted."

3. You can write every integer subtraction problem as an addition problem.

   EXAMPLE: $^{-}9 - 3 = {}^{-}9 + {}^{-}3 = {}^{-}12$

   How could you write these subtraction problems as addition problems?

   A $3 - 7$ B $8 - {}^{-}1$ C $^{-}9 - {}^{-}3$ D $^{-}7 - 6$

4. Explain or show how to solve each equation. Think about finding missing sums or addends. The replacement set is the set of integers.

   EXAMPLE: $x - {}^{-}2 = {}^{-}3$

   SOLUTION: $^{-}3 + {}^{-}2 = x$, $x = {}^{-}5$

   A $y - 3 = {}^{-}7$ B $r - {}^{-}2 = 6$ C $7 - k = 10$

1. Solve the equations. The replacement set is the set of integers.

A $4 - {}^{-}3 = 4 + n$
B $5 - {}^{-}2 = 5 + n$
C $8 - {}^{-}5 = 8 + n$
D ${}^{-}3 - 4 = n + {}^{-}4$
E ${}^{-}2 - 7 = {}^{-}2 + n$
F ${}^{-}7 - 4 = {}^{-}7 + n$
G ${}^{-}1 - 3 = n + {}^{-}3$
H ${}^{-}4 - {}^{-}6 = {}^{-}4 + n$
I ${}^{-}12 - {}^{-}4 = {}^{-}12 + n$
J ${}^{-}15 - 6 = n + {}^{-}6$
K $10 - {}^{-}6 = 10 + n$
L ${}^{-}20 - {}^{-}4 = {}^{-}20 + n$

2. Rewrite each subtraction problem as an addition problem. Then find the sum.

A $6 - 3$
B ${}^{-}2 - 8$
C ${}^{-}4 - {}^{-}9$
D ${}^{-}10 - 1$
E $17 - {}^{-}3$
F ${}^{-}8 - 8$
G $4 - 14$
H $0 - 5$
I $0 - {}^{-}6$
J ${}^{-}70 - {}^{-}70$
K ${}^{-}100 - 43$
L ${}^{-}201 - {}^{-}179$

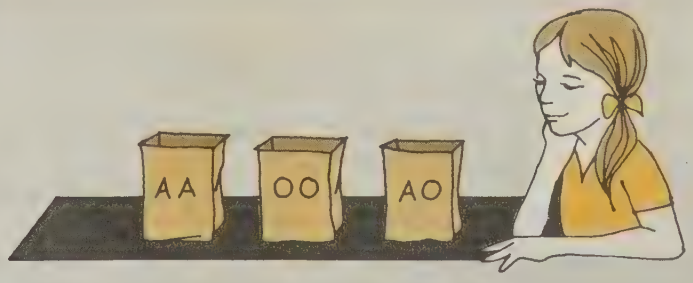
3. Solve the equations. You may find it helpful to replace all subtraction operations with addition.

A $(3 - 7) + {}^{-}1 = r$
B $({}^{-}2 - {}^{-}1) + 6 = t$
C $({}^{-}8 + 4) - 2 = s$
D $3 - ({}^{-}8 + 9) = p$
E $(1 - 2) - 3 = x$
F $({}^{-}4 - 4) + 4 = n$
G $10 - (6 - 4) = b$
H $({}^{-}1 - {}^{-}1) - {}^{-}1 = q$
I ${}^{-}6 + ({}^{-}3 - 7) = h$
J $({}^{-}1 - {}^{-}9) + {}^{-}5 = j$
K $(4 - {}^{-}2) + ({}^{-}3 - 1) = v$
L $({}^{-}2 - {}^{-}3) - ({}^{-}7 - 4) = m$

4. Solve the equations. The replacement set is the set of integers.

A $n - {}^{-}3 = 7$
B $z - 4 = 9$
C $k - {}^{-}10 = 13$
D $x - {}^{-}7 = 1$
E $r - {}^{-}2 = {}^{-}2$
F ${}^{-}3 - s = 5$
G $y - 2 = {}^{-}3$
H $t - 10 = 4$
I ${}^{-}19 - p = {}^{-}2$

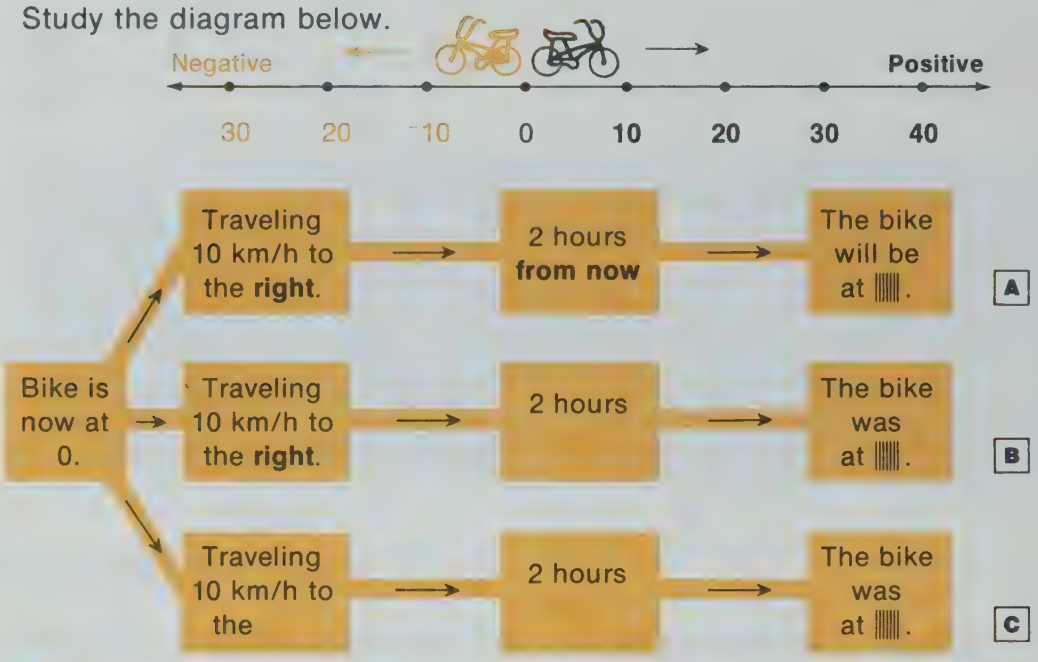
think



In one sack, there are two apples; in another sack, there are two oranges. In a third sack, there are an apple and an orange. **Each sack is incorrectly labelled** *AA, OO,* or *AO.* By looking at just one piece of fruit from one of the sacks, you are to tell what is in each sack. How can you do this?

Investigating the Ideas

Study the diagram below.



Can you find the missing numbers for **A**, **B**, and **C**?

Discussing the Ideas

1. The examples above suggest a model for multiplication of integers. Some agreements must be made in the use of the diagram above.

1. Speeds to the **right** are represented by positive integers.
2. Speeds to the left are represented by negative integers.
3. Hours **from now** are represented by positive integers.
4. Hours ago are represented by negative integers.

Example **A** could be illustrated by the multiplication equation $2 \cdot 10 = 20$.

A What multiplication equation is represented by **B**?

B What is the multiplication equation suggested by **C**?

2. Suppose the bike is now at 0 and traveling 10 km/h to the left.

A Where will it be 2 hours from now?

B What multiplication equation does this suggest?

1.

Think about trips along the number line starting at zero. Complete the table below. Part **A** has been completed for you as an example.

	SPEED	DIRECTION	TIME	FINAL POINT	MULTIPLICATION EQUATION
A	5 km/h	left	3 hours from now		
B	5 km/h	right	3 hours ago		
C	5 km/h	left	3 hours ago		
D	10 km/h	left	4 hours from now		
E	10 km/h	right	4 hours ago		
F	10 km/h	left	4 hours ago		
G	15 km/h	right	2 hours from now		
H	15 km/h	left	4 hours ago		
I	12 km/h	right	5 hours ago		

2. Find the products. Use the number line model if necessary.

A $3 \cdot {}^{-}4$ **D** ${}^{-}5 \cdot {}^{-}2$ **G** $10 \cdot {}^{-}7$ **J** $9 \cdot {}^{-}10$

B $2 \cdot {}^{-}8$ **E** $6 \cdot {}^{-}4$ **H** ${}^{-}2 \cdot {}^{-}9$ **K** ${}^{-}8 \cdot 5$

C ${}^{-}4 \cdot 2$ **F** ${}^{-}8 \cdot {}^{-}2$ **I** $7 \cdot {}^{-}7$ **L** ${}^{-}1 \cdot 10$

3. Find the products. Look for patterns.

A $4 \cdot 3 = \mathbf{P}$ **H** ${}^{-}4 \cdot 3 = \mathbf{P}$

B $4 \cdot 2 = \mathbf{P}$ **I** ${}^{-}4 \cdot 2 = \mathbf{P}$

C $4 \cdot 1 = \mathbf{P}$ **J** ${}^{-}4 \cdot 1 = \mathbf{P}$

D $4 \cdot 0 = \mathbf{P}$ **K** ${}^{-}4 \cdot 0 = \mathbf{P}$

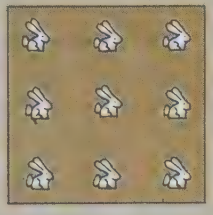
E $4 \cdot {}^{-}1 = \mathbf{P}$ **L** ${}^{-}4 \cdot {}^{-}1 = \mathbf{P}$

F $4 \cdot {}^{-}2 = \mathbf{P}$ **M** ${}^{-}4 \cdot {}^{-}2 = \mathbf{P}$

G $4 \cdot {}^{-}3 = \mathbf{P}$ **N** ${}^{-}4 \cdot {}^{-}3 = \mathbf{P}$

think

Here is a picture of nine rabbits in a pen. Copy this picture and then draw two more squares so that each rabbit will be in a separate pen.



Discussing the Ideas

We accept the same principles for multiplication of integers as for multiplication of whole numbers. Using the principles, we can justify the products which were found by using physical models in the previous lesson.

THE BASIC PRINCIPLES

One Principle
For each integer a, $a \cdot 1 = a$

Commutative Principle
For each pair of integers a and b,
$a \cdot b = b \cdot a$

Associative Principle
For each three integers a, b, and c,
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive Principle
For each three integers a, b, and c,
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$

1. A Which principle can be used to help you solve these equations?

$^-5 \cdot 1 = r$ $\quad$ $^-73 \cdot t = {}^-73$

B What are the solutions?

2. If $^-2 \cdot 8 = {}^-16$, what is $8 \cdot {}^-2$? What principle is involved?

3. A What is the integer for m?

B What is the integer for n?

C What principle do these two equations suggest?

$(^-12 \cdot 4) \cdot 3 = {}^-12 \cdot (4 \cdot m)$

$(^-6 \cdot n) \cdot {}^-7 = {}^-6 \cdot (3 \cdot {}^-7)$

4. The example at the right illustrates how the distributive principle can be used to find the product $^-2 \cdot 6$.

Give the missing number for each step.

$2 + {}^-2 =$ ▒

$6 \cdot (2 + {}^-2) =$ ▒

$(6 \cdot 2) + (6 \cdot {}^-2) =$ ▒

$12 + 6 \cdot {}^-2 =$ ▒

This is the number that adds to 12 and gives zero. → $6 \cdot {}^-2 =$ ▒

$^-2 \cdot 6 =$ ▒

5. Give the missing number for each step in finding the product $^-6 \cdot {}^-2$.

$6 + {}^-6 =$ ▒

$^-2 \cdot (6 + {}^-6) =$ ▒

$(^-2 \cdot 6) + (^-2 \cdot {}^-6) =$ ▒

$^-12 + {}^-2 \cdot {}^-6 =$ ▒

This is the number that adds to $^-12$ and gives zero. → $^-2 \cdot {}^-6 =$ ▒

$^-6 \cdot {}^-2 =$ ▒

1. Use the one principle and find the integer for ***a***. Use the commutative principle and find the integer for ***b***.

A $19 \cdot 1 = a$ $1 \cdot 19 = b$

B $^{-}8 \cdot 1 = a$ $1 \cdot {}^{-}8 = b$

C $^{-}24 \cdot 1 = a$ $1 \cdot {}^{-}24 = b$

D $86 \cdot 1 = a$ $1 \cdot 86 = b$

2. Give the integer for ***a***. Then give the integer for ***b***.

A $^{-}6 + {}^{-}6 = a$ $2 \cdot {}^{-}6 = b$

B $^{-}9 + {}^{-}9 = a$ $2 \cdot {}^{-}9 = b$

C $^{-}18 + {}^{-}18 = a$ $2 \cdot {}^{-}18 = b$

D $^{-}1 + {}^{-}1 = a$ $2 \cdot {}^{-}1 = b$

E $^{-}63 + {}^{-}63 = a$ $2 \cdot {}^{-}63 = b$

F $^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 = a$ $4 \cdot {}^{-}3 = b$

3. Give the integer for ***a***. Use the commutative principle and give the integer for ***b***.

A $2 \cdot {}^{-}8 = a$ $^{-}8 \cdot 2 = b$

B $3 \cdot {}^{-}1 = a$ $^{-}1 \cdot 3 = b$

C $2 \cdot {}^{-}42 = a$ $^{-}42 \cdot 2 = b$

D $3 \cdot {}^{-}9 = a$ $^{-}9 \cdot 3 = b$

E $4 \cdot {}^{-}96 = a$ $^{-}96 \cdot 4 = b$

F $7 \cdot {}^{-}1 = a$ $^{-}1 \cdot 7 = b$

G $6 \cdot {}^{-}8 = a$ $^{-}8 \cdot 6 = b$

H $5 \cdot {}^{-}72 = a$ $^{-}72 \cdot 5 = b$

4. Copy the equations, giving the correct number for each ▒. Each equation can be completed by thinking about the equation above it and a basic principle or multiplication fact.

A

$3 + {}^{-}3 = ▒$

$4 \cdot (3 + {}^{-}3) = ▒$

$(4 \cdot 3) + (4 \cdot {}^{-}3) = ▒$

$12 + 4 \cdot {}^{-}3 = ▒$

$4 \cdot {}^{-}3 = ▒$

This number adds to 12 to give zero. It must be what integer?

B

$5 + {}^{-}5 = ▒$

$2(5 + {}^{-}5) = ▒$

$(2 \cdot 5) + (2 \cdot {}^{-}5) = ▒$

$10 + 2 \cdot {}^{-}5 = ▒$

$2 \cdot {}^{-}5 = ▒$

C

$4 + {}^{-}4 = ▒$

$0 \cdot (4 + {}^{-}4) = ▒$

$(0 \cdot 4) + (0 \cdot {}^{-}4) = ▒$

$0 + 0 \cdot {}^{-}4 = ▒$

$0 \cdot {}^{-}4 = ▒$

5. Find the products.

A $8 \cdot {}^{-}9$

B $^{-}7 \cdot 6$

C $^{-}5 \cdot {}^{-}9$

D $^{-}17 \cdot 0$

E $^{-}23 \cdot 7$

F $8 \cdot {}^{-}56$

G $^{-}7 \cdot {}^{-}98$

H $^{-}30 \cdot {}^{-}40$

I $27 \cdot {}^{-}36$

J $^{-}42 \cdot 58$

K $^{-}29 \cdot {}^{-}97$

L $0 \cdot {}^{-}864$

M $^{-}92 \cdot 10$

N $100 \cdot {}^{-}48$

O $^{-}62 \cdot {}^{-}1000$

P $^{-}100 \cdot {}^{-}100$

Q $({}^{-}15 \cdot 13) \cdot ({}^{-}6 \cdot 0)$

R $^{-}5 \cdot {}^{-}5 \cdot {}^{-}5$

S $4 \cdot {}^{-}4 \cdot 4 \cdot {}^{-}4$

T $^{-}9 \cdot ({}^{-}4 \cdot {}^{-}10)$

6. **A** The product of a positive integer and a negative integer is a ? integer.

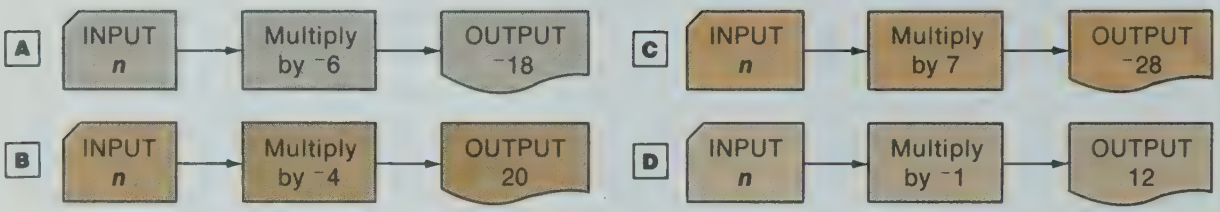
B The product of a negative integer and a negative integer is a ? integer.

C The product of any integer and zero is ?.

More practice, page S-11, Set 19

Investigating the Ideas

Study the flow charts below.



Can you find the number for n in each flow chart?

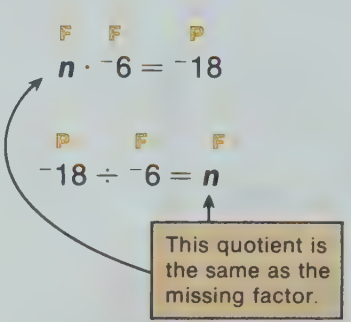
Discussing the Ideas

1. Each flow chart above suggests an equation.

   For flow chart **A**, you can write $n \cdot {}^{-}6 = {}^{-}18$.
   What are the equations for flow charts **B**, **C**, and **D**?

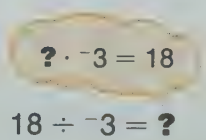
2. When you find the missing factor in a multiplication equation, you have found the **quotient** in an equivalent division equation.

   What is the quotient in the example at the right?



3. Write equivalent division equations for flow charts **B**, **C**, and **D** in the Investigation. Refer to the equations you found in Exercise 1.

4. What is $18 \div {}^{-}3$? Use the diagram to help you.



5. **A** What is $0 \div {}^{-}6$?
   B Can any integer be divided by zero? Explain your answer.

1. Find the missing factor, **F**, in each equation.

A $^{-}15 \div 5 = F$
B $^{-}15 \div {}^{-}5 = F$
C $^{-}15 \div {}^{-}15 = F$
D $15 \div {}^{-}1 = F$
E $0 \div {}^{-}15 = F$
F $^{-}20 \div 4 = F$
G $^{-}39 \div {}^{-}3 = F$
H $64 \div {}^{-}8 = F$
I $0 \div {}^{-}54 = F$
J $^{-}86 \div {}^{-}1 = F$
K $42 \div {}^{-}6 = F$
L $^{-}24 \div 8 = F$
M $^{-}24 \div ({}^{-}6 \div {}^{-}2) = F$
N $({}^{-}24 \div {}^{-}6) \div {}^{-}2 = F$
O $({}^{-}24 \div {}^{-}2) \div {}^{-}6 = F$

2. Write and solve an equivalent division equation which has n as the quotient. Use part **A** as an example.

A $^{-}8 \cdot n = 32$
$32 \div {}^{-}8 = n$
$n = {}^{-}4$
B $^{-}5 \cdot n = {}^{-}20$
C $9 \cdot n = {}^{-}54$
D $n \cdot {}^{-}6 = 6$
E $7 \cdot n = 21$
F $^{-}12 \cdot n = 60$
G $3 \cdot n = {}^{-}27$
H $n \cdot {}^{-}17 = 17$
I $^{-}8 \cdot n = 0$
J $n \cdot {}^{-}9 = 72$
K $^{-}1 \cdot n = 1$
L $^{-}15 \cdot n = 90$
M $6 \cdot n = {}^{-}144$

3. Instead of writing $^{-}12 \div 3$, we can write $\frac{^{-}12}{3}$.
Find the quotients for the following:

A $\frac{^{-}6}{2}$
B $\frac{18}{^{-}3}$
C $\frac{^{-}4}{^{-}2}$
D $\frac{^{-}28}{7}$
E $\frac{63}{^{-}9}$
F $\frac{175}{^{-}25}$
G $\frac{(15 + {}^{-}7)}{^{-}2}$
H $\frac{12}{(^{-}2 + {}^{-}1)}$
I $\frac{(9 - 3)}{(2 - 4)}$
J $\frac{(^{-}2 - {}^{-}1)}{(^{-}1 - {}^{-}2)}$

4. Complete each statement.

A The quotient of two positive integers is a ? integer.
B The quotient of a positive integer and a negative integer is a ? integer.
C The quotient of two negative integers is a ? integer.

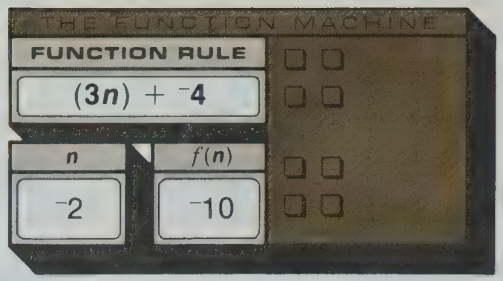
think

The number 1729 is the smallest number that can be expressed in two different ways as the sum of two cubes.
One way is: $1729 = 12^3 + 1^3$
Find another pair of whole numbers for a and b so that $1729 = a^3 + b^3$.

More practice, page S-11, Set 20

Investigating the Ideas

The function machine, with the rule shown, combines the multiplication and addition operations. The function rule can be written as $f(n) = 3n + {}^{-}4$. When we use ${}^{-}2$ for n, the output number is $(3 \cdot {}^{-}2) + {}^{-}4$ or ${}^{-}10$. We could write, "$f({}^{-}2) = {}^{-}10$."

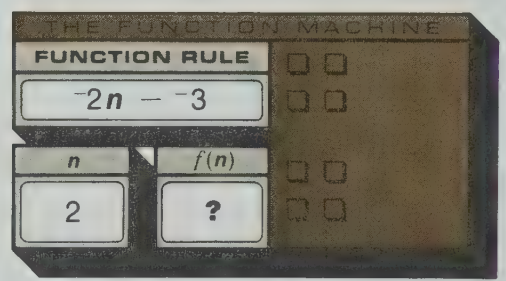


Can you find the output numbers for each of the input numbers in the set $\{{}^{-}5, {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, 4, 5\}$?

Make a table to show your answers.

Discussing the Ideas

1. A What did you find for $f({}^{-}5)$?
   B What is $f(3)$?



2. Suppose a function rule is ${}^{-}2n - {}^{-}3$.
   A What would you do to find $f(2)$?
   B What is $f({}^{-}3)$?
   C What is $f(0)$?

3. What is $f(n)$ for each part of the function table?

Function Rule: $3n + 6$

	n	$f(n)$
A	⁻4	
B	6	
C	⁻7	
D	⁻2	
E	0	
F	⁻30	

4. How would you find the number n in these equations?
   A $({}^{-}14 - 4) \div 3 = n$
   B $(3 \cdot {}^{-}6) \div {}^{-}9 = n$
   C $({}^{-}4 \cdot {}^{-}4) - {}^{-}6 = n$

Complete the function tables for Exercises 1 through 6.

1.

Function Rule: $^-2n + ^-3$

	n	$f(n)$
A	4	
B	$^-6$	
C	0	
D	$^-5$	
E	$^-9$	

2.

Function Rule: $(n \div ^-4) - 3$

	n	$f(n)$
A	28	
B	$^-36$	
C	100	
D	$^-44$	
E	$^-12$	

3.

Function Rule: $^-2n -$ [illegible]

	n	$f(n)$
A	6	
B	$^-5$	
C	4	
D	$^-10$	
E	$^-50$	

4.

Function Rule: $n^2 + 1$

	n	$f(n)$
A	2	
B	$^-2$	
C	5	
D	$^-5$	
E	1	

5.

Function Rule: n^3

	n	$f(n)$
A	2	
B	$^-1$	
C	4	
D	$^-3$	
E		$^-8$

6.

Function Rule: $(^-2)^n$

	n	$f(n)$
A	1	
B	2	
C	6	
D		$^-8$
E		$^-32$

7. Solve the equations. The replacement set is the set of integers.

A $(^-5 + 8) - ^-2 = n$

B $(9 \cdot ^-6) + ^-6 = n$

C $(72 \div ^-8) \cdot ^-8 = n$

D $(^-16 + ^-8) - ^-8 = n$

E $(^-7 - ^-4) \cdot ^-9 = n$

F $(^-9 \cdot ^-6) \div ^-6 = n$

G $(25 - ^-4) + ^-4 = n$

H $(^-4 \cdot 9) - 10 = n$

I $(29 + ^-8) \div ^-7 = n$

★ **8.** Solve the equations. The replacement set is the set of integers.

A $(t + 8) + ^-10 = ^-7$

B $(n + 84) - 85 = ^-93$

C $(^-1b) - 4 = ^-10$

D $^-4x + ^-17 = ^-1$

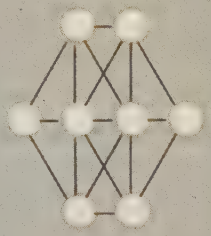
E $(t + ^-7) + ^-7 = 0$

F $(^-5 + 5) + ^-1 = v$

G $2a + 3 = ^-5$

think

Copy the diagram and arrange the numbers 1 through 8 in the circles so that no two consecutive integers are in circles that are directly connected by lines.



Investigating the Ideas

Study the equations below. Can you solve them?

A	$^{-}4 + x = 3$	D	$^{-}7 + x = {}^{-}7$	G	$^{-}1 + x = 1$
B	$^{-}2 + x = {}^{-}5$	E	$4 + x = 10$	H	$^{-}9 + x = {}^{-}12$
C	$6 + x = {}^{-}2$	F	$8 + x = 1$	I	$^{-}10 + x = {}^{-}3$

? In which equations does **x** represent a positive integer?

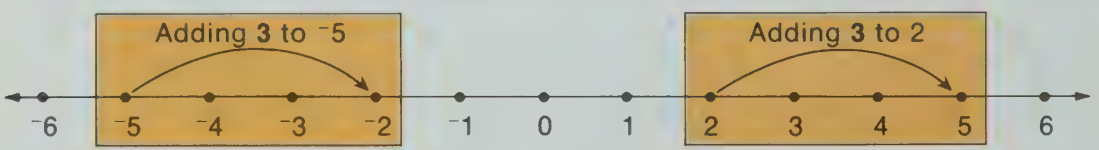
Discussing the Ideas

1. In equation A, you probably found that **x** was the *positive integer* 7. Because **x** is positive, we say that **$^{-}4$ is less than 3** and write this as $^{-}4 < 3$.
Since $^{-}4 < 3$, we also say that **3 is greater than $^{-}4$** and write this as $3 > {}^{-}4$.

   A $^{-}4 + x = 3$
   $^{-}4 +$ $= 3$
   A **positive** integer

   Compare the numbers in the other equations above in this manner.

2. The number line can help you decide which of two integers is the greater. Study the examples.



   A What integer must be added to 2 to give 5?
   B Is $2 < 5$ or is $5 < 2$?
   C What integer must be added to $^{-}5$ to give $^{-}2$?
   D Is $^{-}5 < {}^{-}2$ or is $^{-}2 < {}^{-}5$?

3. Complete the statements.
   A If ***a*** and ***b*** are any integers and $a < b$, then ***a*** is to the ? (**left**, **right**) of ***b*** on the number line.
   B If $b > a$, then ***b*** is to the ? (**left**, **right**) of ***a*** on the number line.

1. Give the integer for each ▒.

A $4 > 2$ because the positive integer ▒ must be added to 2 to give 4.

B $^{-}5 < {}^{-}1$ because the positive integer ▒ must be added to $^{-}5$ to give $^{-}1$.

C $^{-}7 < 0$ because the positive integer ▒ must be added to $^{-}7$ to give 0.

D $^{-}1 > {}^{-}99$ because the positive integer ▒ must be added to $^{-}99$ to give $^{-}1$.

2. Write the correct symbol ($<$ or $>$) for each ▒.

A $9 \;▒\; 0$ **D** $^{-}5 \;▒\; {}^{-}1$ **G** $^{-}8 \;▒\; {}^{-}10$ **J** $^{-}100 \;▒\; 100$

B $1 \;▒\; 0$ **E** $4 \;▒\; {}^{-}4$ **H** $^{-}15 \;▒\; {}^{-}20$ **K** $^{-}2 \;▒\; {}^{-}98$

C $0 \;▒\; {}^{-}1$ **F** $^{-}2 \;▒\; {}^{-}1$ **I** $^{-}75 \;▒\; {}^{-}72$ **L** $4 \;▒\; {}^{-}84$

3. Answer the questions about the set S of integers shown on the number line:

$$S = \{0, 1, {}^{-}1, 2, {}^{-}2, \ldots, 9, {}^{-}9, 10, {}^{-}10\}$$

A What is the smallest integer in S?

B What is the largest integer in S?

C List all integers in S that are less than 0.

D List all integers between 3 and $^{-}3$.

E List all integers less than $^{-}8$.

F List the integers that are both greater than $^{-}4$ and less than 0.

G What is the greatest integer that is less than $^{-}2$?

H List the integers that belong to one or the other or both of these sets: the integers less than $^{-}7$ or the integers greater than 7.

4. **A** Any negative integer is ? than any positive integer.

B Zero is ? than any negative integer.

C One integer is either greater than, equal to, or ? than another integer.

D If a first integer is greater than a second, and the second integer is greater than a third, then the first integer is ? than the third.

E If a first integer is less than a second, and the second integer is less than a third, then the first integer is ? than the third.

Think

This is an **integer magic square.** Complete the square so that the sum in each row, column, and diagonal is the same number.

6			3
	1	0	
$^{-}1$	$^{-}3$	$^{-}4$	
	4		$^{-}9$

Investigating the Ideas

Each statement below refers to a subset of the set of integers S shown on the number line below.

$$S = \{-10, -9, -8, \ldots, 0, \ldots, 8, 9, 10\}$$

How many of the subsets of S described below can you find?	Record the elements in each set.

- **A** The integers that are less than -5.
- **B** The integers that are greater than 4.
- **C** The integers between -4 and 3.
- **D** The integers that are either less than -7 or greater than 6.
- **E** The integers that when multiplied by -2 give a product greater than 10.
- **F** The integers that add to 3 to give a sum that is less than -5.

Discussing the Ideas

1. Sentence A can be shown by the *inequality* $x < -5$
   Using S as the replacement set, the **solution set** for the inequality is $\{-10, -9, -8, -7, -6\}$
   - **A** What inequality could be written for B?
   - **B** What is the solution set?

2. For C, we can write $x > -4$ and $x < 3$ or $-4 < x < 3$
   - **A** What is the solution set for $x > -4$?
   - **B** What is the solution set for $x < 3$?
   - **C** What is the intersection of the sets found in parts **A** and **B**?

3. For D, we can write $x < -7$ or $x > 6$
   - **A** What is the solution set for $x < -7$?
   - **B** What is the solution set for $x > 6$?
   - **C** Is the solution set for D the union or the intersection of these two sets?

4. **A** Describe how you found the solution set for E.
   B Describe how you found the solution set for F.

1. Find the solution set for each part in the table.
The replacement set is $\{^-4, ^-3, ^-2, ^-1, 0, 1, 2, 3, 4\}$.
Parts **C** and **D** are partially completed.

	INEQUALITY	SOLUTION SET
A	$x > ^-3$	
B	$x < 2$	
C	$x > ^-3$ and $x < 2$	$\{^-2, ^-1, 0, 1, 2, 3, 4\}$ $\{^-4, ^-3, ^-2, ^-1, 0, 1\} =$
D	$x < ^-2$ or $x > 3$	$\{^-3, ^-4\}$ $\{4\} =$
E	$x + 3 > 0$	
F	$x + ^-2 < 0$	

2. Find the solution set for the following sentences.
The replacement set is $T = \{^-3, ^-2, ^-1, 0, 1, 2, 3, 4, 5\}$.

A $x < 2$
B $x < ^-1$
C $x > ^-3$
D $x < 0$
E $x + ^-1 < 0$
F $x + 3 < 3$
G $x - 2 > ^-2$
H $^-2 \cdot x > 0$
I $x \div ^-1 < 2$
J $x < ^-1$ and $x > ^-3$
K $x > ^-1$ and $x < 3$
L $x > ^-1$ and $x < 2$
M $x < ^-2$ or $x > 2$
N $x < ^-1$ or $x > 1$
O $x < 0$ or $x > 0$
P $x < 1$ or $x > ^-1$
Q $(2 \cdot x) + 3 < 0$
R $^-4 - x > ^-1$

3. Find the solution set for each part.
The replacement set is

$S = \{0, 1, ^-1, 2, ^-2, 3, ^-3, \ldots, 10, ^-10\}$.

A $x > 7$
B $y < ^-7$
C $a + 2 < ^-7$
D $x + 5 > 0$
E $m - 2 < ^-5$
F $2 \cdot t < ^-4$
G $^-3 \cdot n > 10$
H $y \leq ^-6$
I $z \geq 5$
J $y^2 < 5$
K $t^3 < ^-7$
L $s^2 - s > 6$

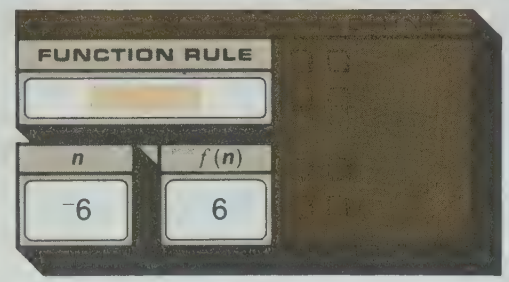
Think

Three men stopped at a hotel and rented a suite of rooms for $60. Each man paid $20. On the way to the rooms, the clerk discovered he had erred and that the suite of rooms was only $55. He found the men in their rooms and returned the $5.00. Not knowing how to split the $5.00, they gave the clerk $2.00 and kept $1.00 apiece. Thus, each man paid $19 for his room or $57 for all three rooms. The clerk got $2.00, making a total of $59. What happened to the other dollar?

Investigating the Ideas

Function table A shows the input-output pairs for an interesting function rule.

Can you discover the function rule and complete function table B?



A

Function Rule	
[redacted]	
n	$f(n)$
$^{-}6$	6
4	4
$^{-}5$	5
0	0
17	17
$^{-}12$	12

B

Function Rule	
[redacted]	
n	$f(n)$
$^{-}4$	[redacted]
5	[redacted]
$^{-}7$	[redacted]
13	[redacted]
$^{-}10$	[redacted]
$^{-}30$	[redacted]

Discussing the Ideas

1. **A** If the input number n is a positive integer, what is $f(n)$?
   B If the input number n is a negative integer, what is $f(n)$?
   C If the input number is 0, what is $f(n)$?

2. The function rule illustrates the **absolute value** function. In this function, the output number is either the input number or its opposite, whichever is nonnegative. Instead of using $f(n)$ to denote the output number, it is standard practice to use the symbol $|n|$. Thus, $|^{-}7| = 7$.
   We say, "The absolute value of $^{-}7$ is 7."
   What is the absolute value of each of these numbers?
   A $^{-}4$ **B** 10 **C** 0 **D** $^{-}11$ **E** $^{-}1$ **F** 15

3. What number is represented by each of the following:
   A $|^{-}2|$ **B** $|8|$ **C** $|^{-}23|$ **D** $|^{-}14|$ **E** $|100|$

4. Explain how to find the integer for n in each part.
   A $|^{-}4 + 2| = n$ **B** $|2 \cdot {}^{-}3| = n$ **C** $|^{-}4 \cdot {}^{-}2| = n$ **D** $|6 - 8| = n$

1. Copy and complete the function tables.

A Function Rule: $|n|$

n	$f(n)$
$^-8$	
$^-2$	
20	
53	
$^-3$	
16	

B Function Rule: $|n|$

n	$f(n)$
$^-3 + {}^-1$	
$^-3 - {}^-1$	
$^-3 + 1$	
$3 \cdot {}^-1$	
$^-3 - 1$	
$^-3 \cdot 2$	

C Function Rule: $|n|$

n	$f(n)$
$2 + {}^-3$	
$^-2 + 3$	
$^-2 \cdot 3$	
$^-2 \div 1$	
$^-2 \div {}^-1$	
$^-2 \cdot {}^-3$	

2. What integer is represented by each of the following?

A $|^-5|$ **B** $|5|$ **C** $|^-18|$ **D** $|2|$ **E** $|^-47|$ **F** $|7 - 3|$ **G** $|3 - 7|$ **H** $|^-9 + 6|$ **I** $|^-4 + {}^-2|$ **J** $|^-5 \cdot 4|$

3. Solve the equations.

A $x = |^-5|$ **B** $|6| = a$ **C** $|^-2 + {}^-3| = b$ **D** $n = |0 - 5|$ **E** $|^-4 \cdot 2| = v$ **F** $s = |^-24 + 5|$ **G** $t = |15 - 6|$ **H** $|8 \div {}^-2| = y$

4. Give all solutions for each equation.

EXAMPLE: $|x| = 6$

SOLUTION: Since $|6| = 6$ and $|^-6| = 6$, then $x = 6$ or $x = {}^-6$.

A $|x| = 3$ **B** $|x| = 8$ **C** $|x| = 4$ **D** $|x| = 10$ **E** $|x^2| = 25$ **F** $|x + 1| = 4$ **G** $|x - 1| = 3$ **H** $|^-3 \cdot x| = 9$

★ **5.** Give the integer for each ▒.

A $|x| = 7$ and $x < 0 \rightarrow x =$ ▒

B $|x| = 5$ and $x > 0 \rightarrow x =$ ▒

C $|2x| = 6$ and $x < 0 \rightarrow x =$ ▒

D $|x| = 2$ and $x > 0 \rightarrow x =$ ▒

E $|x| = 2$ and $x < 0 \rightarrow x =$ ▒

F $|x| = 13$ and $x < 0 \rightarrow x =$ ▒

think

You have nine coins and a pan balance scale. The coins are alike in appearance, but you know that just one of them is counterfeit and lighter than or heavier than the others, but you do not know which. In three weighings, how could you find the bad coin and determine whether it is a heavier or a lighter coin?

Discussing the Ideas

1. Using the number line, you can think of $|n|$ as asking the question, "How far is n from zero on the number line?"

For $|^-6|$, we think $^-6$ is 6 units from zero

A Using the number line, how far is 7 from zero?

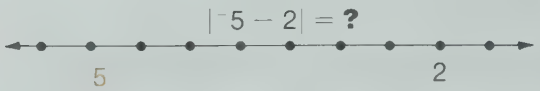
B What is the absolute value of 7?

C How far is $^-20$ from zero?

D What is $|^-20|$?

2. The statement $|n| = 9$ means that a certain number n is 9 units from zero on the number line. Give two possible values for n.

3. You can think of $|a - b|$ as asking the question, "How far is a from b on the number line?"



We think: $^-5$ is 7 units from 2.
We write: $|^-5 - 2| = 7$.
We say: "The absolute value of $^-5 - 2$ is 7."

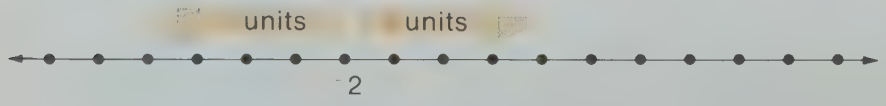
A How far is $^-8$ from $^-3$ on the number line?

B What is $|^-8 - {}^-3|$?

C How far is 6 from 13 on the number line?

D What is $|6 - 13|$?

4. The equation $|^-2 - x| = 4$ can be solved by thinking, "What numbers are 4 units from $^-2$ on the number line?"



A How many solutions does the equation have?

B What are the solutions?

1. Copy the equations, giving the correct integer for each ▒. (It may help you to use the number line.)

A How far is it from $^-8$ to 0? → $|^-8| =$ ▒

B How far is it from 12 to 0? → $|12| =$ ▒

C How far is it between $^-3$ and 5? → $|^-3 - 5| =$ ▒

D How far is it between 5 and $^-7$? → $|5 - {}^-7| =$ ▒

E $|^-15| =$ ▒ **F** $|2| =$ ▒ **G** $|49| =$ ▒ **H** $|^-100| =$ ▒

I $|^-2 - 12| =$ ▒ **J** $|12 - {}^-2| =$ ▒ **K** $|^-15 - 0| =$ ▒ **L** $|9 - 20| =$ ▒

2. Think about the number line and give all the possible numbers for b in each equation.

A $|6 - b| = 6$ **B** $|4 - b| = 4$ **C** $|^-2 - b| = 2$ **D** $|^-9 - b| = 9$ **E** $|6 - b| = 5$ **F** $|4 - b| = 3$ **G** $|^-2 - b| = 1$ **H** $|^-9 - b| = 10$

3. Give the correct symbol ($<$, $>$, or $=$) for each ▒.

A $|6|$ ▒ $|^-6|$ **B** $|5 - (^-5)|$ ▒ 5 **C** $|^-6 - 0|$ ▒ $|^-6|$ **D** $|^-7|$ ▒ $^-7$ **E** $|6 - 1|$ ▒ $|6| - |1|$ **F** $|6 - {}^-1|$ ▒ $|6| - |^-1|$

4. Solve each inequality. The replacement set is $S = \{^-10, ^-9, ^-8, \ldots, 0, \ldots, 8, 9, 10\}$.

A $|x| > 8$ Solution set: $\{9, 10, ^-9, ^-10\}$ **B** $|x| < 4$ **C** $|x - 2| < 3$ **D** $|x - 3| > 5$

think

1. If you were offered a job and could choose between Salary Plans A and B, which would you choose?
2. What would your third-year salary be under Plan A? Under Plan B?
3. What would your total earnings for two years be under Plan A? Under Plan B?
4. If you were an employer choosing between two equally qualified men, would you choose the one who accepted Plan A or the one who accepted Plan B? Explain your choice.

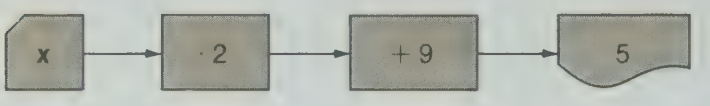
A Start at $8000 per year with a $400 increase on the previous salary at the end of each year

B Start at $4000 per each half year with a $100 increase on the previous salary at the end of each half year

More practice, page S-12, Set 21

Investigating the Ideas

The flow chart below suggests an equation with the set of all integers for the replacement set.



x $\quad 2x \quad 2x + 9 \quad 2x + 9 = 5$

Can you solve the equation by finding the input number for the flow chart?

Discussing the Ideas

1. Tony said he thought about an inverse flow chart for the one in the Investigation.

   What would the rest of the flow chart be like?

2. Kitty said she thought about inverse operations to solve the equation. What are the missing numbers in her solution?

If $2x + 9 = 5$
$2x = 5 -$ ▮
then $2x =$ ▮
and then $x =$ ▮

3. Instead of writing → If $3x + 2 = 17$,
   then $3x = 15$,
   and then $x = 5$,

   you can indicate the same reasoning by using arrows:

   $$3x + 2 = 17 \implies 3x = 15 \implies x = 5.$$

   The arrow $\implies$ can be read as "implies."

   What are the missing numbers in each step below?

   $$4x - 3 = {}^{-}15 \implies 4x = \blacksquare \implies x = \blacksquare$$

4. How can you check the number you found for x in Exercise 3 to see if it is a solution for the equation?

The replacement set for all equations on this page is the set of integers.

1. Give the correct number for each ▒.

A $x + 4 = 1 \Rightarrow x =$ ▒

B $x + {}^-6 = 2 \Rightarrow x =$ ▒

C $3x = {}^-15 \Rightarrow x =$ ▒

D $x \div {}^-4 = 2 \Rightarrow x =$ ▒

E $x - {}^-7 = {}^-3 \Rightarrow x =$ ▒

F $^-7x = {}^-28 \Rightarrow x =$ ▒

2. Give the missing number in each step.

A If $3x + 2 = {}^-7$,
then $3x =$ ▒,
and $x =$ ▒.

B If $2x - 4 = {}^-10$,
then $2x =$ ▒,
and $x =$ ▒.

C If $4x + 5 = 1$,
then $4x =$ ▒,
and $x =$ ▒.

3. Give the missing integers in each equation.

A $2x + 3 = 13 \Rightarrow 2x =$ ▒ $\Rightarrow x =$ ▒

B $3x + 7 = 1 \Rightarrow 3x =$ ▒ $\Rightarrow x =$ ▒

C $^-2x + 1 = 9 \Rightarrow {}^-2x =$ ▒ $\Rightarrow x =$ ▒

D $2(x + 3) = {}^-6 \Rightarrow x + 3 =$ ▒ $\Rightarrow x =$ ▒

E $^-3(x + 4) = {}^-9 \Rightarrow x + 4 =$ ▒ $\Rightarrow x =$ ▒

4. Solve the equations.

A $6x + {}^-5 = 7$

B $^-4x + 1 = 17$

C $(x + 3) \div 2 = {}^-3$

D $(x \div {}^-5) + 2 = 0$

E $^-2x + {}^-9 = {}^-3$

F $(x + {}^-7) - 2 = {}^-4$

G $4(x + 5) = {}^-12$

H $^-5(x + {}^-7) = 10$

I $^-3(x + 6) = {}^-12$

5. Copy the equations, replacing each by the correct numbers. Part **A** is completed for you.

A If $|2x| = 6$,
then $2x = 6$, or $2x = {}^-6$,
and $x =$, or $x =$.

B If $|3x| = 12$,
then $3x =$ ▒, or $3x =$ ▒,
and $x =$ ▒, or $x =$ ▒.

C If $|x + 3| = 5$,
then $x + 3 =$ ▒, or $x + 3 =$ ▒,
and $x =$ ▒, or $x =$ ▒.

Think

Using a certain code,
the word ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓
becomes

Using the same code, what words do
and represent?

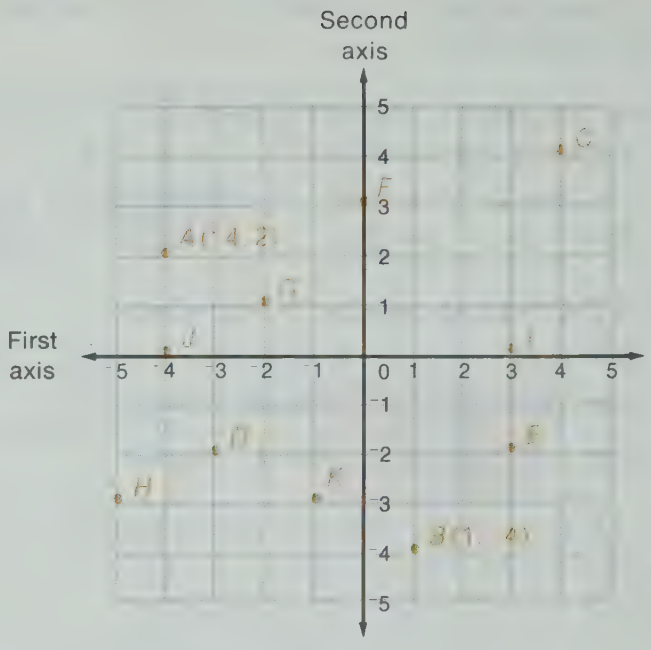
More practice, page S-12, Set 22

Investigating the Ideas

Point A can be associated with the **ordered pair** of integers $(^-4, 2)$.

The **co-ordinates** of point A are $(^-4, 2)$.

The co-ordinates of point B are $(1, ^-4)$.



Can you give the coordinates of the other lettered points?

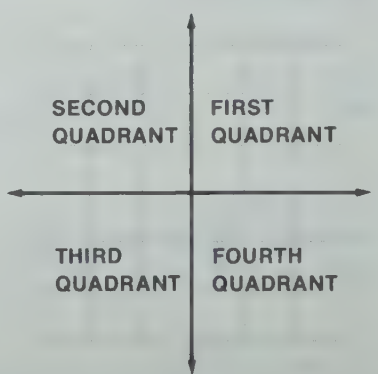
Discussing the Ideas

1. The two crossed number lines are called the **co-ordinate axes.** What are the co-ordinates of the point of intersection of the axes? (This point is called the **origin** of the co-ordinate axes.)

2. Explain how to locate the point for any ordered pairs of integers.

3. Jack said, "I am thinking about a point on the first axis." What can you say about the co-ordinates of his point?

4. The two axes divide the co-ordinate plane into four regions called **quadrants**, which are numbered as shown at the right.



   - **A** In which quadrant do all points have positive co-ordinates?
   - **B** In which quadrant is the first number in each pair negative and the second number positive?
   - **C** In which quadrant is the first number positive and the second number negative?
   - **D** In which quadrant do all points have negative numbers for both co-ordinates?

1. Graph these number pairs in the co-ordinate plane and label them with the letter indicated.

A $(7,6)$, *A*	**D** $(6,^{-}4)$, *D*	**G** $(6,0)$, *G*	**J** $(3,3)$, *J*
B $(^{-}4,5)$, *B*	**E** $(^{-}7,0)$, *E*	**H** $(0,6)$, *H*	**K** $(3,^{-}3)$, *K*
C $(^{-}5,^{-}6)$, *C*	**F** $(0,^{-}7)$, *F*	**I** $(^{-}3,3)$, *I*	**L** $(^{-}3,^{-}3)$, *L*

For Exercises 2 through 9, draw the co-ordinate axes (indicating the integers between $^{-}10$ and 10). Then graph the set of integer pairs indicated.

2. The set of integer pairs in which the first number is equal to the second number.

3. The set of integer pairs in which the first number is one greater than the second number.

4. The set of integer pairs in which the second number is one greater than the first number.

5. The set of integer pairs in which the sum of the first number and the second number is zero.

6. The set of integer pairs in which the second number is zero.

7. The set of integer pairs in which the first number is zero.

8. The set of integer pairs in which the second number is three times the first number.

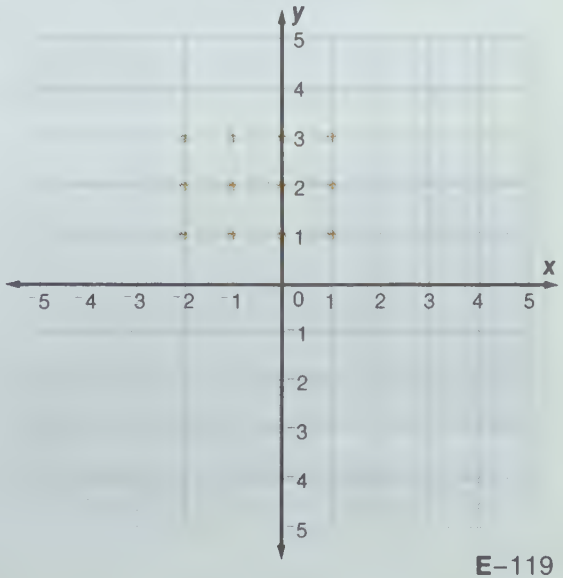
★ 9. The set of integer pairs in which the second number is the absolute value of the first number.

10. If ***(x, y)*** represents the co-ordinates of the points in the co-ordinate plane, then the points graphed at the right are the set of points (x,y) such that $^{-}2 \leq x \leq 1$ and $1 \leq y \leq 3$. Draw graphs of these inequalities.

A $^{-}5 \leq x \leq ^{-}1$ and $1 \leq y \leq 4$

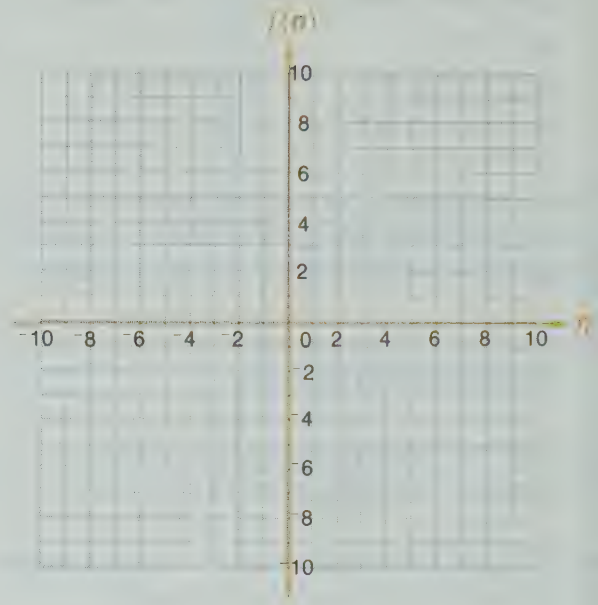
B $x = 3$ and $^{-}3 \leq y \leq 3$

C $^{-}2 \leq x \leq 2$ and $y = 2$



Investigating the Ideas

Given a function rule and a set of input numbers for the function, you can find the corresponding output numbers. When you graph the points for the input-output pairs, you have **graphed the function**.



Can you graph the function whose rule is $f(n) = 9 - n^2$, if the set of input numbers is $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$?

Discussing the Ideas

1. For which input number n is $f(n)$ the largest?

2. For which input number or numbers is $f(n)$ a negative integer?

3. For which integers n is $f(n) = 0$?

4. Do the points that you graphed for the function fall into a pattern? Describe the pattern.

5. A function rule is $f(n) = n^2 + n - 1$.
   If $n = -4$, then $f(-4) = (-4)^2 + -4 - 1$
   $= 16 + -4 - 1 = 11$.
   Give the other output numbers in the table below.

n	-4	-3	-2	-1	0	1	2	3
$f(n)$	11							

6. Draw a graph of the function in Exercise 5 using the number pairs in the table.

1. Complete each function table. Then graph the function for the ordered pairs in the table.

A

Function Rule: $2n + {}^{-}3$	
n	$f(n)$
$^{-}2$	$^{-}7$
$^{-}1$	$^{-}5$
0	
1	
2	
3	
4	
5	

B

Function Rule: $n^2 - 4n + 4$	
n	$f(n)$
5	9
4	
3	
2	
1	
0	
$^{-}1$	
$^{-}2$	

C

Function Rule: $12 \div n$	
n	$f(n)$
12	1
6	
4	
3	
2	
1	
$^{-}1$	
$^{-}2$	
$^{-}3$	
$^{-}4$	
$^{-}6$	

2. Draw the graph of a function whose rules are as follows:
$f(n) = 3$, if $n > 0$,
$f(n) = 0$, if $n = 0$,
$f(n) = {}^{-}2$, if $n < 0$.
Use the integers $^{-}5 \leq n \leq 5$ as input numbers.

★ **3.** Draw a graph of the absolute value function $f(n) = |n|$. Use the integers $^{-}5 \leq n \leq 5$ as input numbers.

★ **4.** **A** Draw graphs of $f(n) = n^2$ and $f(n) = n + 2$ on the same grid. Use integer inputs from $^{-}4$ to 4.

B What points do the two graphs have in common?

think

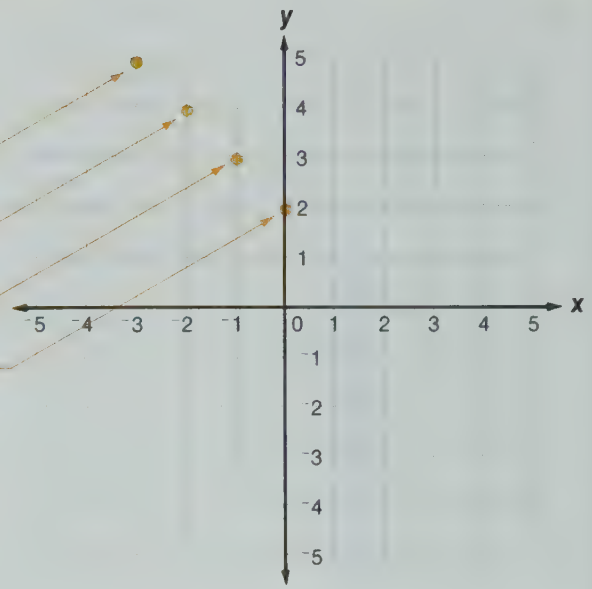
Can you complete the square to form a **multiplication magic square**? All the products in each column, row, and diagonal must be the same number.

256		
	32	
16		4

Investigating the Ideas

Some ordered pairs of integers (x, y) have been graphed for the equation $x + y = 2$.

$x + y = 2$	(x, y)
$^-3 + 5 = 2$	$(^-3, 5)$
$^-2 + 4 = 2$	$(^-2, 4)$
$^-1 + 3 = 2$	$(^-1, 3)$
$0 + 2 = 2$	$(0, 2)$
⋮	⋮



Can you list five more ordered pairs of integers for the equation?

Discussing the Ideas

1. Locate the points for the ordered pairs that you found for the Investigation question. Do the points follow the pattern of the points already graphed?

2. To find the integer pairs (x, y) for an equation, you can try some integers for x in order to determine what y must be. Study the example at the right. Explain how you can find some other ordered pairs for the equation.

EQUATION: $3x - y = 2$

If $x = 0$, then $3 \cdot 0 - y = 2$

$0 - y = 2$

$y = ^-2$

$(0, ^-2)$ is one ordered pair of integers for the equation.

3. Complete the table to find some pairs (x, y) for the equation $3x - y = 2$.

$3x - y = 2$

x	0	1	2	3	$^-1$	$^-2$	$^-3$
y	$^-2$	1					

4. Graph the ordered pairs (x, y) in Exercise 3.

1. Complete the table of ordered pairs (x, y) for the equation $x + y = 0$.

x	-5	5	-4	4	-3	3	-2	-1	1	0
y										

2. Graph the order pairs in the table of Exercise 1. This is part of the graph of $x + y = 0$.

3. Complete the table of ordered pairs (x, y) for the equation $x + 2y = 8$.

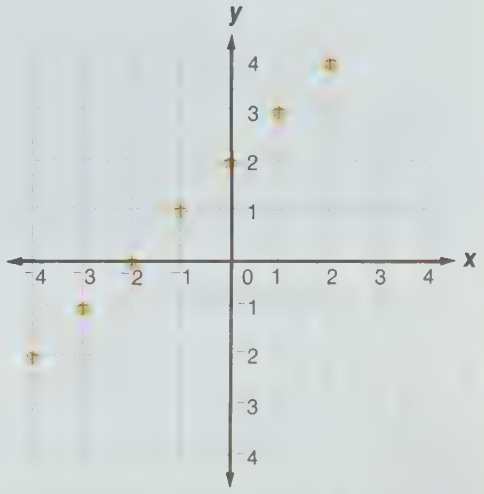
x	0	2	4	6	-2	-4	-6
y							

4. Graph the ordered pairs of Exercise 3. This is part of the graph of $x + 2y = 8$.

5. Make a table of ordered pairs (x, y) for the equation $x - y = 0$. Use both positive and negative integers.

6. Draw the graph of the equation $x - y = 0$.

★ **7.** Part of the graph of an equation is shown. What is the equation?

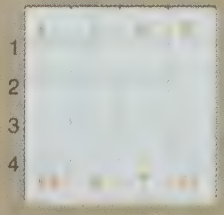


★ **8.** Make a table of ordered pairs (x, y) for each equation. Then draw a graph for each equation.

- **A** $x + y = 6$
- **B** $x - y = 4$
- **C** $x = 2y$
- **D** $y = 2x$
- **E** $2x + 3y = 12$

think

Change [redacted] into [redacted] in four steps by changing one letter at each step to get a new four-letter word each time.



REVIEWING THE IDEAS

1. Give the opposite of each integer.

A $^{-}5$ **B** 8 **C** $^{-}12$

2. The sum of any integer and its opposite is ?.

3. Find the sums.

A $18 + {}^{-}9$ **B** $6 + {}^{-}13$ **C** $^{-}8 + {}^{-}23$

4. Find the missing addends.

A $0 - {}^{-}2 = \mathbf{A}$ **C** $8 - {}^{-}2 = \mathbf{A}$

B $^{-}5 - {}^{-}3 = \mathbf{A}$ **D** $^{-}5 - {}^{-}5 = \mathbf{A}$

5. Solve each equation and give the name of the basic principle the equation illustrates.

A $16 + 0 = a$ **B** $^{-}9 \cdot 1 = b$

C $c + 8 = 0$ **D** $^{-}9 + d = 17 + {}^{-}9$

E $(8 + 4) + {}^{-}4 = e + (4 + {}^{-}4)$

F $^{-}5 \cdot (2 + 1) = (^{-}5 \cdot f) + (^{-}5 \cdot 1)$

G $(^{-}11 \cdot {}^{-}20) \cdot {}^{-}5 = {}^{-}11 \cdot (h \cdot {}^{-}5)$

6. Find the products.

A $7 \cdot 6$ **B** $^{-}7 \cdot 6$ **C** $^{-}7 \cdot {}^{-}6$

7. Find the missing factors.

A $^{-}54 \div 6 = \mathbf{F}$ **C** $^{-}72 \div {}^{-}9 = \mathbf{F}$

B $54 \div {}^{-}9 = \mathbf{F}$ **D** $0 \div {}^{-}8 = \mathbf{F}$

8. Give each solution set.
Replacement set:
$R = \{^{-}5, {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, 4, 5\}$.

A $x < {}^{-}1$ **C** $x < 3$ and $x > {}^{-}2$

B $x > {}^{-}1$ **D** $x > 3$ or $x < {}^{-}4$

9. **A** What is the absolute value of 9?
B What is the absolute value of $^{-}9$?

10. Solve the equations.

A $|{}^{-}13| = a$ **D** $|{}^{-}6 - {}^{-}9| = a$

B $|27| = a$ **E** $|a - 2| = 1$

C $|5 - {}^{-}1| = a$ **F** $|4 - a| = 2$

11. Give the integer for each screen.

A $2(x + 5) = 14 \Rightarrow x + 5 =$ ▒
$\Rightarrow x =$ ▒

B $6 + (^{-}8 + y) = 0 \Rightarrow {}^{-}8 + y =$ ▒
$\Rightarrow y =$ ▒

C $6(x - 2) = 18 \Rightarrow x - 2 =$ ▒
$\Rightarrow x =$ ▒

D $^{-}2(5 + n) = {}^{-}2 \Rightarrow 5 + n =$ ▒
$\Rightarrow n =$ ▒

E $(-3 + a) \div 4 = {}^{-}2 \Rightarrow {}^{-}3 + a =$ ▒
$\Rightarrow a =$ ▒

12. Draw the co-ordinate axes (indicating points for the integers between $^{-}6$ and 6) and graph the integer pairs, using the indicated letters.

A $(4, 2)$, A **E** $(4, 0)$, E

B $(^{-}3, 5)$, B **F** $(0, 3)$, F

C $(^{-}2, {}^{-}4)$, C **G** $(0, {}^{-}3)$, G

D $(5, {}^{-}3)$, D **H** $(^{-}5, 0)$, H

13. Graph the set of integer pairs in which the second number is one more than the first number.

14. Give the missing integers. Then graph the functions.

A

Function Rule	
$2n - 1$	
n	$f(n)$
$^{-}3$	$^{-}7$
$^{-}2$	▒
$^{-}1$	$^{-}3$
0	▒
1	▒

B

Function Rule	
$n^2 - 4$	
n	$f(n)$
$^{-}3$	5
$^{-}2$	▒
$^{-}1$	▒
0	▒
1	▒

15. Which integer is larger, a or b, if

A $a + 4 = b$? **C** $a = b - 7$?

B $a - b = 6$? **D** $b - a = 5$?

TEST YOURSELF

1. Give the opposite of each integer.
   A $^{-}5$ B 16 C 0

2. Find the sums.
   A $^{-}7 + ^{-}2$ C $9 + ^{-}4$
   B $^{-}8 + 5$ D $^{-}3 + ^{-}6$

3. Find the differences.
   A $8 - 10$ C $4 - ^{-}2$
   B $^{-}5 - 3$ D $^{-}2 - ^{-}7$

4. Find the products.
   A $6 \cdot ^{-}3$ B $^{-}8 \cdot 4$ C $^{-}5 \cdot ^{-}3$

5. Find the quotients.
   A $16 \div ^{-}4$ C $^{-}30 \div 5$
   B $^{-}24 \div ^{-}3$ D $^{-}18 \div ^{-}6$

6. Write the correct symbol ($<$ or $>$) for each ▒
   A 5 ▒ $^{-}3$ C $^{-}6$ ▒ $^{-}2$
   B $^{-}1$ ▒ 1 D $^{-}9$ ▒ $^{-}15$

7. What is the solution set for $x > ^{-}4$ and $x < 3$?

8. If $|x - 2| = 5$, what are all the possible integers for x?

9. Solve the equation $3x + 8 = ^{-}13$.

10. Complete the function rule table.

Function Rule

[illegible]

	[illegible]
2	5
1	1
0	A
$^{-}1$	B
$^{-}2$	C
$^{-}3$	D

RESEARCH PROJECTS

A Find how to make a **nomograph**. It is a device that can be used to find sums and differences of integers. Then construct your own nomograph. (See *Making and Using Graphs and Nomographs* by Richard A. Denholm; Pasadena, California: Franklin Publications, 1968, pp. 45-47; Grolier Ltd.)

B It is possible to represent any integer using a negative number base. For example, using base *negative three*, $21_{(^{-}3)}$ means $2 \cdot (^{-}3) + 1 = ^{-}5$, $112_{(^{-}3)}$ means $1 \cdot (^{-}3)^2 + 1 \cdot (^{-}3) + 2 = 8$, and $2111_{(^{-}3)}$ means $2 \cdot (^{-}3)^3 + 1 \cdot (^{-}3)^2 + 1 \cdot (^{-}3) + 1 = ^{-}47$. Choose some negative integer as a base and show how to represent both positive and negative integers in this base.

CUMULATIVE REVIEW

1. **A** Write 1893 using Roman numerals.

B Write MDCXVII using Hindu-Arabic numerals.

2. Use exponents to express $7 \cdot 7 \cdot 7 \cdot 7$.

3. Solve the equations.

A $10^2 \cdot 10^4 = n$ **C** $10^4 \div 10^2 = n$

B $2^5 \cdot 2^2 = n$ **D** $10^3 \div 10^3 = n$

4. **A** Write the Hindu-Arabic numeral for $(5 \cdot 10^6) + (4 \cdot 10^3) + (3 \cdot 10^1) + (9)$.

B Write 78 463 097 using expanded notation.

5. Round 746 813 to the nearest

A thousand.

B ten thousand.

C hundred thousand.

6. Estimate the product.

$$\begin{array}{r} 421 \\ \times\ 77 \\ \hline \end{array}$$

7. Estimate the quotient.

$$394\overline{)287\,449}$$

8. Compute.

A $87 + 287 + 778 + 19 + 5$

B $9000 - 576$

C $678 \cdot 932$

D $21\,544 \div 12$

9. Write each numeral in base two.

A $5_{(ten)}$ **B** $25_{(ten)}$ **C** $77_{(ten)}$

10. Compute as indicated and give the result as a binary numeral.

A
$$\begin{array}{r} 101_{(2)} \\ 11_{(2)} \\ +\ 111_{(2)} \\ \hline \end{array}$$

B
$$\begin{array}{r} 1001_{(2)} \\ -\ \ 11_{(2)} \\ \hline \end{array}$$

C
$$\begin{array}{r} 101_{(2)} \\ \times\ 11_{(2)} \\ \hline \end{array}$$

11. $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

A Find a subset of X which contains all of the odd numbers in X.

B Find a subset of X which contains all of the even numbers in X.

C What is the union of the sets in parts **A** and **B**?

12. Sets R and S are disjoint. Which statement below is true?

A $R \subset S$ **B** $R \cap S = \phi$

13. Evaluate the expression $3x + 7$ when $x = 5$.

14. Find the solution set of each open sentence. The set $\{0, 1, 2, 3, 4, 5\}$ is the replacement set.

A $3 + n = 7$ **B** $n < 4$

C $n \cdot 2 > n + 2$ **D** $n^2 > 7$

15. Give the algebraic expression for each ▥.

A $5x + 2x =$ ▥ **C** $3x + 4x + 5x =$ ▥

B $9x + 6x =$ ▥ **D** $9x + x =$ ▥

16. Write the equation suggested by this flow chart.

17. What is the whole number that is the solution for the equation in Exercise 16?

18. Find the solution set for each inequality. The replacement set is the set of whole numbers.

A $x > 14$ and $x < 19$

B $x = 5$ *or* $x = 3$

C x is divisible by 3 and $x < 10$.

D $2x < 8$ and x is odd.

19. **A** What are the first 10 prime numbers?
B What are the first 10 composite numbers?

20. Give the prime factorization of each number.
A 98 **B** 102 **C** 625

21. Give the GCF of
A 16 and 30. **B** 100 and 30.

22. Give the LCM of
A 8 and 10. **B** 15 and 35.

23. Which of the following numbers are divisible by 2? By 3? By 5? By 9? By 10?
A 15 **B** 330 **C** 78 **D** 225

24. Write the first five terms of a sequence whose n^{th} term is $n^2 + 3$.

25. Give the opposite of each number.
A $^-46$ **B** 97 **C** $^-1000$

26. Find the sums.
A $23 + {}^-5$ **B** $^-17 + 9$

27. Find the differences.
A $13 - 19$ **C** $7 - {}^-5$
B $^-8 - 5$ **D** $^-10 - {}^-35$

28. Find the products.
A $^-4 \cdot 8$ **B** $^-5 \cdot {}^-9$ **C** $^-7 \cdot 0$

29. Find the quotients.
A $45 \div {}^-9$ **C** $^-200 \div {}^-4$
B $^-27 \div 3$ **D** $(^-72 \div {}^-8) \div {}^-3$

30. Solve the equations.
A $^-9 \cdot h = {}^-27$ **C** $k \div 5 = {}^-45$
B $w \cdot {}^-4 = 24$ **D** $^-8 = 32 \div y$

31. Complete each table.

A

Function Rule $n = 3$	
n	$f(n)$
2	
$^-5$	
	7
	$^-3$

B

Function Rule $n^2 - 3n$	
n	$f(n)$
1	
$^-2$	
	0
8	

32. Give the correct symbol ($<$ or $>$) for each ▒.
A 5 ▒ $^-7$ **C** $^-4$ ▒ 4
B $^-5$ ▒ 9 **D** 0 ▒ $^-17$

33. Solve the inequalities. The replacement set is $\{^-4, {}^-2, 0, 2, 4, 6\}$.
A $x + 1 > 3$ **C** $2x < {}^-1$
B $x - 3 < 6$ **D** $^-2x > 1$

34. Give the integer for each.
A $|{}^-4|$ **C** $|{}^-81 + 13|$ **E** $|8 - 8|$
B $|8 + 4|$ **D** $|{}^-13 + 81|$ **F** $|{}^-3 - {}^-2|$

35. Give the whole number for each ▒.
A If $4(x - 7) = 0$, then $x - 7 =$ ▒ and then $x =$ ▒
B If $9(x - 4) = 9$, then $x - 4 =$ ▒ and then $x =$ ▒

36. **A** Complete the table for $x + y = 4$.

x	0	1	2	3	$^-1$	$^-2$	$^-3$	$^-4$
y	4	3	▒	▒	▒	▒	▒	▒

B Draw a graph of the ordered pairs in part **A**.

TOOTHPICK PATTERNS

Suppose that you have a box of 500 toothpicks.

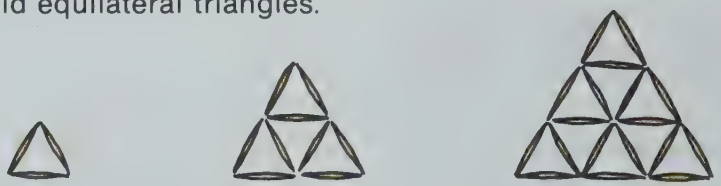
Using the toothpicks, you could build stair-patterned sets of squares like these:

4 toothpicks 10 toothpicks 18 toothpicks

A How many squares are in the largest complete stair-pattern that you could make using as many of the 500 toothpicks as possible?

B How many toothpicks would you have left?

Try the same problem, but this time build equilateral triangles.



3 toothpicks 9 toothpicks 18 toothpicks

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A 5; B $^{-}16$; C 0 **2.** A $^{-}9$; B $^{-}3$; C 5; D $^{-}9$ **3.** A $^{-}2$; B $^{-}8$; C 6; D 5

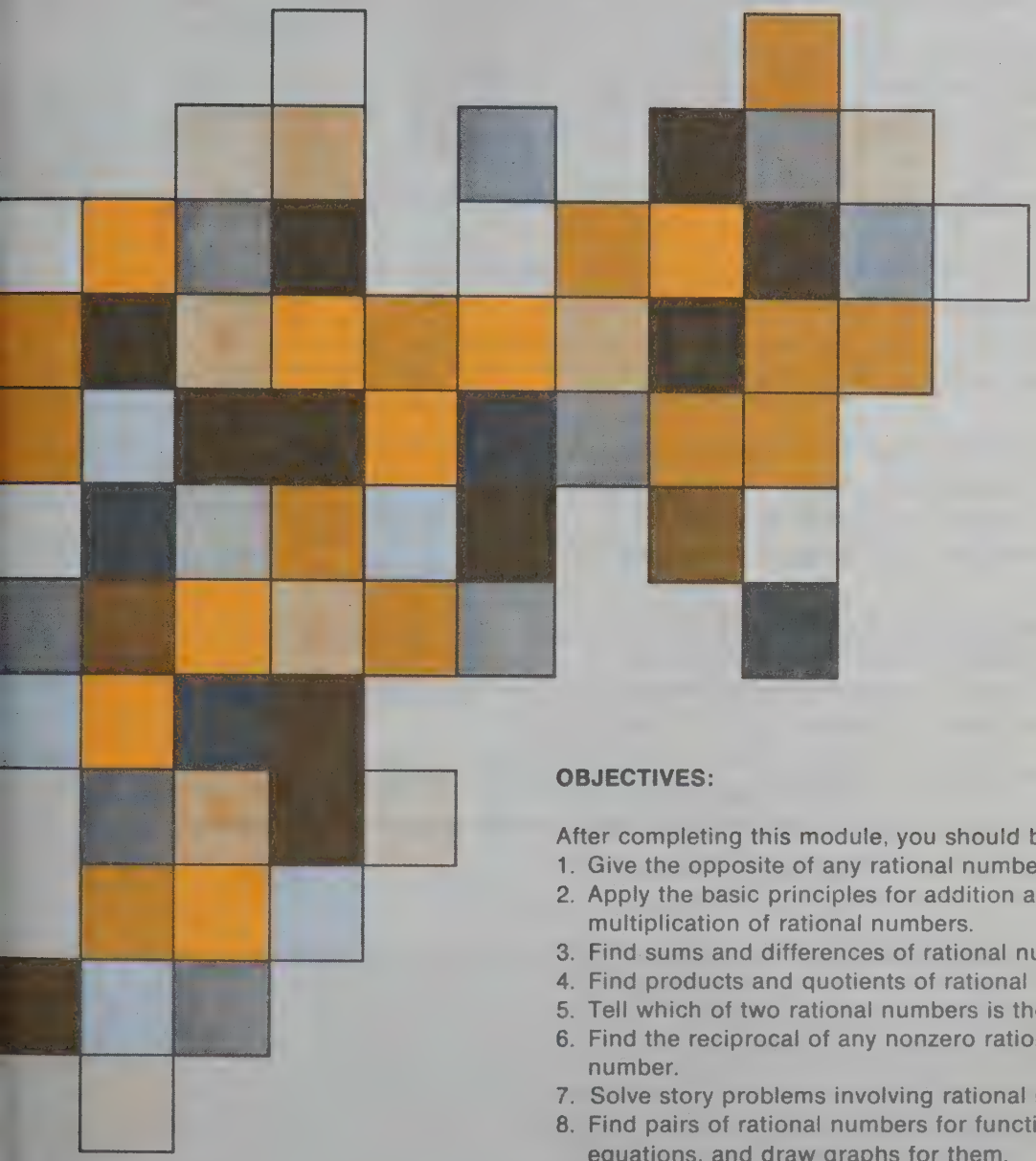
4. A $^{-}18$; B $^{-}32$; C 15 **5.** A $^{-}4$; B 8; C $^{-}6$; D 3 **6.** A $>$; B $<$; C $<$; D

7. $\{^{-}3, ^{-}2, ^{-}1, 0, 1, 2\}$ **8.** $x = 7, x = ^{-}3$ **9.** $x = ^{-}7$

10. A $^{-}3$; B $^{-}7$; C $^{-}11$; D $^{-}15$

UNIT F: *Rational Numbers*

MODULE 1: *Rational Numbers*



OBJECTIVES:

After completing this module, you should be able to:

1. Give the opposite of any rational number.
2. Apply the basic principles for addition and multiplication of rational numbers.
3. Find sums and differences of rational numbers.
4. Find products and quotients of rational numbers.
5. Tell which of two rational numbers is the greater.
6. Find the reciprocal of any nonzero rational number.
7. Solve story problems involving rational numbers.
8. Find pairs of rational numbers for functions and equations, and draw graphs for them.

Investigating the Ideas

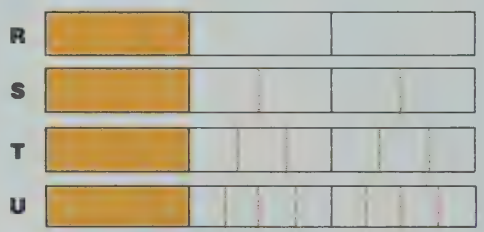
One third of this region is shaded.

How many other statements (using fractions) can you write about this region?

Discussing the Ideas

1. **A** How many different fractions did you use to describe the shaded portion of the region above?
   B How many different fractions can you find for the unshaded portion of the region?

2. Fractions are used to compare one set with another set or to compare one region with another region. What fraction compares the shaded part of region R to the whole region?



3. The figures R, S, T, and U suggest that different fractions may be used to compare the same two regions. Give the fractions suggested by S, T, and U.

4. The sequence of fractions $\left\{\frac{2\cdot 1}{3\cdot 1}, \frac{2\cdot 2}{3\cdot 2}, \frac{2\cdot 3}{3\cdot 3}, \frac{2\cdot 4}{3\cdot 4}, \ldots, \frac{2\cdot n}{3\cdot n}, \ldots\right\}$ where n is a nonzero integer is a sequence of **equivalent fractions**.
   A What is the 5th fraction in the sequence?
   B What is the 10th fraction in the sequence?
   C What is the 100th fraction in the sequence?

5. A fraction is in **lowest terms** if the greatest common factor (GCF) of the numerator and denominator is 1.

$\frac{3}{7} = \frac{3\cdot 1}{7\cdot 1} = \frac{3}{7}$ (GCF: 1; Lowest-terms fraction: $\frac{3}{7}$)

$\frac{10}{15} = \frac{2\cdot 5}{3\cdot 5} = \frac{2}{3}$ (GCF: 5; Lowest-terms fraction: $\frac{2}{3}$)

$\frac{16}{56} = \frac{2\cdot 8}{7\cdot 8} = \frac{2}{7}$ (GCF: 8; Lowest-terms fraction: $\frac{2}{7}$)

Explain how you would find the lowest-terms fraction for each of these fractions.

A $\frac{9}{15}$ **B** $\frac{10}{25}$ **C** $\frac{18}{36}$ **D** $\frac{90}{100}$ **E** $\frac{340}{1000}$

1. Write three fractions which compare the shaded portion of the rectangle to the whole rectangle.

2. Write the next three fractions for each set of equivalent fractions.

A $\left\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots\right\}$ **C** $\left\{\frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \ldots\right\}$ **E** $\left\{\frac{5}{7}, \frac{10}{14}, \frac{15}{21}, \ldots\right\}$

B $\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots\right\}$ **D** $\left\{\frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \ldots\right\}$ **F** $\left\{\frac{9}{8}, \frac{18}{16}, \frac{27}{24}, \ldots\right\}$

3. Write two fractions equivalent to each given fraction.

A $\frac{1}{5}$ **C** $\frac{5}{8}$ **E** $\frac{5}{3}$ **G** $\frac{7}{8}$ **I** $\frac{5}{6}$ **K** $\frac{7}{16}$

B $\frac{4}{3}$ **D** $\frac{3}{10}$ **F** $\frac{1}{8}$ **H** $\frac{5}{12}$ **J** $\frac{11}{10}$ **L** $\frac{7}{20}$

4. For each fraction, give the GCF of the numerator and denominator. Then give the lowest-terms fraction.

A $\frac{12}{18}$ **D** $\frac{42}{49}$ **G** $\frac{20}{12}$ **J** $\frac{0}{10}$ **M** $\frac{48}{64}$

B $\frac{8}{12}$ **E** $\frac{10}{4}$ **H** $\frac{100}{150}$ **K** $\frac{63}{70}$ **N** $\frac{39}{52}$

C $\frac{24}{24}$ **F** $\frac{16}{24}$ **I** $\frac{25}{40}$ **L** $\frac{45}{60}$ **O** $\frac{12}{12}$

5. For each fraction, write an equivalent fraction with a denominator of 24.

A $\frac{2}{3}$ **B** $\frac{1}{2}$ **C** $\frac{5}{12}$ **D** $\frac{3}{8}$ **E** $\frac{1}{4}$ **F** $\frac{5}{6}$ **G** $\frac{7}{8}$

6. A fraction whose numerator is smaller than its denominator is a **proper fraction**. Otherwise, it is an **improper fraction**. Which fractions below are proper and which are improper?

A $\frac{12}{5}$ **B** $\frac{7}{9}$ **C** $\frac{1}{10}$ **D** $\frac{7}{7}$ **E** $\frac{123}{719}$ **F** $\frac{100}{33}$

think

Suppose 623 teams are entered in a basketball tournament. A team is eliminated from the tournament as soon as it loses one game. How many tournament games must be played before the basketball championship is determined?

More practice, page S-13, Set [illegible]

Fractional Numbers and the Number Line

Investigating the Ideas

For each set of equivalent fractions there is one fractional number and one point on the number line.

The number line below is divided into twelfths.
The letters *A* through *G* name sets of equivalent fractions.

$A = \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \ldots\}$

$B = \{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\}$

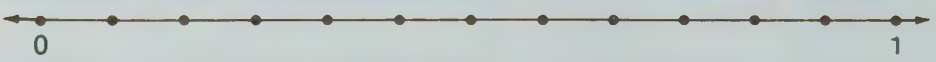
$C = \{\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots\}$

$D = \{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots\}$

$E = \{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots\}$

$F = \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots\}$

$G = \{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots\}$

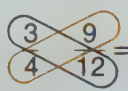


Can you find the point for each set of fractions on a copy of this number line?	Label each point that you find with the correct letter.

Discussing the Ideas

1. Any fraction from the set of equivalent fractions can be used to name the fractional number.
   - **A** Give two fractions to name the number for point *D*.
   - **B** Give two fractions to name the point between *A* and *B* on your number line.

2. The figure at the right suggests a shortcut that can be used to see whether two fractions are equivalent. Explain how to use the shortcut for these pairs of numbers.



$\Rightarrow 3 \cdot 12 = 4 \cdot 9 \Rightarrow \frac{3}{4}$ and $\frac{9}{12}$ are equivalent fractions.

A $\frac{2}{3}, \frac{9}{12}$ **B** $\frac{12}{10}, \frac{6}{5}$ **C** $\frac{8}{3}, \frac{14}{6}$ **D** $\frac{0}{6}, \frac{0}{7}$ **E** $\frac{7}{1}, \frac{14}{2}$

3. When two fractions such as $\frac{2}{3}$ and $\frac{4}{6}$ name the same number, we write $\frac{2}{3} = \frac{4}{6}$. To find x in an equation such as $\frac{3}{4} = \frac{x}{16}$, you can think $4 \cdot x = 3 \cdot 16$, so $4 \cdot x = 48$, and $x = 12$.

   Find x for each of these.

A $\frac{1}{2} = \frac{x}{10}$ **B** $\frac{2}{3} = \frac{x}{15}$ **C** $\frac{x}{8} = \frac{9}{24}$ **D** $\frac{x}{100} = \frac{3}{5}$ **E** $\frac{5}{4} = \frac{x}{24}$

1. Give the number for each lettered point on the number line. Use lowest-terms fractions.

2. Give the missing numbers. Then tell whether the fractions are equivalent.

A $\frac{3}{5} \quad \frac{9}{15} \Rightarrow 3 \cdot 15 = \blacksquare, 5 \cdot 9 = \blacksquare$

B $\frac{4}{12} \quad \frac{2}{6} \Rightarrow 4 \cdot 6 = \blacksquare, 12 \cdot 2 = \blacksquare$

C $\frac{6}{10} \quad \frac{15}{20} \Rightarrow 6 \cdot 20 = \blacksquare, 10 \cdot 15 = \blacksquare$

3. Tell whether the pair of fractions are equivalent or not equivalent.

A $\frac{1}{2}, \frac{8}{16}$ **B** $\frac{3}{4}, \frac{7}{8}$ **C** $\frac{3}{1}, \frac{12}{4}$ **D** $\frac{4}{6}, \frac{14}{21}$ **E** $\frac{20}{30}, \frac{200}{300}$ **F** $\frac{3}{8}, \frac{8}{3}$ **G** $\frac{10}{12}, \frac{15}{18}$ **H** $\frac{0}{11}, \frac{0}{41}$

4. Find the number for **x** in each equation.

A $\frac{1}{8} = \frac{x}{24}$ **B** $\frac{5}{6} = \frac{x}{18}$ **C** $\frac{3}{2} = \frac{x}{30}$ **D** $\frac{x}{12} = \frac{2}{3}$ **E** $\frac{3}{4} = \frac{x}{16}$ **F** $\frac{6}{24} = \frac{x}{4}$ **G** $\frac{5}{4} = \frac{x}{20}$ **H** $\frac{20}{3} = \frac{x}{9}$ **I** $\frac{8}{28} = \frac{x}{7}$ **J** $\frac{x}{78} = \frac{11}{13}$ **K** $\frac{9}{10} = \frac{x}{150}$ **L** $\frac{1}{8} = \frac{x}{1000}$

think

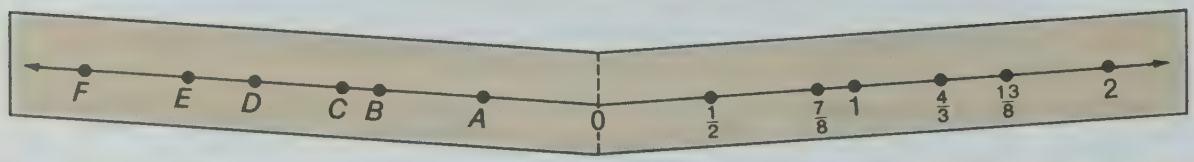
The Z-shaped region is called a **tetromino**. The figure shows how seven of these tetrominos may be fitted inside a 6 by 6 square.

Show how eight of the tetrominos can be fitted inside the square.

More practice, page S-13, Set 24

Investigating the Ideas

Think about folding a number line at zero so that the two "halves" touch.

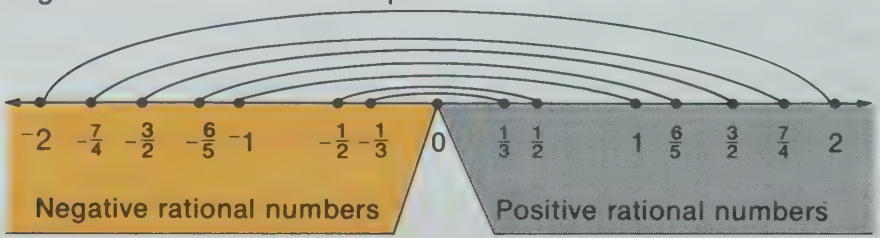


Can you give the letter that each of the labelled points (to the right of zero) would touch?

Discussing the Ideas

1. Suppose each fractional number has an opposite found by "touching" number line points. The number for the point where $\frac{1}{2}$ touches is called **negative** $\frac{1}{2}$. Give the name of each of the other points above.

2. The figure below shows two sets of numbers and the symbols we use for the negatives. These two sets together with zero make up the set of **rational numbers**.


-2 $-\frac{7}{4}$ $-\frac{3}{2}$ $-\frac{6}{5}$ -1 $-\frac{1}{2}$ $-\frac{1}{3}$ 0 $\frac{1}{3}$ $\frac{1}{2}$ 1 $\frac{6}{5}$ $\frac{3}{2}$ $\frac{7}{4}$ 2
Negative rational numbers
Positive rational numbers

Write symbols for each of the letters *A* through *F* in the Investigation.

3. We accept the following principle for rational numbers:

The Opposites Principle
Each rational number has an opposite. The sum of any rational number and its opposite is zero.

A What is the opposite of $\frac{2}{3}$?

B What is the sum $\frac{2}{3} + \left(-\frac{2}{3}\right)$?

1. Sometimes the term **additive inverse** is used instead of *opposite*. Thus, the additive inverse of $\frac{3}{4}$ is $-\frac{3}{4}$, and $\frac{3}{4} + \left(-\frac{3}{4}\right) = 0$.
Give the additive inverse of each rational number.

A $\frac{3}{8}$ **B** $-\frac{5}{6}$ **C** $-\frac{1}{4}$ **D** $\frac{11}{5}$ **E** $-\frac{37}{45}$

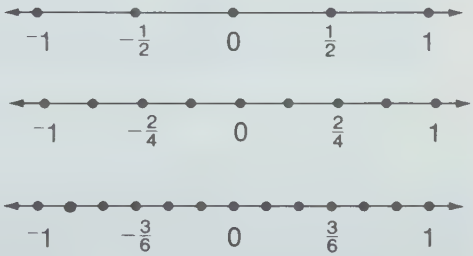
2. Does zero have an additive inverse? What is it?

3. Give the rational number for each lettered point.

4. Give the rational number for n.

A $\frac{3}{4} + n = 0$
B $n + \left(-\frac{5}{8}\right) = 0$
C $\frac{11}{4} + n = 0$
D $n + \left(-\frac{7}{8}\right) = 0$
E $n + \left(-\frac{11}{6}\right) = 0$
F $\frac{8}{3} + n = 0$
G $2 + n = 0$
H $1\frac{3}{4} + n = 0$
I $-\frac{2}{3} + n = 0$
J $\frac{7}{8} + \left(-\frac{7}{8}\right) = n$
K $-\frac{5}{4} + n = 0$
L $1\frac{5}{6} + \left(-\frac{11}{6}\right) = n$
M $-4 + n = 0$
N $-\frac{13}{8} + 1\frac{5}{8} = n$
O $n + \frac{5}{2} = 0$

5. The number lines show that $-\frac{1}{2}$, $-\frac{2}{4}$, and $-\frac{3}{6}$ are equivalent fractions. Give three more fractions equivalent to these three.



6. Complete each sentence.

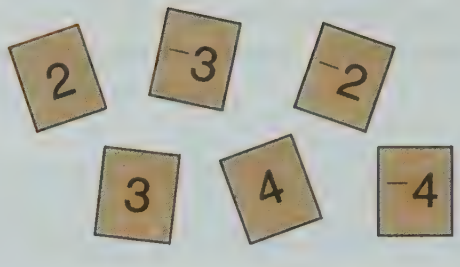
A The opposite of a positive rational number is a ? rational number.
B The opposite of a negative rational number is a ? rational number.
C The opposite of zero is ? .

think

Trace this figure without lifting your pencil from the paper. You will have to trace some segments twice. What is the shortest possible path for the tracing?

Investigating the Ideas

Suppose we assume that the quotient of every pair of nonzero integers is a rational number. Give as many rational numbers as you can (written in fraction form) using the integers on these cards.



EXAMPLES: $\frac{-2}{3}, \frac{4}{-2}, \frac{-3}{-4}, \frac{2}{3}$

How many of your numbers can you locate on the number line above?

Discussing the Ideas

1. Assuming the basic principles hold for all the rational numbers, can you find this sum? $\frac{2}{3} + \frac{-2}{3} = n$

2. A Find this sum. $\frac{2}{3} + \left(-\frac{2}{3}\right) = n$
   B Now can you explain why $\frac{-2}{3} = -\frac{2}{3}$?

3. Can you tell what rational number $2 \div {}^-3$ might be? HINT: Write it as a fraction and multiply the numerator and denominator by $^-1$.

4. Explain why $\frac{-3}{-4}$ and $\frac{3}{4}$ are symbols for the same rational number. (Use the hint in Exercise 3.)

5. Since you cannot divide by zero, there is no rational number with denominator zero. What rational number do you think $\frac{0}{-2}$ or $\frac{0}{-3}$ might be?

6. Each rational number below is either positive $\frac{5}{8}$ or negative $\frac{5}{8}$. Tell which.
   A $5 \div 8$ B $^-5 \div {}^-8$ C $5 \div 8$ D $\frac{-5}{8}$ E $\frac{5}{-8}$ F $\frac{-5}{-8}$

1. Write each quotient as a fraction with a positive denominator.

EXAMPLE: $\frac{2}{-3} = \frac{2 \cdot (-1)}{-3 \cdot (-1)} = \frac{-2}{3}$

A $4 \div -3$ **B** $-5 \div -6$ **C** $3 \div -5$ **D** $-6 \div -10$ **E** $0 \div -7$

2. Answer **T** (true) or **F** (false) for each part.

A $\frac{-2}{3} = -\frac{2}{3}$ **B** $\frac{2}{-3} = \frac{2}{3}$ **C** $\frac{-2}{-3} = -\frac{2}{3}$ **D** $\frac{-3}{-4} = \frac{3}{4}$ **E** $\frac{3}{-4} = -\frac{3}{4}$

3. Give the point P or Q for each rational number below.

A $\frac{-8}{5}$ **B** $\frac{-8}{-5}$ **C** $8 \div -5$ **D** $8 \div 5$ **E** $-8 \div -5$ **F** $\frac{8}{5}$

4. Give the point (R, S, T, or U) for each number below.

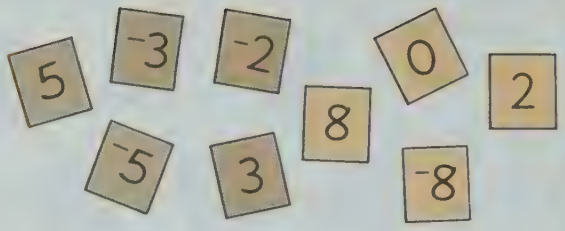
A $\frac{7}{-10}$ **B** $\frac{-10}{7}$ **C** $-10 \div -7$ **D** $-7 \div 10$ **E** $\frac{-7}{10}$ **F** $-10 \div 7$ **G** $\frac{-7}{-10}$ **H** $7 \div 10$ **I** $\frac{10}{-7}$ **J** -0.7 **K** $\frac{-7}{10}$ **L** $\frac{10}{-7}$

think

Turn over any two adjacent coins at a time in arrangement **A**. What is the fewest number of moves to get arrangement **B**?

Investigating the Ideas

Suppose all the usual basic principles and rules hold for the set of rational numbers. That is, you can use the same rules for adding rational numbers as you used for positive rational numbers.



?	How many ways can you find to place the slips of paper above on the shaded spaces below to form equations?	Record each equation that you find.

$$\frac{\square}{10} + \frac{\square}{10} = \frac{\square}{10}$$

Discussing the Ideas

1. **A** Would the placement of the slips of paper be any different if the denominator were 15 instead of 10?
   B Explain how adding positive and negative rational numbers is much like adding positive and negative integers.

2. In the Investigation, each denominator was positive. Can you write a fraction for any rational number so that it has a positive denominator?

3. How would you change these problems before trying to add?

   A $\frac{-5}{12} + \frac{1}{-12} = x$ **B** $\frac{-3}{-8} + \frac{-2}{-8} = n$ **C** $-\frac{3}{7} + \frac{2}{7} = d$

4. What is the sum in each part of Exercise 3?

5. Explain why the sum in **A** and **B** is the same. **A** $\frac{-7}{10} + \frac{5}{10}$ **B** $\frac{7}{-10} + \frac{-5}{-10}$

6. Give the missing numerator.
   If $\frac{a}{b}$ and $\frac{c}{b}$ are any two rational numbers with $b > 0$, then $\frac{a}{b} + \frac{c}{b} = \frac{\text{III}}{b}$.

1. Find the sums.

A $\frac{3}{6} + \frac{2}{6} = n$

B $\frac{3}{10} + \frac{-2}{10} = r$

C $\frac{-3}{8} + \frac{2}{8} = m$

D $\frac{-3}{4} + \frac{-2}{4} = x$

E $\frac{7}{12} + \frac{5}{12} = c$

F $\frac{7}{10} + \frac{-5}{10} = y$

G $\frac{-7}{100} + \frac{5}{100} = s$

H $\frac{-7}{10} + \frac{-5}{10} = d$

I $\frac{-7}{8} + \frac{3}{8} = p$

J $\frac{7}{5} + \frac{-3}{5} = t$

K $\frac{7}{6} + \frac{3}{6} = q$

L $\frac{-7}{9} + \frac{-3}{9} = a$

2. Find the sums. HINT: Change to positive denominators before adding.

A $\frac{3}{7} + \frac{-4}{-7}$

B $\frac{-1}{-4} + \frac{3}{4}$

C $\frac{4}{-10} + \frac{3}{-10}$

D $-\frac{6}{8} + \frac{1}{8}$

E $\frac{5}{6} + \frac{-1}{-6}$

F $\frac{-2}{-5} + \frac{-4}{-5}$

G $\frac{-7}{-12} + \frac{5}{12}$

H $-\frac{1}{4} + \frac{-5}{-4}$

I $\frac{6}{-10} + \frac{-3}{10}$

J $\frac{-3}{-8} + \frac{-3}{-8}$

K $\frac{12}{100} + \frac{7}{100}$

L $\frac{-2}{9} + \frac{-4}{-9}$

3. Find the sums.

A $\frac{3}{4} + \frac{-1}{4} + \frac{5}{4}$

B $\frac{-1}{8} + \frac{-3}{8} + \frac{7}{8}$

C $\frac{11}{10} + \frac{-3}{10} + \frac{-7}{10}$

D $1 + \frac{-1}{3} + \frac{-2}{3}$

E $-1 + \frac{1}{4} + \frac{1}{4}$

F $-\frac{1}{6} + \frac{-5}{6} + \frac{-1}{6}$

G $\frac{-9}{10} + \frac{4}{10} + \frac{3}{10}$

H $\frac{11}{3} + \frac{-8}{3} + \frac{-2}{3}$

I $\frac{-3}{-4} + \frac{-2}{-4} + \frac{9}{4}$

4. Solve the equations. The replacement set is the set of rational numbers.

A $\frac{-3}{4} + x = \frac{1}{4}$

B $\frac{11}{6} + r = \frac{5}{6}$

C $y + \frac{-1}{3} = -\frac{2}{3}$

D $\frac{5}{8} + z = \frac{-1}{8}$

E $\frac{-1}{-2} + s = \frac{1}{2}$

F $\frac{9}{10} + t = 0$

G $\frac{-17}{20} + \frac{-2}{20} = x$

H $\frac{1}{7} + \frac{-8}{7} = y$

I $n + \frac{-4}{15} = 0$

J $\frac{-13}{10} + k = \frac{-3}{10}$

K $\frac{7}{6} + s = 1$

L $m + \frac{-3}{4} = \frac{3}{4}$

M $\frac{-1}{6} + q = \frac{-5}{6}$

N $2 + \frac{-1}{2} = t$

O $h + \frac{3}{10} = \frac{-7}{10}$

P $\frac{37}{100} + r = \frac{50}{100}$

Q $\frac{-1}{10} + j = \frac{-4}{10}$

R $p + \frac{-63}{100} = \frac{-29}{100}$

think

Find the missing base in each equation.

A $13_{(\text{ten})} = 31_{(a)}$

B $23_{(7)} = 32_{(b)}$

C $21_{(3)} = 12_{(c)}$

D $12_{(9)} = 21_{(d)}$

Investigating the Ideas

Can you match the equations so that the one on the right will help you solve the one on the left?

1. $\frac{3}{4} + \frac{-2}{3} = n$
2. $\frac{-4}{5} + \frac{1}{3} = r$
3. $\frac{-3}{8} + \frac{-1}{6} = t$
4. $\frac{5}{6} + \frac{-3}{4} = a$
5. $\frac{-7}{10} + \frac{23}{100} = y$

A $\frac{-9}{24} + \frac{-4}{24} = \frac{-13}{24}$

B $\frac{-70}{100} + \frac{23}{100} = \frac{-47}{100}$

C $\frac{9}{12} + \frac{-8}{12} = \frac{1}{12}$

D $\frac{-12}{15} + \frac{5}{15} = \frac{-7}{15}$

E $\frac{10}{12} + \frac{-9}{12} = \frac{1}{12}$

Discussing the Ideas

1. Explain your matchings for each part of the Investigation.

2. Give the **least common denominator** (LCD) that you would use for adding each pair of fractions.

   A $\frac{\square}{4} + \frac{\square}{3}$ B $\frac{\square}{2} + \frac{\square}{6}$ C $\frac{\square}{5} + \frac{\square}{4}$ D $\frac{\square}{2} + \frac{\square}{10}$ E $\frac{\square}{10} + \frac{\square}{100}$

3. Explain how to find the missing numerators.

   A $\frac{5}{8} = \frac{\square}{24}$ B $\frac{-1}{3} = \frac{\square}{12}$ C $\frac{2}{3} = \frac{\square}{30}$ D $\frac{-3}{4} = \frac{\square}{12}$ E $\frac{-3}{10} = \frac{\square}{30}$

4. Study the example: $\underbrace{\frac{2}{3} + \frac{-1}{2}} = \frac{2 \cdot 2}{3 \cdot 2} + \frac{-1 \cdot 3}{2 \cdot 3} = \frac{4}{6} + \frac{-3}{6} = \frac{4 + -3}{6} = \frac{1}{6}$

   The LCD is ...

   What is the sum $\frac{-2}{3} + \frac{1}{4}$?

5. When adding several positive rational numbers using mixed numerals, it is helpful to arrange your work as shown in the example.

   A Explain how $29\frac{23}{12}$ was found.

   B Why does $29\frac{23}{12}$ represent the same number as $30\frac{11}{12}$?

EXAMPLE:

$$\begin{array}{rr} 16\frac{2}{3} & 16\frac{8}{12} \\ 3\frac{3}{4} \rightarrow & 3\frac{9}{12} \\ +10\frac{1}{2} & +10\frac{6}{12} \\ \hline & 29\frac{23}{12} = 30\frac{11}{12} \end{array}$$

1. Copy the equations and give the numbers for ***a***, ***b***, ***c***.

A $\frac{1}{4}+\frac{1}{3}=\frac{a}{12}+\frac{b}{12}=\frac{c}{12}$

B $\frac{1}{6}+\frac{3}{8}=\frac{a}{24}+\frac{b}{24}=\frac{c}{24}$

C $\frac{-3}{10}+\frac{7}{100}=\frac{a}{100}+\frac{b}{100}=\frac{c}{100}$

D $\frac{-3}{16}+\frac{-5}{12}=\frac{a}{48}+\frac{b}{48}=\frac{c}{48}$

2. Find the sums.

A $\frac{5}{7}+\frac{-1}{7}$

B $\frac{-3}{4}+\frac{1}{4}$

C $\frac{3}{10}+\frac{-7}{10}$

D $\frac{-1}{2}+\frac{-1}{4}$

E $\frac{-1}{3}+\frac{1}{2}$

F $\frac{1}{4}+\frac{-5}{8}$

G $\frac{-1}{5}+\frac{-3}{10}$

H $\frac{1}{6}+\frac{-3}{2}$

I $\frac{3}{4}+\frac{3}{8}$

J $\frac{1}{4}+\frac{1}{5}$

K $\frac{-1}{2}+\frac{1}{3}$

L $\frac{1}{3}+\frac{-1}{4}$

M $\frac{-5}{8}+\frac{-1}{6}$

N $\frac{2}{3}+\frac{3}{4}$

O $\frac{5}{6}+\frac{-1}{2}$

P $\frac{-5}{8}+\frac{3}{20}$

3. Find the sums.

A $42\frac{5}{6} + 21\frac{1}{3} + 37\frac{2}{3}$

B $9\frac{3}{8} + 7\frac{1}{4} + 5\frac{3}{16}$

C $16\frac{1}{2} + 15\frac{1}{3} + 14\frac{1}{6}$

D $23\frac{1}{6} + 18\frac{1}{4} + 12\frac{1}{2}$

E $31\frac{5}{8} + 17\frac{1}{2} + 19\frac{1}{4}$

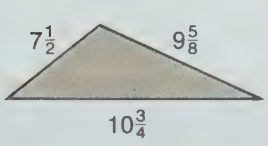
F $123\frac{1}{10} + 243\frac{3}{10} + 366\frac{3}{5}$

G $23\frac{1}{2} + 76\frac{9}{10} + 84\frac{17}{100}$

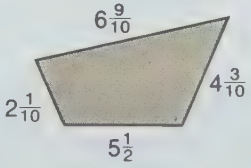
H $243\frac{3}{10} + 184\frac{8}{25} + 769\frac{61}{100}$

4. Find the perimeter (distance around) of each region. The units of length for each side are marked on the figure.

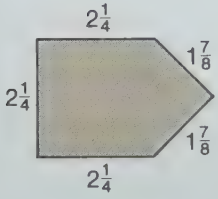
A



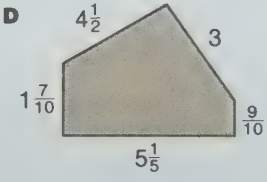
B



C



D



think

Give the missing numbers so that this will be a magic square.

7	$^{-}7$	$^{-}6$	4
$^{-}4$	2	1	
	$^{-}2$		3
		6	

More practice, page S-14, Set 25

Investigating the Ideas

Select one of these sets of numbers. Write two addition and two subtraction equations using your numbers.

$\{^{-}7, 2, ^{-}5\}$

$\{8, ^{-}2, 6\}$

$\{^{-}3, ^{-}4, ^{-}7\}$

Can you use your numbers as numerators in the two addition and the two subtraction equations below?

$\frac{\square}{10} + \frac{\square}{10} = \frac{\square}{10}$ $\quad\quad$ $\frac{\square}{10} - \frac{\square}{10} = \frac{\square}{10}$

$\frac{\square}{10} + \frac{\square}{10} = \frac{\square}{10}$ $\quad\quad$ $\frac{\square}{10} - \frac{\square}{10} = \frac{\square}{10}$

Discussing the Ideas

1. Find these differences. Explain how to check your answers by adding.

   A $6 - {}^{-}2 = n$ $\quad$ B $^{-}4 - 3 = s$ $\quad$ C $^{-}2 - {}^{-}6 = x$

2. Now try to find these differences. Remember, you should be able to check your answers by adding.

   A $\frac{6}{10} - \frac{^{-}2}{10} = m$ $\quad$ B $\frac{^{-}4}{10} - \frac{3}{10} = t$ $\quad$ C $\frac{^{-}2}{10} - \frac{^{-}6}{10} = y$

3. Give the missing numerator.

   If $\frac{a}{b}$ and $\frac{c}{b}$ are any two rational numbers with $b > 0$, then $\frac{a}{b} - \frac{c}{b} = \frac{\square}{b}$.

4. Explain how to find these differences.

   A $\frac{3}{4} - \frac{^{-}2}{3} = q$ $\quad$ B $\frac{^{-}5}{10} - \frac{1}{2} = d$ $\quad$ C $\frac{^{-}3}{8} - \frac{^{-}1}{4} = r$

1. Find the differences.

A $\frac{5}{8} - \frac{-3}{8}$ **D** $\frac{-7}{10} - \frac{3}{10}$ **G** $\frac{5}{6} - \frac{-1}{6}$ **J** $\frac{3}{8} - \frac{-7}{8}$ **M** $\frac{27}{100} - \frac{11}{100}$

B $\frac{-2}{10} - \frac{6}{10}$ **E** $\frac{3}{8} - \frac{-5}{8}$ **H** $\frac{-7}{12} - \frac{5}{12}$ **K** $\frac{-6}{10} - \frac{9}{10}$ **N** $\frac{-139}{100} - \frac{-257}{1000}$

C $\frac{4}{5} - \frac{-8}{5}$ **F** $\frac{-1}{4} - \frac{-3}{4}$ **I** $\frac{-8}{10} - \frac{-4}{10}$ **L** $\frac{1}{16} - \frac{-7}{16}$ **O** $\frac{3}{10} - \frac{7}{10}$

2. Find the differences.

A $\frac{7}{8} - \frac{1}{2}$ **C** $\frac{5}{6} - \frac{1}{4}$ **E** $\frac{8}{10} - \frac{3}{5}$ **G** $\frac{9}{10} - \frac{3}{4}$ **I** $\frac{63}{100} - \frac{1}{2}$

B $\frac{3}{4} - \frac{1}{3}$ **D** $\frac{7}{10} - \frac{1}{2}$ **F** $\frac{5}{12} - \frac{1}{3}$ **H** $\frac{7}{8} - \frac{5}{6}$ **J** $\frac{257}{1000} - \frac{-1}{10}$

3. Find the differences.

A $\frac{7}{8} - \frac{-1}{4}$ **D** $\frac{2}{3} - \frac{-1}{6}$ **G** $\frac{-9}{10} - \frac{2}{5}$ **J** $\frac{-3}{8} - \frac{3}{6}$ **M** $\frac{2}{3} - \frac{-1}{2}$

B $\frac{-6}{10} - \frac{2}{5}$ **E** $\frac{7}{4} - \frac{3}{8}$ **H** $\frac{1}{10} - \frac{-1}{4}$ **K** $\frac{2}{5} - \frac{-1}{4}$ **N** $\frac{2}{3} - \frac{1}{2}$

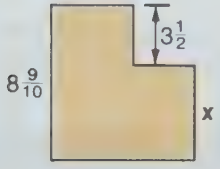
C $\frac{5}{12} - \frac{3}{4}$ **F** $\frac{5}{6} - \frac{-6}{10}$ **I** $\frac{-4}{5} - \frac{-3}{10}$ **L** $\frac{-4}{10} - \frac{-1}{2}$ **O** $\frac{-2}{3} - \frac{-1}{2}$

4. Find the differences.

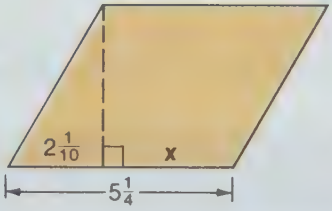
A $16\frac{5}{6} - 11\frac{1}{4}$ **B** $43\frac{1}{2} - 19\frac{5}{16}$ **C** $66\frac{9}{10} - 44\frac{3}{5}$ **D** $107\frac{1}{3} - 94\frac{1}{2}$ **E** $412\frac{1}{8} - 365\frac{1}{4}$

5. Find the distance **x** for each figure.

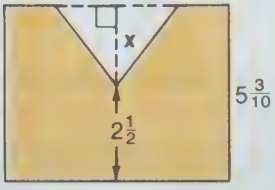
A



B



C



think

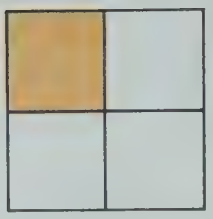
1. Arrange five ones (1) and four negative ones ($^-1$) in the nine squares so that the product of the numbers in each row, each column, and the two diagonals is 1.

2. Arrange four ones and five negative ones in the nine squares so that the product of the numbers in each row, each column, and the two diagonals is $^-1$.

More practice, page S-14, Set 26

Discussing the Ideas

1. The fraction $\frac{1}{4}$ compares the shaded portion of the square region to the whole region. The fraction $\frac{1}{4}$ is an example of a unit fraction. A **unit fraction** is a fraction whose numerator is 1 and whose denominator is a positive integer.



What unit fractions are suggested by these regions?

A

B

C

2. The regions and the number line below suggest multiplication with unit fractions. Study each figure and complete each equation.

A

$\frac{1}{3}$ of a region → $\frac{1}{2}$ of $\frac{1}{3}$ → $\frac{1}{6}$ of a region

We write: $\frac{1}{2} \cdot \frac{1}{3} =$ ▨.

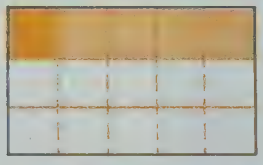
3 jumps of $\frac{1}{5}$

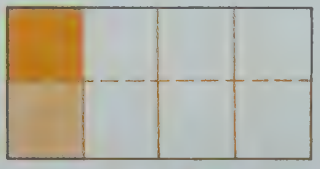
We write: $3 \cdot \frac{1}{5} =$ ▨.

3. Discussion Exercise 2 suggests the following generalizations concerning unit fractions.

Unit Fraction Principles	
For every pair of unit fractions, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$.	If a is a non-negative integer, then $a \cdot \frac{1}{b} = \frac{a}{b}$.

Give the multiplication fact illustrated by each region or number line.

A 

B 

C

D

1. Solve the equations.

A $\frac{1}{4} \cdot \frac{1}{3} = n$ **C** $\frac{1}{2} \cdot n = \frac{1}{10}$ **E** $\frac{1}{7} \cdot \frac{1}{6} = n$ **G** $\frac{1}{100} \cdot n = \frac{1}{1000}$

B $\frac{1}{8} \cdot \frac{1}{2} = n$ **D** $n \cdot \frac{1}{4} = \frac{1}{12}$ **F** $\frac{1}{10} \cdot n = \frac{1}{100}$ **H** $n \cdot \frac{1}{100} = \frac{1}{200}$

2. Find the products.

A $4 \cdot \frac{1}{5}$ **B** $6 \cdot \frac{1}{8}$ **C** $0 \cdot \frac{1}{8}$ **D** $6 \cdot \frac{1}{6}$ **E** $12 \cdot \frac{1}{3}$

3. If a is a nonzero whole number, then $a \cdot \frac{1}{a} =$ ▥.

4. Assume that the basic principles hold for the set of rational numbers. Give the principle illustrated by each equation.

A $\frac{2}{7} \cdot \frac{8}{8} = \frac{2}{7}$ **C** $\frac{-3}{4} \cdot \frac{7}{6} = \frac{7}{6} \cdot \frac{-3}{4}$ **E** $\frac{2}{3} \cdot \left(\frac{4}{5} + \frac{3}{4}\right) = \left(\frac{4}{5} + \frac{3}{4}\right) \cdot \frac{2}{3}$

B $\frac{-7}{5} \cdot 1 = \frac{-7}{5}$ **D** $\frac{-1}{2} \cdot \left(\frac{1}{3} \cdot \frac{1}{4}\right) = \left(\frac{-1}{2} \cdot \frac{1}{3}\right) \cdot \frac{1}{4}$ **F** $\left(\frac{3}{4} \cdot \frac{2}{3}\right) + \left(\frac{3}{4} \cdot \frac{1}{5}\right) = \frac{3}{4} \cdot \left(\frac{2}{3} + \frac{1}{5}\right)$

5. The example at the right proves that $-1 \cdot \frac{2}{3} = \frac{-2}{3}$. Find the products.

A $-1 \cdot \frac{1}{4}$ **D** $-1 \cdot \frac{3}{8}$

B $-1 \cdot \frac{3}{10}$ **E** $-1 \cdot \frac{5}{6}$

C $-1 \cdot \frac{2}{5}$ **F** $-1 \cdot \frac{17}{100}$

EXAMPLE:

$$-1 \cdot \frac{2}{3} + \frac{2}{3} = -1 \cdot \frac{2}{3} + 1 \cdot \frac{2}{3}$$
$$= (-1 + 1) \cdot \frac{2}{3}$$
$$= 0 \cdot \frac{2}{3} = 0$$

Since $\frac{-2}{3} + \frac{2}{3} = 0$ and $-1 \cdot \frac{2}{3} + \frac{2}{3} = 0$, then $-1 \cdot \frac{2}{3} = \frac{-2}{3}$.

6. The product of a positive rational number and negative 1 is the ? of the rational number.

7. Give the missing number for each ▥.

A $-1 \cdot \left(\frac{-1}{3}\right) = -1 \cdot \left(-1 \cdot \frac{1}{3}\right) = (-1 \cdot -1) \cdot \frac{1}{3} =$ ▥

B $-1 \cdot \left(\frac{-3}{8}\right) = -1 \cdot \left(-1 \cdot \frac{3}{8}\right) = (-1 \cdot -1) \cdot \frac{3}{8} =$ ▥

C $-1 \cdot \left(\frac{-9}{10}\right) = -1 \cdot \left(-1 \cdot \frac{9}{10}\right) =$ ▥

D $-1 \cdot \left(\frac{-3}{4}\right) =$ ▥

8. Give the number for n in each sentence.

A $-1 \cdot \frac{1}{4} = n$ **B** $\frac{1}{10} \cdot -1 = n$ **C** $-1 \cdot n = \frac{-7}{8}$ **D** $n \cdot \frac{5}{6} = \frac{-5}{6}$

9. Study the examples then find the products.

A $\frac{-1}{3} \cdot \frac{1}{2}$ **D** $\frac{-1}{5} \cdot \frac{-1}{2}$

B $\frac{1}{8} \cdot \frac{-1}{3}$ **E** $\frac{1}{10} \cdot \frac{-1}{100}$

C $\frac{-1}{6} \cdot \frac{-1}{3}$ **F** $\frac{-1}{10} \cdot \frac{-1}{10}$

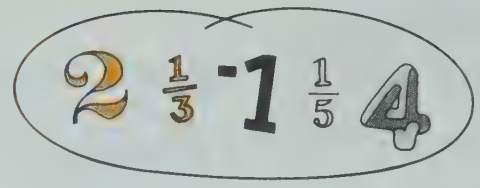
EXAMPLES:

$$\frac{-1}{2} \cdot \frac{1}{3} = -1 \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right) = -1 \cdot \frac{1}{6} = \frac{-1}{6}$$

$$\frac{-1}{4} \cdot \frac{-1}{2} = \left(-1 \cdot \frac{1}{4}\right) \cdot \left(-1 \cdot \frac{1}{2}\right) = (-1 \cdot -1) \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

Investigating the Ideas

Using the numbers in this set as factors, many different products can be found.



How many different products can you find for each part below?	Record the factors and products.

- **A** The product of any two of the factors.
- **B** The product of any three of the factors.
- **C** The product of any four of the factors.
- **D** The product of all five factors.

Discussing the Ideas

1. Explain how you could arrange and group the factors $^-1$, $\frac{1}{3}$, 4, and $\frac{1}{5}$ to find their product easily.

2. The unit fraction principles and the basic principles can be used to justify the rule for multiplying two positive rational numbers. Study the example below.

$$\frac{2}{3}\cdot\frac{4}{5} = \left(2\cdot\frac{1}{3}\right)\cdot\left(4\cdot\frac{1}{5}\right)$$ Unit fraction principle

$$= (2\cdot 4)\cdot\left(\frac{1}{3}\cdot\frac{1}{5}\right)$$ Associative and commutative principles of multiplication for rational numbers

$$= 8\cdot\frac{1}{15}$$ Multiplication fact for integers; Unit fraction principle

$$= \frac{8}{15}$$ Unit fraction principle

A How would you find the product $\frac{3}{5}\cdot\frac{3}{8}$?

B Notice that $\frac{2}{3}\cdot\frac{4}{5} = \frac{8}{15}$ and $\frac{5}{6}\cdot\frac{7}{8} = \frac{35}{48}$. What shortcut for multiplying two rational numbers do these examples suggest?

3. Study the example Then explain how to find the products below.

A $\frac{2}{3}\cdot\frac{^-4}{5}$ **B** $\frac{^-7}{8}\cdot\frac{^-3}{5}$

$$\frac{^-3}{5}\cdot\frac{4}{7} = \left(^-1\cdot\frac{3}{5}\right)\cdot\frac{4}{7}$$
$$= {}^-1\cdot\left(\frac{3}{5}\cdot\frac{4}{7}\right) = {}^-1\cdot\frac{12}{35}$$
$$= \frac{^-12}{35}$$

4. Explain why the product $\frac{^-5}{3}\cdot\frac{^-3}{4}$ is the same as $\frac{5}{3}\cdot\frac{3}{4}$.

1. Find the products.

A $\frac{2}{5} \cdot \frac{4}{3}$ **B** $\frac{5}{7} \cdot \frac{1}{3}$ **C** $\frac{9}{10} \cdot \frac{5}{3}$ **D** $\frac{3}{8} \cdot \frac{3}{4}$ **E** $\frac{4}{5} \cdot \frac{5}{8}$ **F** $\frac{7}{3} \cdot \frac{5}{7}$ **G** $\frac{9}{4} \cdot \frac{4}{9}$ **H** $\frac{15}{16} \cdot \frac{8}{5}$ **I** $\frac{2}{3} \cdot \frac{4}{4}$ **J** $\frac{5}{6} \cdot \frac{3}{10}$ **K** $\frac{3}{8} \cdot \frac{0}{5}$ **L** $4 \cdot \frac{5}{9}$ **M** $\frac{2}{3} \cdot 5$ **N** $\frac{3}{5} \cdot \frac{3}{2}$ **O** $\frac{9}{10} \cdot \frac{3}{10}$

2. Find the products.

A $\frac{-3}{8} \cdot \frac{3}{2}$ **B** $\frac{5}{7} \cdot \frac{-1}{3}$ **C** $\frac{-1}{2} \cdot \frac{-1}{2}$ **D** $\frac{-2}{3} \cdot \frac{2}{5}$ **E** $\frac{-1}{4} \cdot \frac{-1}{3}$ **F** $-1 \cdot \frac{-1}{2}$ **G** $\frac{3}{8} \cdot \frac{-8}{3}$ **H** $5 \cdot \frac{-2}{5}$ **I** $\frac{-7}{8} \cdot \frac{-2}{7}$ **J** $\frac{-2}{3} \cdot \frac{-2}{3}$ **K** $\frac{5}{2} \cdot -1$ **L** $-4 \cdot \frac{-5}{9}$ **M** $12 \cdot \frac{-1}{2}$ **N** $-30 \cdot \frac{7}{100}$ **O** $\frac{-3}{10} \cdot 25$ **P** $-6 \cdot \frac{7}{10}$ **Q** $\frac{-5}{3} \cdot -27$ **R** $\frac{-7}{2} \cdot \frac{-2}{7}$ **S** $\left(\frac{-1}{2} \cdot \frac{1}{4}\right) \cdot -8$ **T** $\left(\frac{9}{10} \cdot \frac{-1}{2}\right) \cdot -140$

3. Complete each sentence with one of the words **positive** or **negative.**

A The product of two positive rational numbers is a ? rational number.

B The product of two negative rational numbers is a ? rational number.

C The product of a positive rational number and a negative rational number is a ? rational number.

4. Solve the equations.

A $6 \cdot \left(2 + \frac{1}{3}\right) = 6 \cdot 2 + 6 \cdot \frac{1}{3} = n$

B $-4 \cdot \left(3 + \frac{1}{2}\right) = n$

C $10 \cdot \left(1 + \frac{1}{5}\right) = n$

D $-8 \cdot 3\frac{1}{8} = n$

E $4 \cdot 2\frac{1}{4} = n$

F $-6 \cdot -2\frac{2}{3} = n$

5. Write each mixed numeral as an improper fraction. Then find the product.

A $2\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{3} \cdot \frac{4}{5} =$ ▒▒▒

B $1\frac{1}{2} \cdot \frac{7}{8} =$ ▒▒▒

C $-2\frac{1}{2} \cdot \frac{4}{5} =$ ▒▒▒

D $\frac{5}{6} \cdot -4\frac{2}{3} =$ ▒▒▒

E $-1\frac{1}{2} \cdot -2\frac{1}{2} =$ ▒▒▒

F $6\frac{2}{3} \cdot 6 =$ ▒▒▒

G $33\frac{1}{3} \cdot \frac{-3}{4} =$ ▒▒▒

H $-10 \cdot -2\frac{1}{2} =$ ▒▒▒

I $-3\frac{3}{4} \cdot -5\frac{1}{3} =$ ▒▒▒

think

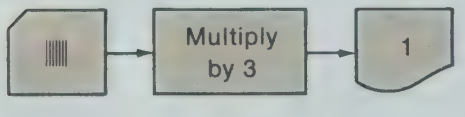
A postman is to deliver letters to each of 10 houses located all in a row and equally spaced. If he can start at any house, what is the **longest** path he can take?

More practice, page S-15, Set 27

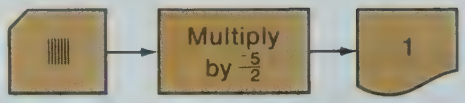
Investigating the Ideas

Each of these flow charts has an output number of 1.

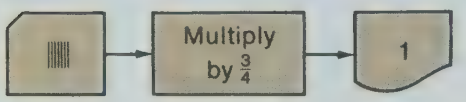
A



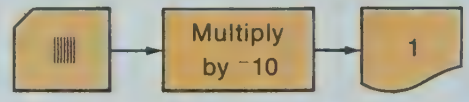
C



B



D



How many of the input numbers can you find?

Discussing the Ideas

1. If the product of two numbers is 1, each number is the **multiplicative inverse** or **reciprocal** of the other.
For example, $\frac{3}{4}$ is the reciprocal of $\frac{4}{3}$ because $\frac{3}{4} \cdot \frac{4}{3} = \frac{12}{12} = 1$
and $\frac{-7}{2}$ is the multiplicative inverse of $\frac{-2}{7}$ because $\frac{-7}{2} \cdot \frac{-2}{7} = \frac{14}{14} = 1$.

   Give the reciprocal of each number. Explain your answer.

   A $\frac{11}{10}$ **B** $\frac{-1}{2}$ **C** $^{-}3$ **D** $2\frac{1}{5}$ **E** 1

2. One of the basic differences in the structure of the set of rational numbers and the set of integers is that every rational number, except zero, has a multiplicative inverse in the set of rational numbers. However, in the set of integers, only 1 and $^{-}1$ have multiplicative inverses.

   a is a reciprocal of ***b***, and ***b*** is a reciprocal of ***a*** if $a \cdot b = 1$.

   A Explain why zero does not have a reciprocal.

   B What kind of a number is the reciprocal of a unit fraction?

3. If a rational number is negative, will its reciprocal be a positive or a negative rational number?

4. Since $2\frac{1}{4} = \frac{9}{4}$, the reciprocal of $2\frac{1}{4}$ is $\frac{4}{9}$.
Explain how to find the reciprocal of $3\frac{7}{10}$.

1. Give the reciprocal of each number.

A $\frac{2}{5}$ **B** $\frac{-1}{3}$ **C** $\frac{7}{8}$ **D** $\frac{16}{15}$ **E** -4 **F** $\frac{3}{7}$ **G** 1 **H** $\frac{-11}{6}$ **I** $3\frac{1}{3}$ **J** $7\frac{2}{5}$ **K** 10 **L** $\frac{-1}{100}$

2. Solve the equations.

A $\frac{3}{8} \cdot n = 1$ **B** $\frac{5}{4} \cdot n = 1$ **C** $n \cdot \frac{8}{7} = 1$ **D** $\frac{6}{5} \cdot n = 1$ **E** $\frac{12}{7} \cdot n = 1$ **F** $\frac{4}{9} \cdot n = 1$ **G** $10 \cdot n = 1$ **H** $\frac{1}{6} \cdot n = 1$ **I** $2\frac{1}{2} \cdot n = 1$ **J** $5\frac{1}{8} \cdot n = 1$ **K** $16\frac{2}{3} \cdot n = 1$ **L** $1 \cdot n = 1$

3. Solve each equation.

A $(5 \cdot 3) \cdot x = 1$ **B** $(-4 \cdot 7) \cdot x = 1$ **C** $\left(\frac{1}{2} \cdot \frac{1}{4}\right) \cdot x = 1$ **D** $\left(-2 \cdot \frac{1}{5}\right) \cdot x = 1$ **E** $\frac{3}{4} \cdot \left(\frac{-1}{2} \cdot x\right) = 1$ **F** $\frac{-7}{10} \cdot \left(\frac{-1}{3} \cdot x\right) = 1$

4. Study the example. Then solve the equations.

A $\frac{5}{8} \cdot \left(\frac{2}{3} \cdot n\right) = \frac{5}{8}$
B $\frac{3}{4} \cdot \left(\frac{10}{9} \cdot n\right) = \frac{3}{4}$
C $\frac{1}{2} \cdot \left(\frac{5}{6} \cdot n\right) = \frac{5}{6}$
D $\frac{-7}{3} \cdot \left(\frac{8}{3} \cdot n\right) = \frac{7}{3}$
E $\left(\frac{3}{4} \cdot \frac{5}{7}\right) \cdot n = \frac{3}{4}$
F $\left(\frac{6}{5} \cdot \frac{7}{10}\right) \cdot n = \frac{6}{5}$

EXAMPLE:
Solve the equation: $\frac{3}{4} \cdot \left(\frac{5}{7} \cdot n\right) = \frac{3}{4}$.

This number must be 1 because $\frac{3}{4} \cdot 1 = \frac{3}{4}$.

$\frac{3}{4} \cdot \frac{5}{7} \cdot n = \frac{3}{4}$

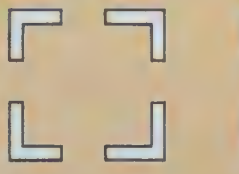
If $\frac{5}{7} \cdot n = 1$, then $n = \frac{7}{5}$.

SOLUTION: $n = \frac{7}{5}$

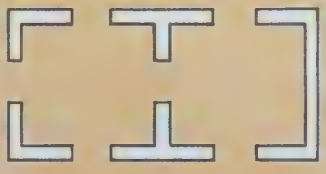
think

Figures **A**, **B**, and **C** represent the floor plans of houses.

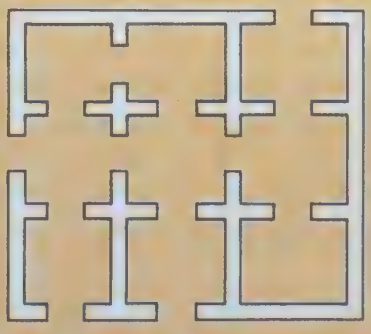
A



B



C



1. Starting inside, can you walk through all the doors of each house only once?

2. Starting outside, can you walk through all the doors of each house only once?

ELECTRICAL ENERGY

Electrical power is measured in **watts**. A **kilowatt** is 1000 watts.
A **kilowatt-hour** (kW.h) is 1000 watts of power in use for one hour.
For example, one kilowatt-hour will run a 100-watt light bulb 10 hours.
(That is, 100 watts × 10 hours = 1000 watt-hours = 1 kilowatt-hour.)

The table below shows the approximate number of kilowatt-hours used by some electrical appliances.

APPLIANCE	kW.h	APPLIANCE	kW.h
Water heater	13 per day	Hi-fi set	$\frac{1}{10}$ per hour
Refrigerator	6 per day	Washing machine	$\frac{1}{3}$ per load
Clothes dryer	3 per load	Toaster	$1\frac{1}{3}$ per hour
Kitchen range	1 per meal	Disposal	$\frac{1}{100}$ each time
Color television	$\frac{1}{3}$ per hour	Hair dryer	$\frac{1}{3}$ per hour
Iron	$1\frac{1}{4}$ per hour	Vacuum cleaner	$\frac{2}{3}$ per hour

1. If a color television is used 5 hours per day for 30 days, how many kW.h are used?

2. If you listen to a hi-fi while ironing for one hour, how many kW.h are used?

3. Five loads of clothes are washed and dried. It takes 2 hours to iron the clothes. How many kW.h?

4. Suppose a disposal is used about 15 times per day. How many kW.h are used in 30 days?

5. A toaster is in use for 15 minutes at breakfast. How many kW.h?

6. How many kW.h are used to make three meals per day for one week?

7. The average residential consumer uses about 100 kW.h in home lighting in one month's time. How many 100-watt light bulbs burning for one hour would use the same amount of kW.h?

8. If a vacuum cleaner is used for $\frac{1}{2}$ hour, the range is used for a meal, and the disposal used four times, how many kW.h?

9. Suppose the cost of one kW.h is 3 cents. What does it cost to do the following?
   - **A** Heat water for 30 days.
   - **B** Watch a 90-minute color television show.
   - **C** Run a refrigerator for a year.
   - **D** Go on vacation for a week with a 100-watt light bulb left burning.

Improving Computational Skills

1. Find the sums.

A $16\frac{3}{10} + 29\frac{1}{2}$

B $8\frac{3}{4} + 11\frac{3}{4}$

C $123\frac{7}{10} + 96\frac{9}{10}$

D $83\frac{1}{3} + 16\frac{2}{3}$

E $110\frac{5}{6} + 88\frac{1}{5}$

F $\frac{1}{2} + \frac{1}{3} + \frac{1}{5}$

G $\frac{3}{10} + \frac{37}{100} + \frac{377}{1000}$

H $\frac{1}{4} + \frac{2}{3} + \frac{3}{10}$

I $\frac{11}{10} + \frac{5}{2} + \frac{13}{100}$

J $\frac{1}{2} + \frac{1}{20} + \frac{1}{200}$

2. Find the differences.

A $\frac{7}{10} - \frac{14}{100}$

B $\frac{3}{4} - \frac{27}{100}$

C $\frac{2}{3} - \frac{3}{5}$

D $\frac{13}{10} - \frac{746}{100}$

E $\frac{5}{6} - \frac{8}{45}$

F $15\frac{9}{10} - 7\frac{7}{10}$

G $28 - 19\frac{1}{4}$

H $35\frac{7}{10} - 28\frac{3}{5}$

I $72\frac{1}{2} - 26\frac{9}{10}$

J $100 - 8\frac{7}{8}$

3. Find the products.

A $\frac{7}{10} \cdot \frac{2}{5}$

B $\frac{5}{4} \cdot \frac{2}{15}$

C $\frac{3}{8} \cdot \frac{5}{6}$

D $\frac{1}{3} \cdot 52$

E $45 \cdot \frac{4}{9}$

F $2\frac{1}{2} \cdot 6\frac{1}{10}$

G $8\frac{2}{3} \cdot \frac{3}{2}$

H $42\frac{1}{2} \cdot 5\frac{1}{2}$

I $6 \cdot 16\frac{2}{3}$

J $9\frac{1}{2} \cdot 9\frac{1}{3}$

think

Can you find these products quickly? Look for some shortcuts.

A $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdot \frac{8}{9} =$ ▮▮▮

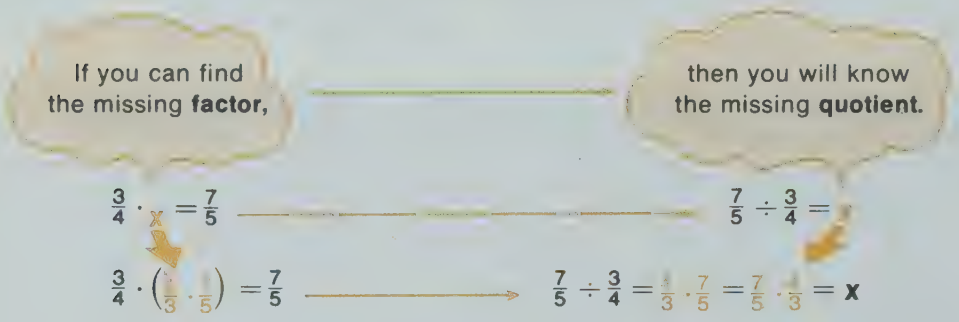
B $\frac{-4}{2} \cdot \frac{-3}{2} \cdot \frac{-2}{2} \cdot \frac{-1}{2} \cdot \frac{0}{2} \cdot \frac{1}{2} \cdot \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{4}{2} =$ ▮▮▮

C $\frac{7}{8} \cdot \frac{3}{10} \cdot \frac{6}{13} \cdot 1\frac{4}{7} \cdot \frac{1}{2} \cdot \frac{7}{11} \cdot \frac{13}{6} \cdot \frac{10}{3} \cdot 1\frac{1}{7} =$ ▮▮▮

D $\left(1+\frac{1}{2}\right) \cdot \left(1+\frac{1}{3}\right) \cdot \left(1+\frac{1}{4}\right) \cdot \left(1+\frac{1}{5}\right) \cdot \left(1+\frac{1}{6}\right) \cdot \left(1+\frac{1}{7}\right) =$ ▮▮▮

Discussing the Ideas

1. Finding missing factors in multiplication problems may help you to understand division of rational numbers. Study the example and then give the number for x.


If you can find the missing **factor,**
then you will know the missing **quotient.**
$\frac{3}{4} \cdot x = \frac{7}{5}$
$\frac{7}{5} \div \frac{3}{4} =$
$\frac{3}{4} \cdot \left(\frac{4}{3} \cdot \frac{7}{5}\right) = \frac{7}{5}$
$\frac{7}{5} \div \frac{3}{4} = \frac{4}{3} \cdot \frac{7}{5} = \frac{7}{5} \cdot \frac{4}{3} = x$

2. If the divisor is not zero, the same argument may be made for any pair of rational numbers. Study the example. Then give the missing fraction for ▥.

$\frac{c}{d} \cdot x = \frac{a}{b}$ $\qquad$ $\frac{a}{b} \div \frac{c}{d} = x$

$\frac{c}{d} \cdot \left(\frac{d}{c} \cdot \frac{a}{b}\right) = \frac{a}{b}$ $\qquad$ $\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \cdot \frac{a}{b} = \frac{a}{b} \cdot \frac{d}{c}$

Therefore, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot$ ▥.

3. The example shows how dividing with negative rational numbers is carried out.

$\frac{-2}{5} \div \frac{3}{4} = \frac{-2}{5} \cdot \frac{4}{3} = \frac{-8}{15}$

Try these division problems.

A $\frac{-1}{2} \div \frac{5}{4}$ **B** $\frac{-3}{8} \div \frac{-2}{5}$ **C** $\frac{10}{3} \div \frac{-5}{2}$ **D** $\frac{1}{10} \div \frac{-1}{5}$

4. To subtract one rational number from another, you can add the additive inverse.

$\frac{3}{4} - \frac{2}{3} = \frac{3}{4} + \frac{-2}{3}$

Notice the division example and then complete this statement:

$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \cdot \frac{3}{2}$

To divide one rational number by another, you can multiply ? ? ?.

1. Solve the equations.

A $\frac{3}{5} \div \frac{2}{3} = \frac{3}{5} \cdot n$

B $\frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \cdot n$

C $\frac{7}{10} \div \frac{5}{2} = \frac{7}{10} \cdot n$

D $\frac{1}{4} \div \frac{2}{3} = \frac{1}{4} \cdot n$

E $\frac{5}{6} \div \frac{-1}{2} = \frac{5}{6} \cdot n$

F $\frac{-2}{3} \div \frac{-3}{8} = \frac{-8}{3} \cdot n$

G $\frac{15}{8} \div \frac{5}{4} = \frac{4}{5} \cdot n$

H $\frac{9}{2} \div \frac{3}{4} = \frac{4}{3} \cdot n$

I $\frac{-12}{5} \div \frac{3}{5} = \frac{5}{3} \cdot n$

2. Find the quotients.

A $\frac{1}{3} \div \frac{1}{2}$

B $\frac{5}{6} \div \frac{1}{3}$

C $\frac{9}{10} \div \frac{3}{5}$

D $\frac{7}{12} \div \frac{3}{4}$

E $\frac{1}{2} \div \frac{-1}{3}$

F $\frac{1}{4} \div \frac{-1}{3}$

G $\frac{3}{5} \div \frac{-3}{5}$

H $\frac{6}{5} \div \frac{3}{10}$

I $\frac{3}{8} \div \frac{1}{4}$

J $\frac{11}{10} \div \frac{11}{4}$

K $\frac{2}{3} \div 1$

L $\frac{-9}{10} \div 3$

3. Write the mixed numerals as improper fractions and then find the quotients.

A $1\frac{2}{3} \div \frac{1}{2}$

B $4\frac{1}{2} \div 2\frac{1}{2}$

C $6\frac{1}{4} \div 3\frac{1}{2}$

D $1\frac{1}{8} \div \frac{1}{2}$

E $\frac{1}{3} \div -1\frac{1}{3}$

F $-10 \div 3\frac{1}{3}$

G $-2\frac{1}{2} \div -1\frac{1}{4}$

H $9\frac{1}{3} \div 3\frac{1}{2}$

4. Use division to find the number for r.

EXAMPLE: $\frac{3}{8} \cdot r = \frac{5}{2} \rightarrow r = \frac{5}{2} \div \frac{3}{8} = \frac{5}{2} \cdot \frac{8}{3} = \frac{40}{6} = \frac{20}{3}$

A $\frac{3}{5} \cdot r = \frac{1}{2}$

B $\frac{1}{10} \cdot r = \frac{5}{2}$

C $\frac{1}{4} \cdot r = \frac{3}{4}$

D $\frac{-1}{2} \cdot r = \frac{3}{8}$

E $\frac{-2}{3} \cdot r = \frac{-5}{6}$

F $\frac{3}{4} \cdot r = \frac{-1}{2}$

G $\frac{-1}{2} \cdot r = \frac{2}{7}$

H $\frac{1}{3} \cdot r = \frac{-5}{6}$

5. Find the number for x in each equation.

A $\left(\frac{3}{4} + \frac{1}{2}\right) \div \frac{2}{3} = x$

B $\left(\frac{2}{5} \div \frac{1}{2}\right) + \left(\frac{1}{3} \div \frac{1}{2}\right) = x$

C $\left(\frac{2}{5} + \frac{1}{3}\right) \div \frac{1}{2} = x$

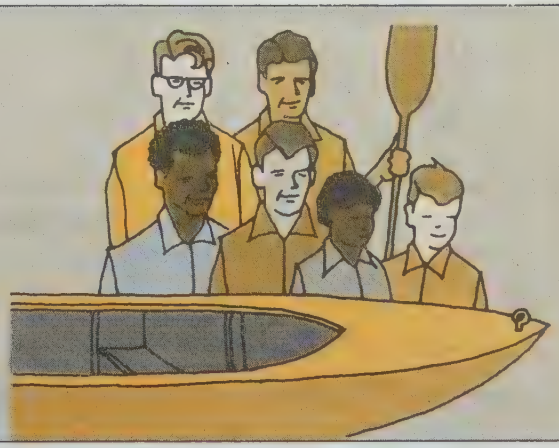
D $\left(\frac{3}{4} \div \frac{2}{3}\right) + \left(\frac{1}{2} \div \frac{2}{3}\right) = x$

E $\frac{2}{3} \div \left(\frac{1}{4} + \frac{1}{3}\right) = x$

F $\left(\frac{2}{3} \div \frac{1}{4}\right) + \left(\frac{2}{3} \div \frac{1}{3}\right) = x$

think

Four men and two boys must cross a river in a canoe that will hold only one of the men or the two boys at one time. It will not carry one man and one boy. Tell how the six people can get across the river. What is the smallest number of times the canoe must cross the river?



More practice, page S-15, Set 28

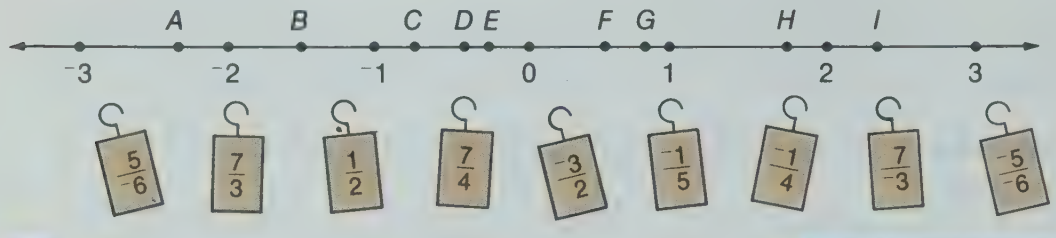
Investigating the Ideas

Give the missing numbers.

A Since $\frac{3}{2}$ is between 1 and 2, then $\frac{-3}{2}$ must be between ? and ?.

B Since $\frac{7}{3}$ is between 2 and 3, then $\frac{-7}{3}$ must be between ? and ?.

C Since $\frac{15}{4}$ is between 3 and 4, then $\frac{-15}{4}$ must be between ? and ?.



Can you match the tags with the proper points?

Discussing the Ideas

1. The smallest rational number named by any of the fraction tags was the number that goes with point *A*, the point farthest to the left. What was that number?

2. The largest number goes with point *I*, the point farthest to the right. What was that number?

3. Using the definition at the right, which symbol ($<$ or $>$) should be placed between each pair of fractions?

   A $\frac{7}{3}$ ▒ $\frac{7}{4}$ C $\frac{-3}{2}$ ▒ $\frac{-1}{5}$

   B $\frac{-5}{-6}$ ▒ $\frac{1}{2}$ D $\frac{-5}{6}$ ▒ $\frac{-5}{-6}$

> If the point for $\frac{a}{b}$ lies to the left of the point for $\frac{c}{d}$ on the number line, we say that $\frac{a}{b}$ is less than $\frac{c}{d}$, $\frac{a}{b}$ < $\frac{c}{d}$.
> If $\frac{a}{b} < \frac{c}{d}$, then $\frac{c}{d}$ is greater than $\frac{a}{b}$, $\frac{c}{d}$ > $\frac{a}{b}$.

4. Instead of using the number line, you can write each fraction with a **positive common denominator** and then compare the numerators. Study the examples.

   A $\frac{-5}{-6} = \frac{5}{6} =$ ▒ and $\frac{7}{4} =$ ▒

   Since 10 < ▒, then $\frac{-5}{-6}$ ▒ $\frac{7}{4}$.

   B $\frac{-1}{5} = \frac{6}{30}$ and $\frac{-5}{-6} = \frac{5 \cdot -5}{-6 \cdot -5} = \frac{-25}{30}$

   Since $^-6 > ^-25$, then $\frac{-1}{5} > \frac{-5}{-6}$.

   Compare these pairs of rational numbers by this method.

   A $\frac{3}{8}$ and $\frac{5}{12}$ B $\frac{-4}{3}$ and $\frac{-5}{4}$ C $\frac{-1}{2}$ and $\frac{2}{3}$ D $\frac{2}{-3}$ and $\frac{-3}{5}$

1. Give the integer for n.

A $\frac{1}{8} = \frac{n}{16}$ **C** $\frac{-2}{3} = \frac{n}{12}$ **E** $\frac{7}{-3} = \frac{35}{n}$ **G** $\frac{9}{n} = \frac{45}{35}$ **I** $\frac{n}{3} = \frac{20}{30}$

B $\frac{3}{4} = \frac{9}{n}$ **D** $\frac{12}{n} = \frac{24}{10}$ **F** $\frac{-5}{-2} = \frac{n}{10}$ **H** $\frac{-4}{-11} = \frac{28}{n}$ **J** $\frac{-8}{28} = \frac{-2}{n}$

2. Give the numbers for a and b. Then give the correct symbol ($<$ or $>$) for each ◍.

A $\frac{1}{5} = \frac{a}{20}$, $\frac{1}{4} = \frac{b}{20}$ $\Rightarrow \frac{1}{5}$ ◍ $\frac{1}{4}$

C $\frac{-6}{5} = \frac{a}{20}$, $\frac{-5}{4} = \frac{b}{20}$ $\Rightarrow \frac{-6}{5}$ ◍ $\frac{-5}{4}$

E $\frac{-20}{-3} = \frac{a}{21}$, $\frac{40}{7} = \frac{b}{21}$ $\Rightarrow \frac{-20}{-3}$ ◍ $\frac{40}{7}$

B $\frac{5}{12} = \frac{a}{24}$, $\frac{3}{8} = \frac{b}{24}$ $\Rightarrow \frac{5}{12}$ ◍ $\frac{3}{8}$

D $\frac{7}{11} = \frac{a}{44}$, $\frac{3}{4} = \frac{b}{44}$ $\Rightarrow \frac{7}{11}$ ◍ $\frac{3}{4}$

F $\frac{5}{16} = \frac{a}{48}$, $\frac{-5}{12} = \frac{b}{48}$ $\Rightarrow \frac{5}{16}$ ◍ $\frac{-5}{12}$

3. Copy each problem and give the correct symbol ($<$, $=$, or $>$) for each ◍.

A $\frac{2}{3}$ ◍ $\frac{3}{4}$ **E** $\frac{-5}{12}$ ◍ $\frac{-7}{16}$ **I** $\frac{1}{11}$ ◍ $\frac{1}{10}$ **M** $\frac{-1}{8}$ ◍ $\frac{-1}{9}$

B $\frac{3}{10}$ ◍ $\frac{1}{3}$ **F** $\frac{-11}{15}$ ◍ $\frac{-7}{10}$ **J** $\frac{7}{3}$ ◍ $\frac{-5}{2}$ **N** $\frac{2}{9}$ ◍ $\frac{3}{9}$

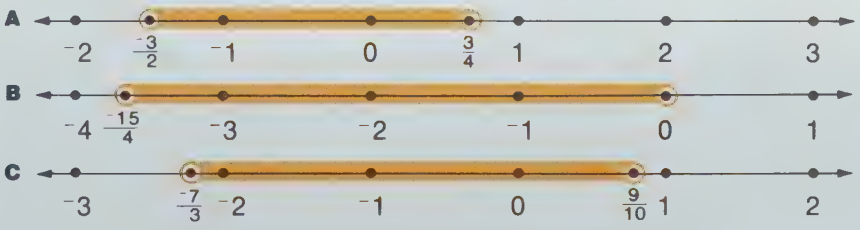
C $\frac{3}{8}$ ◍ $\frac{1}{6}$ **G** $\frac{3}{8}$ ◍ $\frac{-2}{5}$ **K** $\frac{-7}{10}$ ◍ $\frac{-5}{8}$ **O** $\frac{-11}{-14}$ ◍ $\frac{11}{15}$

D $\frac{7}{3}$ ◍ $\frac{9}{4}$ **H** $\frac{-11}{-4}$ ◍ $\frac{22}{8}$ **L** $\frac{30}{100}$ ◍ $\frac{3}{10}$ **P** $\frac{7}{8}$ ◍ $\frac{5}{6}$

4. The graph in the example illustrates the idea that we can think about all rational numbers x such that $x < \frac{9}{16}$ and $x > \frac{1}{8}$. Recall that we write such inequalities as $\frac{1}{8} < x < \frac{9}{16}$.

EXAMPLE:

Write similar inequalities for each number line.

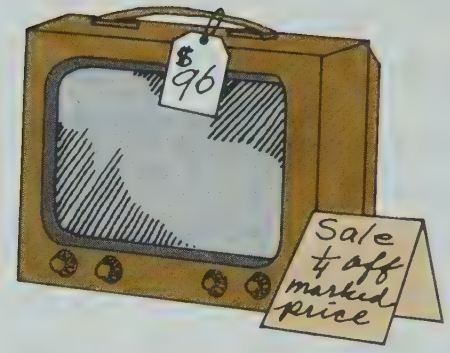


5. Illustrate each inequality by using a number line.

A $\frac{3}{8} < x < \frac{7}{8}$ **B** $\frac{-5}{3} < x < 0$ **C** $\frac{-3}{4} < x < \frac{-1}{2}$ **D** $\frac{-3}{10} < x < \frac{1}{10}$

Investigating the Ideas

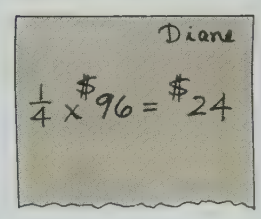
Study the advertisement at the right.



How much would you have to pay for the television set if you bought it during the sale?

Discussing the Ideas

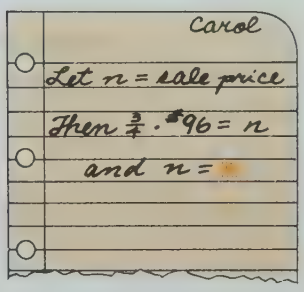
1. Diane found that $\frac{1}{4}$ of \$96 was \$24. Is this the sale price of the television set or the amount that the marked price is reduced?



2. How can you use Diane's work to find the sale price?

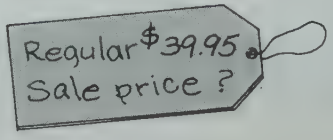
3. Carol thought about the problem differently. She thought about it this way: "Since there is $\frac{1}{4}$ off the marked price, you must pay $\frac{3}{4}$ of the marked price during the sale."

   What is the solution to Carol's equation?



4. **A** How much would you save on the television set if it were a "$\frac{1}{3}$-off" sale?
   B What would be the sale price of the television set?

5. Prices are reduced for a sale by $\frac{1}{5}$. How would you find the sale price for this tag?



1. A 10-speed bicycle regularly priced at \$109 is on sale during a "$\frac{1}{3}$-off" sale. What is the sale price of the bicycle?

2. A reclining chair regularly priced at \$150 is reduced in price by $\frac{1}{10}$ for a sale. What is the sale price of the chair?

3. Jane works in a dress shop which is having a "$\frac{1}{4}$-off" sale. What should she mark on the tag as the sale price of a pair of slacks that is regularly marked \$12?

4. A Estimate the cost of $39\frac{7}{10}$ litres of gasoline at $11\frac{9}{10}$¢ per litre.
   B Compute the actual cost.

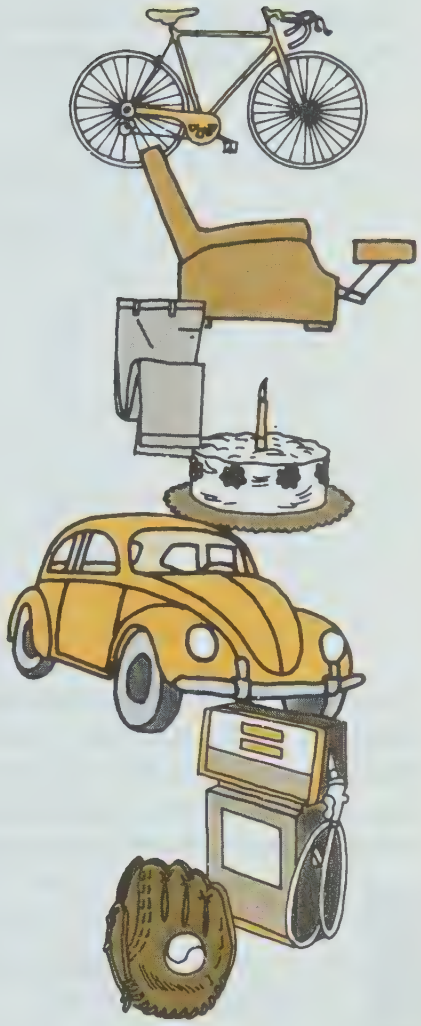
5. When Anne's sister had her twelfth birthday, Anne said, "You are $\frac{3}{4}$ as old as I am." How old is Anne?

6. A car was driven 90 kilometres in $1\frac{1}{2}$ hours. What was the average speed in kilometres per hour?

7. A tank is $\frac{3}{4}$ full of fuel oil. After 250 litres are used, the tank is $\frac{1}{2}$ full. How many litres does the tank hold?

8. A school hockey team won $\frac{2}{3}$ of its games. It won 3 more games than it lost. How many games did the team play?

9. The average of two rational numbers is $\frac{7}{8}$. One number is $\frac{3}{2}$. What is the other number?



think

A bought an article for \$10 and sold it to B for $\frac{1}{5}$ more than he paid for it. B sold the article to C for $\frac{1}{6}$ more than he paid for it. Then C sold it to D for $\frac{1}{7}$ more than he paid for it.

A How much did D pay for the article?

B How much profit did A make? B? C?

More practice, page S-16, Set 29

Investigating the Ideas

The set of all input numbers for a function is called the **domain** of the function. The set of all output numbers is called the **range** of the function.

Function Rule

$n + 1$	
n	$f(n)$
0	1
$\frac{1}{4}$	$1\frac{1}{4}$
$\frac{1}{2}$	$1\frac{1}{2}$
$\frac{-1}{2}$	$\frac{1}{2}$
$\frac{-3}{4}$	$\frac{1}{4}$
$^{-}1$	0
$\frac{-3}{2}$	$\frac{-1}{2}$

Domain:
All rational numbers

Co-ordinates

$\rightarrow (0, 1)$

$\rightarrow \left(\frac{1}{4}, 1\frac{1}{4}\right)$

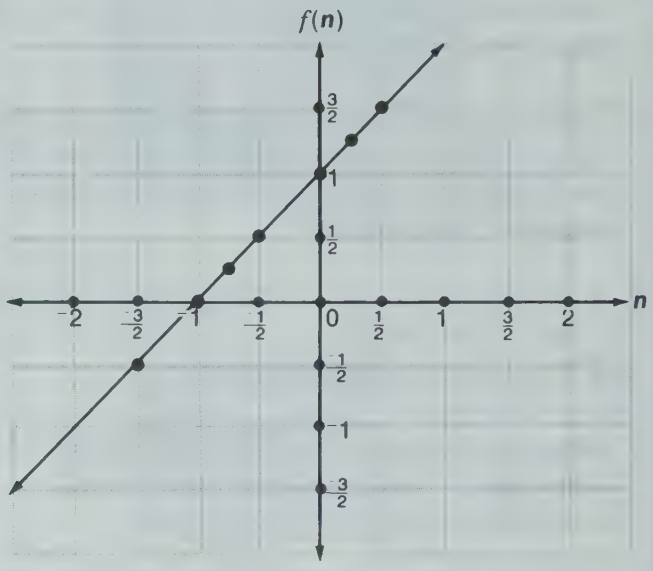
$\rightarrow \left(\frac{1}{2}, 1\frac{1}{2}\right)$

$\rightarrow \left(\frac{-1}{2}, \frac{1}{2}\right)$

$\rightarrow \left(\frac{-3}{4}, \frac{1}{4}\right)$

$\rightarrow (^{-}1, 0)$

$\rightarrow \left(\frac{-3}{2}, \frac{-1}{2}\right)$



If the domain of the function above is the set of all rational numbers, we cannot list all number pairs for the function. Instead, we find a few number pairs and locate the points with those co-ordinates on the co-ordinate plane. Then we connect the points with a line to show where other points for the function must be located.

Can you prepare a function table and draw the graph for the function whose rule is $\frac{1}{2} \cdot n$ when the domain is the set of rational numbers?

Discussing the Ideas

1. Are the pairs that you found for the function rule necessarily the same as those found by another classmate?

2. Will your graph look the same as the graph of another classmate if neither of you has made any mistakes?

3. If the domain of the function graphed in the Investigation had been all rational numbers n such that $^{-}1 \leq n \leq 1$, would the graph have been a *line* or a *segment* or a *ray*?

Complete the function tables for Exercises 1 through 6. Then draw a graph of the function. The domain is the set of all rational numbers.

1.

Function Rule: $n - 1$

	n	$f(n)$
	0	$^{-}1$
	$\frac{1}{2}$	$^{-}\frac{1}{2}$
	1	0
	$1\frac{1}{2}$	$\frac{1}{2}$
A	$^{-}1$	
B	$^{-}1\frac{1}{2}$	

2.

Function Rule: $2 \cdot n$

	n	$f(n)$
	$^{-}2$	$^{-}4$
	$^{-}1\frac{1}{2}$	$^{-}3$
	$^{-}1$	$^{-}2$
A	$^{-}\frac{1}{2}$	
B	0	
C	$1\frac{1}{2}$	

3.

Function Rule: $2 - n$

	n	$f(n)$
	0	2
	1	1
	2	0
A	$^{-}1$	
B	$^{-}2$	
C	$^{-}\frac{1}{2}$	

4.

Function Rule: $(\frac{1}{2} \cdot n) + 1$

	n	$f(n)$
	0	1
	1	$1\frac{1}{2}$
A	2	
B	3	
C	$^{-}1$	

5.

Function Rule: $^{-}3 \cdot n$

	n	$f(n)$
A	0	
B	$\frac{1}{3}$	
C	$\frac{1}{2}$	
D	$^{-}1$	
E	$^{-}\frac{1}{3}$	

6.

Function Rule: $\frac{1}{4} \cdot (n^2)$

	n	$f(n)$
A	0	
B	1	
C	2	
D	3	
E	$^{-}1$	

Complete the function tables for Exercises 7, 8, and 9. Then draw the graph of each function. In each case, the domain is given.

7. Domain: $^{-}3 < n < 3$

Function Rule: $n + 3$

	n	$f(n)$
	$^{-}2\frac{1}{2}$	$\frac{1}{2}$
A	$^{-}2$	
B	$^{-}1$	
C	$^{-}\frac{1}{2}$	
D	0	
E	1	
F	$^{-}1\frac{1}{2}$	$1\frac{1}{2}$

★ **8.** Domain: $^{-}2 \leq n \leq 2$

Function Rule: $4 - n^2$

	n	$f(n)$
	2	0
	1	3
	0	4
A	$^{-}1$	
B	$^{-}2$	
C	$1\frac{1}{2}$	
D	$^{-}1\frac{1}{2}$	

★ **9.** Domain: All rationals except 0

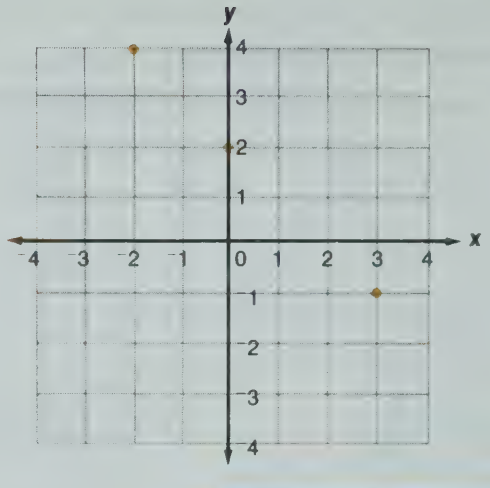
Function Rule: $1 \div n$

	n	$f(n)$
	1	1
	$\frac{1}{2}$	2
A	$\frac{1}{4}$	
B	2	
	$^{-}\frac{1}{2}$	$^{-}2$
C	$^{-}\frac{1}{4}$	
D	$^{-}2$	

Investigating the Ideas

The co-ordinates of three points for the graph of the equation $x + y = 2$ have been found and graphed.

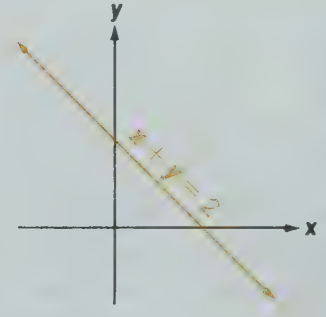
equation $x + y = 2$		co-ordinates
x	y	(x, y)
0	2	→ (0, 2)
3	$^{-}1$	→ (3, $^{-}1$)
$^{-}2$	4	→ ($^{-}2$, 4)



Can you find the co-ordinates of at least six more points for the equation and show them on a grid?

Discussing the Ideas

1. Do all the points that you found seem to lie in one line? If they do, draw a line through them. This shows that the replacement set of **x** and **y** is the set of rational numbers.



2. **A** Is it true that $\frac{5}{4} + \frac{7}{8} = 2$?
   B Is the pair $\left(\frac{5}{4}, \frac{7}{8}\right)$ the co-ordinates of some point on the graph of $x + y = 2$?

3. Is the pair $\left(\frac{^{-}3}{2}, \frac{7}{2}\right)$ the co-ordinates of some point on the graph of $x + y = 2$? How can you tell?

4. The point with co-ordinates $\left(^{-}1\frac{9}{10}, y\right)$ is a point on the graph of $x + y = 2$. What rational number is y?

5. Equations which have graphs that are straight lines are called **linear equations**. Is $x + y = 2$ a linear equation?

6. Find some co-ordinates for the equation $x \cdot y = 12$. Show that $x \cdot y = 12$ is not a linear equation.

For Exercises 1 through 6, complete each table and graph the equation. The replacement set for x and y is the set of rational numbers.

1. equation: $x + y = 3$

	x	y
	1	2
	$^-1$	4
A	2	
B	$^-2$	

2. equation: $x - y = 2$

	x	y
	5	3
	3	1
A	0	
B	$^-2$	

3. equation: $2x - y = 0$

	x	y
	2	4
	$^-3$	$^-6$
A	0	
B	4	

4. equation: $x - 2y = {}^-2$

	x	y
A	0	
B	4	
C	$^-4$	

5. equation: $2y + y = {}^-5$

	x	y
A	0	
B	$^-5$	
C	1	

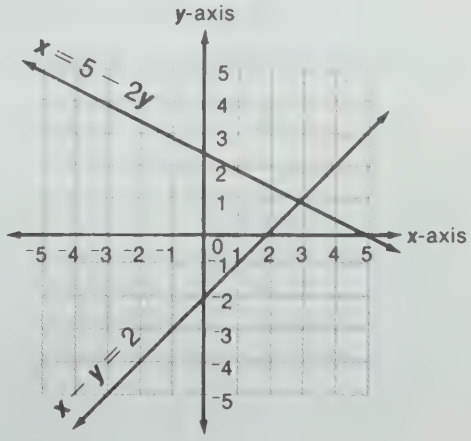
6. equation: $3x - 2y = 8$

	x	y
A	2	
B	0	
C	$^-4$	

7. Make your own table and then graph each equation. Find at least three pairs of co-ordinates for each equation.

A $x + y = 5$ B $x - y = {}^-1$ C $2x - y = 1$

8. Use the figure to give the number pair that is common to the graphs of both equations. **Substitute** this pair of values into each equation to check this choice.



9. Graph both of the following equations on the same grid:

$x + y = 0$ $x - 2y = 15$

Find the co-ordinates of the point that the two graphs have in common. Check to see that this number pair "works" in both equations.

REVIEWING THE IDEAS

1. A The opposite of $\frac{3}{7}$ is ?.

B The opposite of $\frac{-2}{3}$ is ?.

C $\frac{-5}{7}$ is the additive inverse of ?.

D The additive inverse of 0 is ?.

2. Find the sums.

A $\frac{3}{8}+\frac{-3}{8}$

B $\frac{5}{7}+\frac{-2}{7}$

C $\frac{-2}{3}+\frac{-1}{6}$

D $\frac{1}{3}+\frac{-1}{2}$

E $\frac{3}{4}+\frac{-1}{8}$

F $\frac{1}{4}+\frac{-5}{8}$

3. Find the differences.

A $\frac{3}{7}-\frac{4}{7}$

B $\frac{-7}{8}-\frac{1}{8}$

C $\frac{5}{7}-\frac{-1}{7}$

D $\frac{1}{3}-\frac{-4}{3}$

E $\frac{5}{12}-\frac{7}{12}$

F $\frac{-2}{5}-\frac{-4}{5}$

4. Find the products.

A $\frac{3}{5}\cdot\frac{4}{3}$

B $\frac{2}{3}\cdot\frac{-3}{5}$

C $\frac{-5}{6}\cdot\frac{-2}{3}$

D $\frac{3}{8}\cdot\frac{-4}{9}$

E $\frac{-8}{3}\cdot\frac{-6}{5}$

F $\frac{9}{10}\cdot\frac{-5}{3}$

G $3\frac{2}{3}\cdot\frac{3}{11}$

H $1\frac{1}{2}\cdot\frac{-5}{6}$

I $15\cdot\frac{-3}{5}$

5. What is the reciprocal of each number?

A $\frac{3}{4}$

B $\frac{-2}{3}$

C $\frac{-1}{6}$

D $\frac{-3}{7}$

E 4

F -5

G $\frac{-4}{3}$

H $1\frac{2}{5}$

6. Find the quotients.

A $\frac{9}{10}\div\frac{3}{2}$

B $\frac{-1}{4}\div\frac{1}{2}$

C $\frac{1}{2}\div\frac{-3}{8}$

D $\frac{14}{15}\div\frac{2}{3}$

E $\frac{-15}{16}\div\frac{-3}{8}$

F $\frac{5}{12}\div\frac{-5}{4}$

G $3\frac{1}{3}\div\frac{-5}{6}$

H $-2\frac{2}{3}\div 1\frac{1}{2}$

I $\frac{-7}{8}\div 1\frac{2}{5}$

7. Solve the equations.

A $\frac{5}{8}\cdot n=10$

B $\frac{7}{5}-n=\frac{3}{10}$

C $n\div\frac{2}{7}=\frac{3}{2}$

D $\frac{2}{3}+n=\frac{7}{6}$

E $\frac{9}{10}\cdot n=\frac{1}{2}$

F $8\cdot\left(3+\frac{7}{8}\right)=n$

G $10\cdot(2+n)=22$

H $\left(\frac{5}{6}\cdot\frac{1}{2}\right)+\left(\frac{1}{6}\cdot\frac{1}{2}\right)=n$

8. Give the smaller number of each pair.

A $\frac{1}{6}, \frac{3}{8}$

B $\frac{11}{16}, \frac{3}{5}$

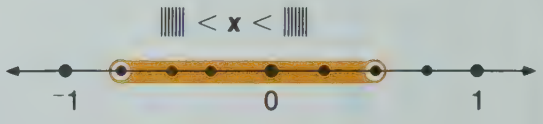
C $\frac{-3}{10}, \frac{-1}{5}$

D $0, \frac{-1}{2}$

E $-1\frac{1}{2}, \frac{-1}{2}$

F $-1\frac{1}{3}, \frac{1}{3}$

9. Give the rational numbers for the inequality suggested by this number line.



10. An article originally priced at \$132 is marked for sale at $\frac{1}{4}$ off. What is the sale price?

11. Complete the function rule table.

Function Rule: $n+\frac{1}{2}$

	n	$f(n)$
	$\frac{3}{4}$	$\frac{5}{4}$
	$\frac{1}{2}$	1
A	$\frac{1}{4}$	
B	$\frac{-1}{4}$	
C	$\frac{-1}{2}$	
D	$\frac{-3}{4}$	

12. Draw a graph of the function for Exercise 11. The domain is the set of rational numbers.

13. Give four number pairs for points on the graph of the equation $2x+y=11$.

14. Give the number pair for the point of intersection of the two lines $x+y=11$, $x-y=3$. Graph both equations.

TEST YOURSELF

1. Give the additive inverse of each rational number.

   A $\frac{3}{4}$ B $\frac{-2}{3}$ C $2\frac{1}{4}$

2. Solve the equations.

   A $\frac{-7}{10} + x = \frac{-7}{10}$ C $\frac{-8}{3} + t = 0$

   B $\frac{5}{8} + \frac{-2}{3} = r + \frac{5}{8}$ D $s \cdot 1 = \frac{-3}{10}$

3. Find the sums.

   A $\frac{3}{5} + \frac{1}{2}$ C $\frac{-7}{3} + \frac{5}{3}$

   B $\frac{-1}{5} + \frac{3}{5}$ D $\frac{-1}{2} + \frac{7}{4}$

4. Find the differences.

   A $\frac{7}{8} - \frac{1}{2}$ B $\frac{-5}{6} - \frac{1}{6}$

5. Find the products.

   A $\frac{1}{5} \cdot \frac{1}{7}$ C $-3 \cdot \frac{1}{4}$

   B $\frac{2}{3} \cdot \frac{-5}{6}$ D $\frac{-3}{2} \cdot \frac{-5}{3}$

6. What is the reciprocal of $\frac{-5}{8}$?

7. Find the quotients.

   A $\frac{1}{2} \div \frac{3}{4}$ C $1\frac{1}{4} \div \frac{-3}{4}$

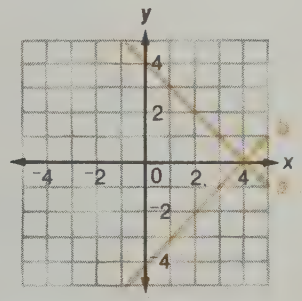
   B $\frac{-1}{4} \div \frac{-1}{8}$ D $\frac{-5}{6} \div 1\frac{1}{4}$

8. Give the correct symbol ($<$, $=$, or $>$) for each ◍.

   A $\frac{-5}{6}$ ◍ $\frac{2}{3}$ B $\frac{-3}{2}$ ◍ $\frac{-2}{3}$

9. An article originally priced at \$124 is marked for sale at $\frac{1}{4}$ off. What is the sale price?

10. Which line, *a* or *b*, is the graph of the equation $x - y = 4$?



RESEARCH PROJECTS

A Design a flow chart that will show the steps in finding the sum of any two rational numbers. Try to make flow charts for the other operations using fractional numbers.

B Read about the historical development of the symbols for fractions. Find out what other notation for fractions has been used. Write a report on your findings. *See *History of Mathematics* by David E. Smith; New York: Dover Publications, 1925, vol. II, pp. 208-228; General Publishing)

C Every rational number can be expressed as a **continued fraction**.

For example, $\frac{5}{12} = \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}$ and

$$\frac{-11}{5} = -3 + \frac{1}{1 + \frac{1}{4}}$$

Find out what continued fractions are and how they can be computed from ordinary fractions. (See *An Introduction To Continued Fractions* by Charles G. Moore: Washington, D.C.: National Council of Teachers of Mathematics, 1964.)

TRIANGULAR PATTERNS

Figure	Number of small triangles	Part shaded	Part unshaded
	1	1	0
	4	$\frac{3}{4}$	$\frac{1}{4}$
	9	$\frac{6}{9}$	$\frac{3}{9}$
⋮	16 ⋮	$\frac{10}{16}$ ⋮	$\frac{6}{16}$ ⋮

1. Draw and shade the next three triangular figures.

2. What fractional part of each region that you drew is shaded?

3. If the large triangular figure contained 100 small triangles, what part of the figure would be shaded? What part of it would be unshaded?

4. If the large triangular figure contained 400 small triangles, what part of the figure would be shaded? What part would be unshaded?

5. If the large triangular figure contained a "very large" number of small triangles, about what part of the figure would be shaded?

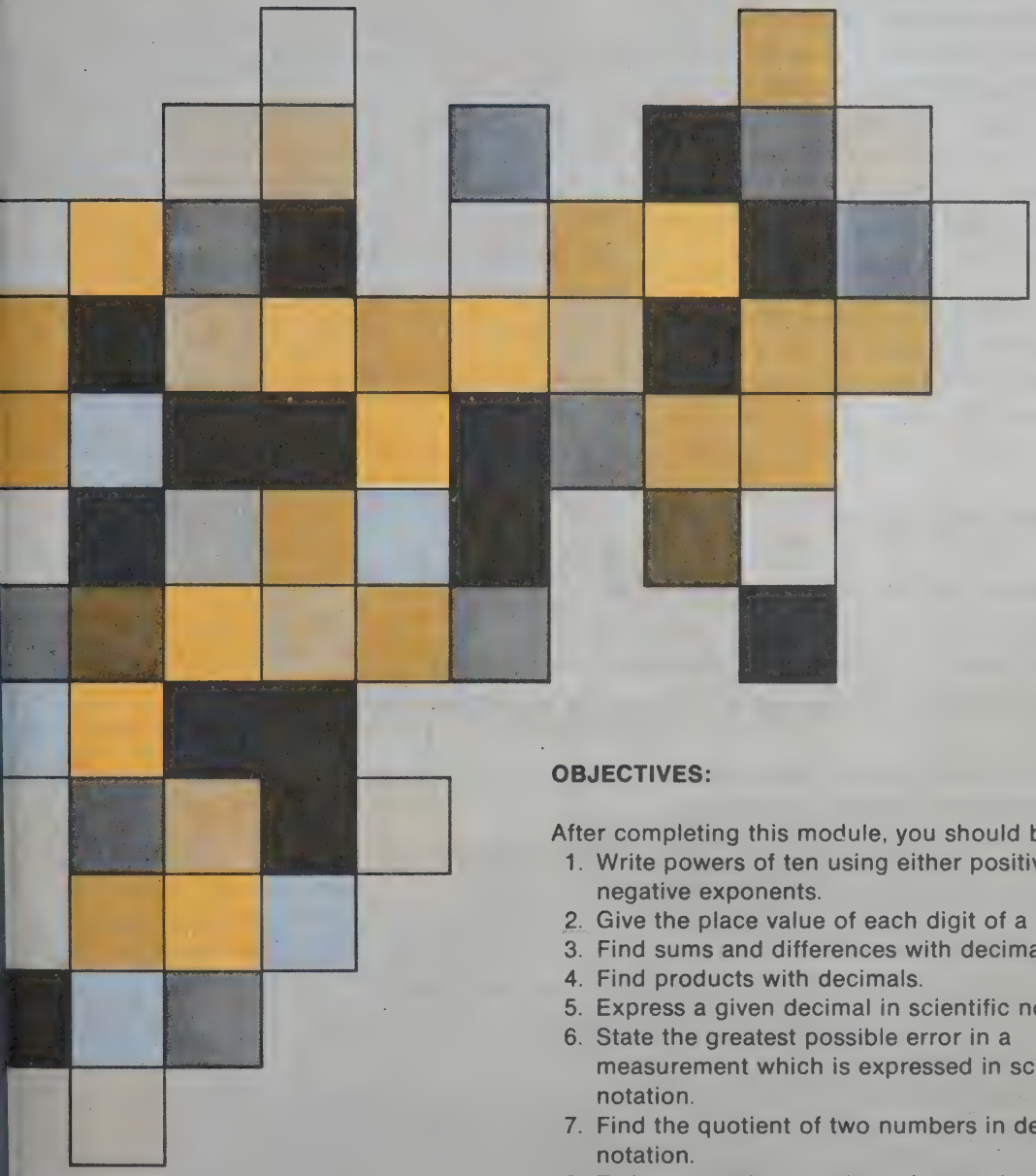
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A $\frac{-3}{4}$; B $\frac{2}{3}$; C $-2\frac{1}{4}$ 2. A $x = 0$; B $r = \frac{-2}{3}$; C $t = \frac{8}{3}$; D $s = \frac{-3}{10}$
3. A $\frac{11}{10}$; B $\frac{2}{5}$; C $\frac{-2}{3}$; D $\frac{5}{4}$ 4. A $\frac{3}{8}$; B -1 5. A $\frac{1}{35}$; B $\frac{-5}{9}$; C $\frac{-3}{4}$; D $\frac{5}{2}$
6. $\frac{-8}{5}$ 7. A $\frac{2}{3}$; B 2; C $\frac{-5}{3}$; D $\frac{-2}{3}$ 8. A $<$; B $<$ 9. \$93 10. *b*

UNIT F: *Rational Numbers*

MODULE 2: *Decimals*



OBJECTIVES:

After completing this module, you should be able to:

1. Write powers of ten using either positive or negative exponents.
2. Give the place value of each digit of a decimal.
3. Find sums and differences with decimals.
4. Find products with decimals.
5. Express a given decimal in scientific notation.
6. State the greatest possible error in a measurement which is expressed in scientific notation.
7. Find the quotient of two numbers in decimal notation.
8. Estimate products and quotients using decimals.
9. Find a decimal for a given rational number in fraction notation.
10. Find a repeating decimal for a given rational number.

Investigating the Ideas

You are already familiar with non-negative integer exponents. The idea of exponents may be extended to include negative integers. Study the table of powers of ten. Notice that each power of ten is $\frac{1}{10}$ as large as the one above it.

POWERS OF TEN

10^4	$= 10\,000$	$\leftarrow \frac{1}{10}$ as large as 10^5
10^3	$= 1000$	$\leftarrow \frac{1}{10}$ as large as 10^4
10^2	$= 100$	$\leftarrow \frac{1}{10}$ as large as 10^3
10^1	$= 10$	$\leftarrow \frac{1}{10}$ as large as 10^2
10^0	$= 1$	$\leftarrow \frac{1}{10}$ as large as 10^1
10^{-1}	$= \frac{1}{10}$	$\leftarrow \frac{1}{10}$ as large as 10^0
10^{-2}	$=$ ▮	$\leftarrow \frac{1}{10}$ as large as 10^{-1}
10^{-3}	$=$ ▮	$\leftarrow \frac{1}{10}$ as large as 10^{-2}
10^{-4}	$=$ ▮	$\leftarrow \frac{1}{10}$ as large as 10^{-3}
10^{-5}	$=$ ▮	$\leftarrow \frac{1}{10}$ as large as 10^{-4}

Can you discover a pattern for the powers of ten in the table and give the number for each ▮?

Discussing the Ideas

1. The pattern of the table suggests that $10^{-1} = \frac{1}{10^1} = \frac{1}{10}$, $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$, ...

   A How could 10^{-6} be rewritten with a positive exponent?

   B How could $\frac{1}{100\,000\,000}$ be expressed as a power of ten?

2. To extend the idea of exponents to include all integers, we make the following definition:

   If b is a positive integer and n is any integer, then $b^{-n} = \frac{1}{b^n}$.

   Thus, $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ and $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$.

   A How can you express 7^{-2} with a positive exponent?

   B How can you express $\frac{1}{2^3}$ with a negative exponent?

3. The example shows how $7 \cdot 10^{-1} + 3 \cdot 10^{-2}$ can be expressed as a rational number. Express each of these sums as a rational number.

   EXAMPLE:
   $7 \cdot 10^{-1} + 3 \cdot 10^{-2} = 7 \cdot \frac{1}{10} + 3 \cdot \frac{1}{100}$
   $= \frac{7}{10} + \frac{3}{100}$
   $= \frac{73}{100}$

   A $2 \cdot 10^{-1} + 3 \cdot 10^{-2}$

   B $6 \cdot 10^{-1} + 2 \cdot 10^{-2} + 9 \cdot 10^{-3}$

   C $3 \cdot 10^1 + 2 \cdot 10^0 + 7 \cdot 10^{-1}$

   D $5 \cdot 10^2 + 5 \cdot 10^{-2}$

1. Rewrite each power using positive exponents.

A 10^{-2} **C** 10^{-5} **E** 6^{-2} **G** 2^{-8} **I** 7^{-3}

B 10^{-4} **D** 10^{-14} **F** 5^{-3} **H** 3^{-4} **J** 8^{-5}

2. Rewrite each numeral using negative exponents.

A $\frac{1}{10^3}$ **C** $\frac{1}{7^5}$ **E** $\frac{1}{10^1}$ **G** $\frac{1}{1000}$ **I** $\frac{1}{2}$

B $\frac{1}{10^4}$ **D** $\frac{1}{3^3}$ **F** $\frac{1}{10^1}$ **H** $\frac{1}{10\,000}$ **J** $\frac{1}{3^5}$

3. Rewrite each numeral as a power of ten.

A 1000 **C** $\frac{1}{100}$ **E** 10 000 **G** 100 000 **I** 1 000 000

B $\frac{1}{1000}$ **D** 100 **F** $\frac{1}{10}$ **H** $\frac{1}{10\,000}$ **J** 1

4. Express each sum as a rational number.

A $6 \cdot 10^{-1} + 4 \cdot 10^{-2}$

B $7 \cdot 10^{-1} + 3 \cdot 10^{-3}$

C $3 \cdot 10^{0} + 5 \cdot 10^{-1}$

D $1 \cdot 10^{0} + 3 \cdot 10^{-1} + 9 \cdot 10^{-2}$

E $8 \cdot 10^{2} + 7 \cdot 10^{0} + 4 \cdot 10^{-1}$

F $5 \cdot 10^{-1} + 6 \cdot 10^{-2} + 3 \cdot 10^{-3} + 1 \cdot 10^{-4}$

G $1 \cdot 10^{1} + 2 \cdot 10^{0} + 3 \cdot 10^{-1} + 7 \cdot 10^{-2}$

H $3 \cdot 10^{3} + 1 \cdot 10^{1} + 6 \cdot 10^{-2}$

5. What rational number is this sum?

$1 \cdot 2^{2} + 1 \cdot 2^{1} + 1 \cdot 2^{0} + 1 \cdot 2^{-1} + 1 \cdot 2^{-2}$

6. Which statement is correct? Try to prove your answer.

A $10^{-1} \cdot 10^{-1} = 10^{1}$

B $10^{-1} \cdot 10^{-1} = 10^{-1}$

C $10^{-1} \cdot 10^{-1} = 10^{-2}$

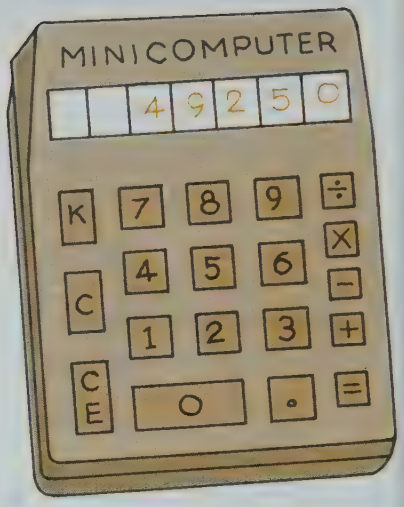
D $10^{-1} \cdot 10^{-1} = 10^{2}$

think

Sixteen pencils are arranged to form five squares. Show how four squares can be formed by moving two of the pencils to new positions. All sixteen pencils must be used, and each must be a side of a square.

Investigating the Ideas

Suppose the minicomputer shows a certain amount of money but does not show the decimal point.



Can you place a decimal point in the numeral so that it shows

- **A** the cost of a new automobile?
- **B** the cost of a table model AM-FM radio?
- **C** the cost of a color television set?

Discussing the Ideas

1. Instead of $\frac{1}{10}$, a minicomputer would show

   Instead of $\frac{1}{100}$, a minicomputer would show

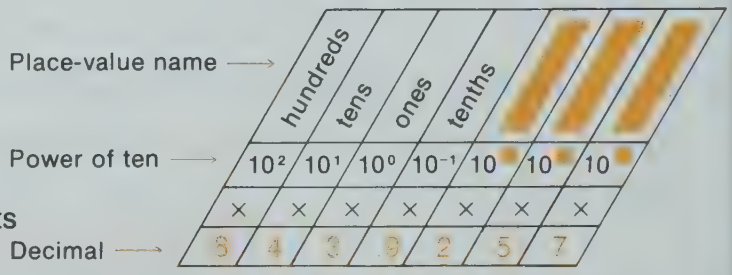
   What decimal would a minicomputer show for each of these?

   A $\frac{1}{1000}$ **B** $\frac{1}{10\,000}$ **C** $\frac{2}{10}$ **D** $\frac{3}{100}$ **E** $\frac{27}{100}$ **F** $\frac{179}{1000}$

2. The diagram will help you review how place value is used in decimal notation.

   A Give the missing place-value names.

   B Give the missing exponents for the power of ten.



3. The decimal in the diagram in Exercise 2 can be written using **expanded notation** (exponent form) as

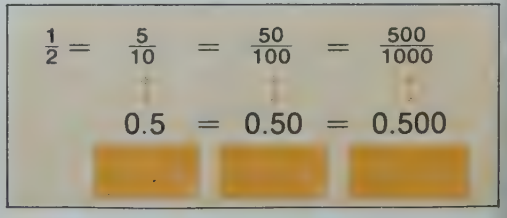
   $8 \cdot 10^2 + 4 \cdot 10^1 + 3 \cdot 10^0 + 9 \cdot 10^{-1} + 2 \cdot 10^{-2} + 5 \cdot 10^{-3} + 7 \cdot 10^{-4}$.

   How would you write each of these decimals in expanded notation?

   A 75.4 **B** 2.708 **C** 129.35 **D** 3.1416

4. The diagram at the right shows how to write a one-place decimal, a two-place decimal, and a three-place decimal for $\frac{1}{2}$. Give one-, two-, and three-place decimals for each of these fractions.

   A $\frac{9}{10}$ **B** $\frac{3}{5}$ **C** $\frac{1}{10}$ **D** $\frac{4}{5}$



1. Give the missing word and number.

EXAMPLE: 32.57: The 7 in the hundredths' place represents 7×10^{-2}

A 325.467: The 3 in the ? place represents ▒.

B 325.467: The 6 in the ? place represents ▒.

C 325.467: The 7 in the ? place represents ▒.

2. Write each fraction in decimal notation.

A $\frac{3}{10} = 0.3$ C $\frac{19}{100}$ E $\frac{9}{10}$ G $\frac{743}{1000}$ I $\frac{39}{100}$ K $\frac{291}{10\,000}$

B $\frac{7}{10}$ D $\frac{239}{1000}$ F $\frac{3}{100}$ H $\frac{6197}{10\,000}$ J $\frac{7}{1000}$ L $\frac{9}{1000}$

3. Write a decimal for each mixed numeral.

A $23\frac{75}{100} = 23.75$ C $56\frac{375}{1000}$ E $25\frac{7}{100}$ G $62\frac{539}{10\,000}$

B $7\frac{142}{1000}$ D $475\frac{3}{10}$ F $27\frac{46}{1000}$ H $3\frac{7}{10\,000}$

4. Write each decimal in expanded notation.

A 3.147 B 298.42 C 0.6129 D 0.0067

5. Write a decimal for each of the following:

A $2 \cdot 10^1 + 3 \cdot 10^0 + 1 \cdot 10^{-1}$ C $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0 + 8 \cdot 10^{-1} + 1 \cdot 10^{-2}$

B $8 \cdot 10^0 + 0 \cdot 10^{-1} + 3 \cdot 10^{-2}$ D $7 \cdot 10^2 + 0 \cdot 10^1 + 0 \cdot 10^0 + 0 \cdot 10^{-1} + 3 \cdot 10^{-2}$

6. For the fraction $\frac{7}{10}$, write

A a one-place decimal. B a two-place decimal. C a three-place decimal.

7. Write a three-place decimal and a four-place decimal for 0.75.

8. Write each minicomputer numeral with a decimal according to the directions given.

A ▒▒▒▒▒▒▒▒ A number between 10 and 100

B ▒▒▒▒▒▒▒▒ A number between 1 and 10

C ▒▒▒▒▒▒▒▒ A number between $\frac{1}{10}$ and $\frac{1}{100}$

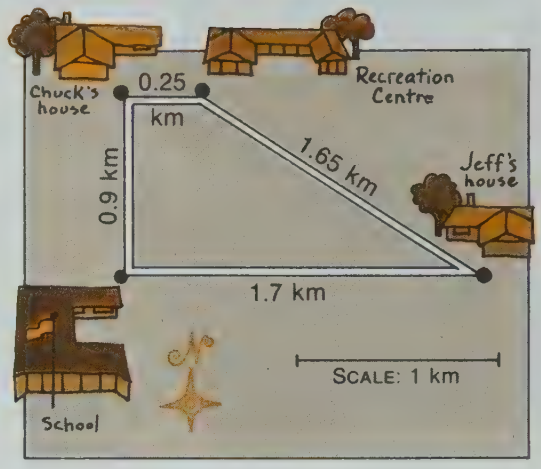
think

Suppose fractional exponents are used according to these examples: $4^2 = 16 \leftrightarrow 16^{\frac{1}{2}} = 4$ and $2^3 = 8 \leftrightarrow 8^{\frac{1}{3}} = 2$
What numbers do these powers represent?

A $25^{\frac{1}{2}}$ B $27^{\frac{1}{3}}$ C $100^{\frac{1}{2}}$ D $49^{\frac{1}{2}}$ E $125^{\frac{1}{3}}$ F $81^{\frac{1}{4}}$

Investigating the Ideas

Study the map and the distances shown on it.



How many of the questions below can you answer?

A How far is it from school to Chuck's house to the Recreation Centre?

B How far is it from Jeff's house to school to Chuck's house?

C How much farther is it from Chuck's house to school than it is from his house to the Recreation Centre?

Discussing the Ideas

1. Suppose a student added 0.9 and 0.25 like this: →

$$\begin{array}{r} 0.9 \\ +\,0.25 \\ \hline 0.34 \end{array}$$

Since $0.9 = \frac{9}{10} = \frac{90}{100}$ and $0.25 = \frac{25}{100}$, explain why the sum shown is incorrect.

2. To subtract 0.25 from 0.9, it is convenient to replace 0.9 by the equivalent two-place decimal 0.90. This is the same as choosing a common denominator in the problem

$$\frac{9}{10} - \frac{25}{100} = \frac{90}{100} - \frac{25}{100}.$$

$$\begin{array}{rcl} 0.9 & = & 0.90 \\ \underline{-\,0.25} & = & \underline{-\,0.25} \end{array}$$

Explain how you would subtract 0.347 from 0.4.

3. Explain how we can think of the sum 0.31 + 0.024 + 0.135 as $\left(\frac{3}{10} + \frac{1}{100}\right) + \left(\frac{2}{100} + \frac{4}{1000}\right) + \left(\frac{1}{10} + \frac{3}{100} + \frac{5}{1000}\right)$.

4. Explain how the associative and commutative principles help us to think of lining up the decimal points to get $\left(\frac{3}{10} + \frac{1}{10}\right) + \left(\frac{1}{100} + \frac{2}{100} + \frac{3}{100}\right) + \left(\frac{4}{1000} + \frac{5}{1000}\right)$. →

$$\begin{array}{r} 0.31 \\ 0.024 \\ +\,0.135 \end{array}$$

1. Find the sums.

A
$$\begin{array}{r} 87.14 \\ 23.87 \\ \hline \end{array}$$

B
$$\begin{array}{r} 0.129 \\ 0.744 \\ 0.668 \\ \hline \end{array}$$

C
$$\begin{array}{r} 135.29 \\ 642.77 \\ 883.56 \\ \hline \end{array}$$

D
$$\begin{array}{r} 9.006 \\ 18.213 \\ 31.747 \\ \hline \end{array}$$

E
$$\begin{array}{r} 0.0714 \\ 0.0068 \\ 0.1429 \\ 0.0009 \\ \hline \end{array}$$

2. Find the differences.

A
$$\begin{array}{r} 74.3 \\ 59.8 \\ \hline \end{array}$$

B
$$\begin{array}{r} 7.05 \\ 1.96 \\ \hline \end{array}$$

C
$$\begin{array}{r} 0.621 \\ 0.583 \\ \hline \end{array}$$

D
$$\begin{array}{r} 109.3 \\ 86.5 \\ \hline \end{array}$$

E
$$\begin{array}{r} 7.002 \\ 0.785 \\ \hline \end{array}$$

3. Write equivalent three-place decimals and then add.

A $5 + 0.6 + 2.13 + 0.255$

B $1.24 + 3.062 + 4 + 2.5$

C $12.46 + 1.5 + 0.262 + 7$

D $14 + 4.32 + 1.256 + 0.7$

EXAMPLE:

$2 + 0.5 + 1.14 + 0.316$

$$\begin{array}{r} 2.000 \\ 0.500 \\ 1.140 \\ +\,0.316 \\ \hline \end{array}$$

4. Line up the decimal points and then add without changing to equivalent decimals.

A $12 + 1.4 + 0.56$

B $37 + 5.8 + 3.14 + 0.226$

C $5.4 + 11.28 + 0.423 + 7$

D $12 + 3.16 + 0.004 + 7.9$

EXAMPLE:

$17 + 0.8 + 4.26$

$$\begin{array}{r} 17. \\ 0.8 \\ +\ 4.26 \\ \hline \end{array}$$

5. Write equivalent decimals and then subtract.

A $4.3 - 1.742$

B $16 - 3.517$

C $8.2 - 4.35$

D $100 - 0.8776$

EXAMPLE:

$5.2 - 1.468$

$$\begin{array}{r} 5.200 \\ -1.468 \\ \hline \end{array}$$

6. Negative rational numbers can be represented by decimals as shown. Find the sums.

EXAMPLE:

$^-0.72 + 0.57$

$$\begin{array}{r} {}^-0.72 \\ +\ 0.57 \\ \hline {}^-0 \end{array}$$

A
$$\begin{array}{r} 18.2 \\ {}^-12.6 \\ \hline \end{array}$$

B
$$\begin{array}{r} {}^-0.765 \\ {}^-0.426 \\ \hline \end{array}$$

C
$$\begin{array}{r} {}^-9.5 \\ {}^-6.4 \\ \hline \end{array}$$

D
$$\begin{array}{r} {}^-0.0068 \\ 0.0724 \\ \hline \end{array}$$

E $^-0.17 + {}^-0.09$

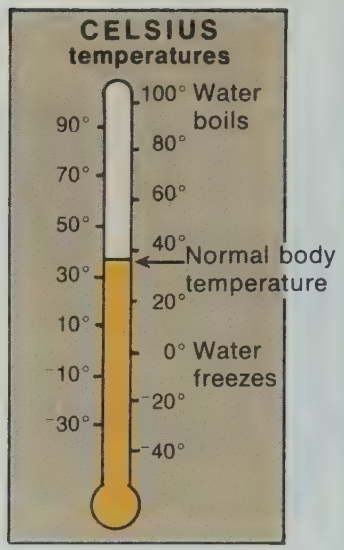
F $^-0.58 + {}^-0.36$

G $^-5.28 + 7.45$

H $^-23.4 + 19.15$

More practice, page S-16, Set 30

CELSIUS TEMPERATURES



1. Normal body temperature for humans is 37° C (Celsius).
   - A How much below normal temperature is 35.7° C?
   - B How much above normal temperature is 39.2° C?

2. The spiny anteater of Australia has a normal body temperature of only 23.2° C. How many degrees lower than normal human body temperature is this?

3. A marathon runner, on a hot day, was found to have a temperature of 40.6° C. How much above normal was his temperature?

4. The lowest recorded temperature for British Columbia is $^{-}$58.9° C. The highest recorded temperature for British Columbia is 46.7° C. How many degrees difference is there in these two temperatures?

5. The lowest recorded temperature in the world was at the Soviet Antarctic station Vostok on July 24, 1960. The temperature was $^{-}$88.3° C. The highest temperature in the world, 57.8° C, was recorded at Azizia, Libya, on September 13, 1922. What is the difference in these temperatures?

6. The record high temperature for San Francisco, California, is 41.1° C. The record low temperature is $^{-}$6.7° C. What is the difference in these temperatures?

7. Water boils at 100° C. **Absolute zero,** the coldest possible temperature, is 373.16° C below the boiling point of water. What is the temperature for absolute zero?

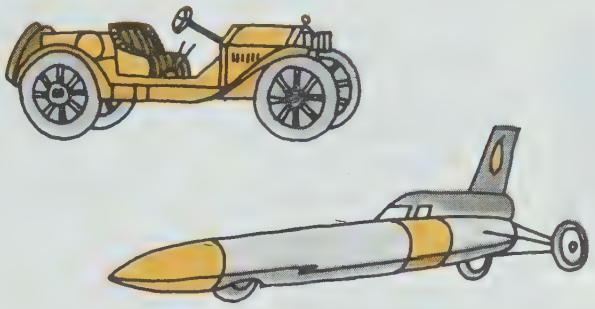
8. "Dry ice" (solid carbon dioxide) will evaporate at temperatures above $^{-}$78.5° C How much above the temperature for absolute zero is this?

9. Mercury becomes a solid instead of a liquid at $^{-}$39° C. Can a mercury thermometer be used to find the temperature of "dry ice"?

10. Over a period of nine years, the temperature at Saipan, Mariana Islands, varied from 19.6° C to 31.4° C. How many degrees variation is this?

Speeeeeeeeeeeeeed Records

1. In 1910, Barney Oldfield set a land speed record of 211.983 kilometres per hour (km/h). In 1970, Gary Gabelich set a new record of 1001.64 km/h. How much faster was Gabelich's speed than Oldfield's speed?



2. The greatest speed at which man has travelled was flown by the astronauts during the Apollo X mission. Their maximum speed was 39 892 kilometres per hour. About how many times faster than Gabelich's land speed record is this?

3. In 1972, Mark Donahue set an automobile record speed of 263.064 km/h at Indianapolis. This was 37.291 km/h faster than Rodger Ward's record in 1962. What was Ward's speed in 1962?

4. In the 1968 Olympics, Lee Evans had an average speed of 9.13 metres per second in the 400-metre race and Jim Hines had a speed of 10.1 metres per second in the 100-metre race. How much faster was Hines' speed?

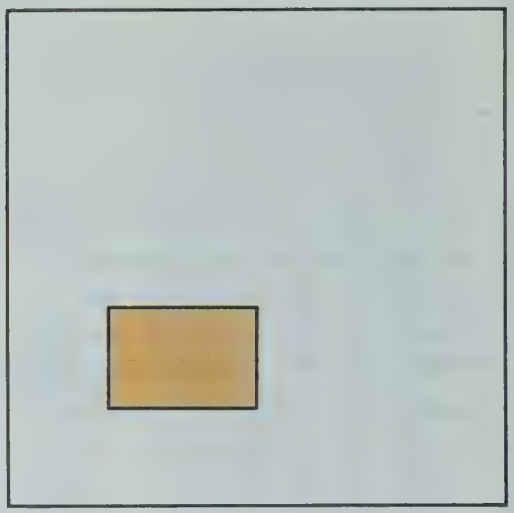
5. A record walking speed was set by Agapor of the USSR in 1969. In a 20 000-metre race, he walked at a rate of 3.84 metres per second. How much slower is this than Lee Evans' running speed? (See Exercise 4.)

6. Mark Spitz had a swimming speed of 1.95 metres per second in setting a record time for the 100-metre swim in the 1972 Olympics. His speed for the 200-metre race was 0.18 metres per second slower. What was his speed in the 200-metre race?

7. Shane Gould set a record for the 100-metre free-style swim by swimming at a rate of 1.71 metres per second. How much slower than Mark Spitz' rate was her rate?

Investigating the Ideas

The shaded rectangular region is 0.3 units long and 0.2 units wide. What part of the entire square region does it cover?



How many rectangular regions can you show on a square grid like this that will cover between 0.06 and 0.3 of the square region?

Discussing the Ideas

1. How long and how wide were the rectangles you found?

2. The small rectangle shown in the Investigation suggests the multiplication equation $0.3 \times 0.2 = x$, where x denotes the shaded portion of the entire square region. To find the product, you can do all the computing as if working with whole numbers. Then, the last step is to locate the decimal point in the product. What is the product?

Tenths times tenths is hundredths. The product is ▮.

$0.3 \times 0.2 = ▮$

$3 \times 2 = 6$

3. What are the missing numbers in each part?

A A two-place decimal times a two-place decimal is a four-place decimal. The product is ▮.

$0.04 \times 0.12 = ▮$

$4 \times 12 = 48$

B A whole number times a three-place decimal is a three-place decimal. The product is ▮.

$100 \times 0.035 = ▮$

$100 \times 35 = 3500$

C A two-place decimal times a three-place decimal is a five-place decimal. The product is ▮.

$2.13 \times 0.045 = ▮$

$213 \times 45 = 9585$

4. Multiplying a decimal by a power of ten can often be done mentally. Study the examples. Then complete the two rules.

$10 \times 0.23 = 10 \times \frac{23}{100} = \frac{230}{100} = 2.3$

$0.1 \times 0.23 = \frac{1}{10} \times \frac{23}{100} = \frac{23}{1000} = 0.023$

To multiply by 10, we move the decimal point one place to the ? .
To multiply by .1, we move the decimal point one place to the ? .

1. Give the missing word.

- **A** 3 tenths times 6 tenths is 18 ? .
- **B** 14 hundredths times 3 tenths is 42 ? .
- **C** 23 thousandths times 31 hundredths is 713 ? .

2. Find the products.

- **A** 21.3×6
- **B** 4.2×0.9
- **C** 318×0.07
- **D** 12.5×0.13
- **E** 6000×0.875
- **F** 7.76×1.9
- **G** 4.28×6.5
- **H** 1.5×3.1416
- **I** 1.6093×48.4

Solve each short story for Exercises 3 through 8.

3. Skirt price: $18.00.
Sales tax: 0.05 times the price.
How much sales tax?

4. Sheet of paper: 0.016 cm thick.
How thick are 75 sheets of paper?

5. Speed of sound in air:
One metre per 0.00301 seconds.
How long for sound to travel one kilometre?

6. Earnings: $120.
Income tax: 0.175 of earnings.
How much income tax?

7. Length of walking pace: 0.84 metre.
How far in 1200 paces?

8. Speed of sound: about 0.3 km/sec.
Speed of light: about one million times as fast.
How fast is the speed of light?

★ **9.** When one or more factors are negative, find the product as if both factors were positive and then decide whether the product is positive or negative.

EXAMPLES: $^{-}0.3 \times 0.7 = {}^{-}0.21$ $\quad {}^{-}2 \times {}^{-}3.4 = 6.8$

Find the products.

- **A** $0.8 \times {}^{-}2.1$
- **B** $^{-}0.04 \times 6$
- **C** $^{-}0.7 \times {}^{-}3.2$
- **D** $^{-}0.13 \times 6.1$
- **E** $12.8 \times {}^{-}0.2$
- **F** $^{-}0.63 \times {}^{-}0.8$
- **G** $0.101 \times {}^{-}31.4$
- **H** $^{-}0.001 \times 68.4$
- **I** $^{-}0.111 \times {}^{-}3.29$

think

The symbol $\sum_{1}^{n}$ means the **sum** of the first n positive integers.

Thus, $\sum_{1}^{5} = 1 + 2 + 3 + 4 + 5 = 15$.

The symbol $n!$ means the **product** of the first n positive integers.
Thus, $5! \ldots 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

What integer is n if $\sum_{1}^{n} = n!$?

More practice, page S-17, Set 31

Investigating the Ideas

Each number in the left column matches one of the products in the right column.

Can you match each number with the correct product?

NUMBER	PRODUCT
A 32 000	H 3.2×10^{6}
B 0.0032	I 3.2×10^{5}
C 0.32	J 3.2×10^{4}
D 320 000	K 3.2×10^{0}
E 3.2	L 3.2×10^{-1}
F 3 200 000	M 3.2×10^{-2}
G 0.032	N 3.2×10^{-3}

Discussing the Ideas

1. Each number in the left column is expressed in **scientific notation** by one of the products in the right column in the Investigation.
How is 32 000 expressed in scientific notation?

2. To express a number in scientific notation, you write it as **a product of a number from 1 to 10, not including 10, and a power of ten.**
Thus, 0.00217 written in scientific notation is 2.17×10^{-3}.

   Which numbers below are written in scientific notation?

   A 6.46×10^{-2} C 31.4×10^{3} E 11.4×10^{-1}

   B 3.8×10^{0} D 9.99×10^{-9} F 6.3×8^{3}

3. Examples A and B below illustrate a shortcut for writing decimals in scientific notation.

   A $27\,600 = 2.76 \times 10^{4}$

   To get the decimal point here

   We need this power of ten.

   B $0.0000395 = 3.95 \times 10^{-5}$

   To get the decimal point here

   We need this power of ten.

   Show how each decimal can be written in scientific notation.

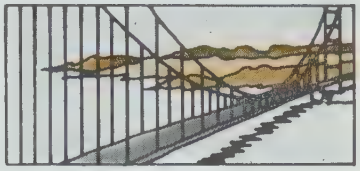
   A 5800 B 0.017 C 10 000 D 3.76 E 0.00008

1. Copy and give the integer for n.

A $52 = 5.2 \times 10^n$
B $430 = 4.3 \times 10^n$
C $0.52 = 5.2 \times 10^n$
D $0.043 = 4.3 \times 10^n$
E $4350 = 4.35 \times 10^n$
F $7\,000\,000 = 7 \times 10^n$
G $320\,000 = 3.2 \times 10^n$
H $0.000000043 = 4.3 \times 10^n$
I $1.53000 = 1.53 \times 10^n$

2. Express the number in each statement in scientific notation.

A A side of the Great Pyramid is 230 metres long.
B The length of the channel span of the Golden Gate Bridge is 1280 metres.
C The height of Mount McKinley is 6190 metres.
D The air distance from Toronto to St. John's is 3029 kilometres.


Golden Gate Bridge, San Francisco

E The circumference of the Earth is about 40 000 kilometres.
F The circumference of the sun is about 4 440 000 kilometres.
G The circumference of a giant star is about 8 700 000 000 kilometres.
H According to Einstein, the possible radius of the universe is 30 000 000 000 000 000 000 000 kilometres.

3. Numbers represented by scientific notation are given below. Give the ordinary decimal symbol for each number.

A 7×10^3
B 6.2×10^{-2}
C 5.45×10^{-6}
D 9.03×10^7
E 6.43×10^0
F 5×10^{-12}
G 1.1×10^{10}
H 3.37×10^{-8}

4. Write an ordinary decimal for each length. The lengths are measured in centimetres.

A Size of very small insects: 2×10^{-2}
B Size of large bacteria: 3.5×10^{-1}
C Red light wave length: 6.4×10^{-5}
D Green light wave length: 5.3×10^{-5}
E Violet light wave length: 4.5×10^{-5}
F Size of a virus: 2.5×10^{-7}
G Diameter of an atom: 1.06×10^{-8}
H Cosmic ray wave length: 3×10^{-19}

5. Give the products in the form $k \times 10^n$ where $1 \leq k < 10$ and k is an integer.

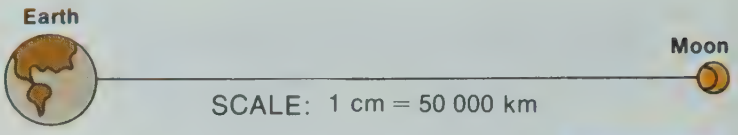
EXAMPLE: $4 \times 10^2 \times 6 \times 10^3 = 24 \times 10^5 = 2.4 \times 10^6$

A $5 \times 10^2 \times 6 \times 10^4$
B $4 \times 10^{10} \times 5 \times 10^{12}$
C $5 \times 10^{-3} \times 4 \times 10^{-2}$
D $8 \times 10^{-8} \times 7 \times 10^5$
E $2 \times 10^3 \times 5 \times 10^4$
F $2 \times 10^{-3} \times 5 \times 10^4$
G $6 \times 10^5 \times 7 \times 10^7$
H $3 \times 10^{-7} \times 2 \times 10^0$
I $2 \times 10^{-3} \times 5 \times 10^{-4}$

More practice, page S-17, Set 32

Investigating the Ideas

Suppose this line segment represents the distance from Earth to the moon.



Can you find the distance from Earth to the moon using the scale drawing?	Express this distance in scientific notation.

Discussing the Ideas

1. Using the scale drawing above, is the moon
   - **A** 300 000 to 400 000 km from Earth?
   - **B** 350 000 to 400 000 km from Earth?
   - **C** 350 000 to 375 000 km from Earth?
   - **D** 375 000 to 400 000 km from Earth?

2. The **significant digits** of a measurement are those digits which are correct when the measurement is carefully made and properly rounded. All nonzero digits are significant digits.

   300 000 has 1 significant digit. **35**0 000 has 2 significant digits.

   - **A** How many significant digits in 384 000?
   - **B** How many significant digits in 283 300?

3. Some zeros are significant. The measure 2.07 metres has 3 significant digits while 0.06 metre has 1 significant digit. Give the number of significant digits for each measure.

   A 6.394 mm **B** 20.7 cm **C** 0.7724 km **D** 0.003 mm

4. When scientific notation is used, all digits in the number times the power of ten are significant.

   EXAMPLES: **A** 3.8×10^5 **B** 6.30×10^6 **C** 1.6093×10^{-4}

   2 significant digits — 3 significant digits — 5 significant digits

   How many significant digits are in 3.05×10^5?

5. When scientific notation is used, you can determine the **greatest possible error** (GPE). For example, if a measure of 3.8×10^5 km is given for a distance, the actual distance is between 3.75×10^5 km and 3.85×10^5 km. The greatest possible error is 0.05×10^5 km or 5000 km.

   - **A** What is the GPE for 3.84×10^5 km?
   - **B** What is the GPE for 3.843×10^5 km?

1. Give the number of significant digits for each measurement.

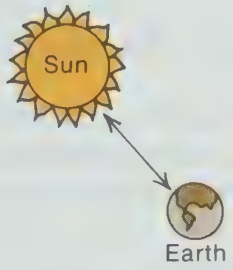
A 5.4×10^3 cm
B 5.42×10^3 cm
C 5.4×10^0 cm
D 5.40×10^{-2} cm
E 5.004×10^{-2} cm
F 5.0004×10^{-2} cm

2. Find the greatest possible error (GPE) for each measurement.

EXAMPLE: 2.4×10^3 m → GPE = $.05 \times 10^3$ m = 50 m

A 3.9×10^4 cm
B 6.28×10^{-2} mm
C 5.3×10^9 year
D 7.04×10^7 kg
E 1.001×10^{-4} cm
F 9.6×10^3 ml
G 3.0×10^6 m
H 5×10^{11} km
I 8.1×10^{-1} cm

3. The average distance from Earth to the sun is 150 000 000 km, to the nearest million km. We can write this as 150 000 000 ± 500 000 km, or in scientific notation as $(1.50 \pm .005) \times 10^8$ km. The greatest possible error in this measurement is $.005 \times 10^8$ km. The **relative error** in this measurement is the ratio of the greatest possible error to the measure.



$$\text{Relative error} = \frac{\text{GPE}}{\text{measure}} = \frac{.005 \times 10^8}{1.50 \times 10^8} = \frac{.005}{1.50} = \frac{5}{1500} = \frac{1}{300}$$

The relative error is about 1 part in 300.

Find the relative error of each measurement.

A $(6.5 \pm .05) \times 10^3$ m
B $(7.2 \pm .05) \times 10^{-2}$ cm
C $(3.0 \pm .05) \times 10^8$ m/s
D $(1.25 \pm .005) \times 10^{-2}$ cm

4. The more **precise** of two measures is the one with the smaller relative error. Choose the more precise measure of each pair.

A $(3.2 \pm .05) \times 10^{-2}$ cm or $(3.24 \pm .005) \times 10^2$ cm
B 1.3×10^{-2} ml or 2.14×10^3 ml
C 4.0×10^3 km or 4×10^{-3} km

Investigating the Ideas

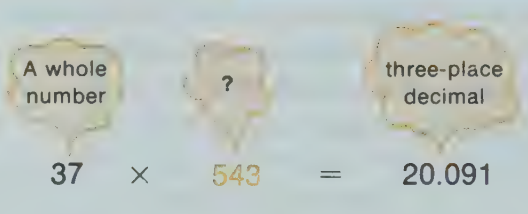
Only one of the quotients in each example below is correct.

A	B	C
$37\overline{)20.091}$	$23.5\overline{)1.1045}$	$0.68\overline{)0.017}$
$37\overline{)20.091}$	$23.5\overline{)1.1045}$	$0.68\overline{)0.017}$
$37\overline{)20.091}$	$23.5\overline{)1.1045}$	$0.68\overline{)0.017}$

Can you use multiplication to decide which is the correct quotient for each part?

Discussing the Ideas

1. How can you use the diagram at the right to help you place the decimal point correctly in 543 for example A above?



2. Explain how to place the decimal point in examples B and C.

3. When dividing by a whole number, the decimal point is placed directly above the decimal point in the dividend. Locate the decimal point in each division problem below.

EXAMPLE: $23\overline{)3.128}$ → $.136$

$23\overline{)3.128}$

23 (82)

69 (138)

138 (0)

A $12\overline{)43.44}$ B $35\overline{)0.08505}$ C $142\overline{)3.266}$

4. Study the example.

EXAMPLE: $1.2\overline{)4.32}$ →

Think: 10 × divisor. → 1.2

Think: 10 × dividend. → $\overline{)4.32}$

Work the problem: $12\overline{)43.2}$

$1.2\overline{)4.32}$ → $12\overline{)43.2}$

Replace each problem by a problem with a whole number divisor which has the same answer. Then divide.

A $0.7\overline{)29.47}$ B $1.1\overline{)0.0352}$ C $0.015\overline{)0.00225}$ D $21.5\overline{)156.95}$

1. Find the quotients.

A $72\overline{)273.6}$ **C** $53\overline{)3.339}$ **E** $23\overline{)0.0529}$ **G** $76\overline{)402.344}$

B $94\overline{)53.58}$ **D** $19\overline{)67.64}$ **F** $18\overline{)0.00270}$ **H** $519\overline{)384.06}$

2. Find the quotients.

A $0.06\overline{)2.142}$ **C** $0.004\overline{).0256}$ **E** $3.01\overline{)1.2341}$ **G** $0.417\overline{)0.7089}$

B $0.21\overline{)7.98}$ **D** $3.18\overline{).19716}$ **F** $9.9\overline{)76.23}$ **H** $0.049\overline{)0.001764}$

Solve the short story problems for Exercises 3 through 8.

3. Paper: 0.008 cm thick. How many sheets are there in a stack 1 cm high?

4. Quarter: about 0.15 cm thick. About how many quarters in a stack 6 cm high?

5. Distance around a tree is about 3.1 times the diameter of the tree. If the distance around the tree is 4.06 m, what is the diameter?

6. Mathematics book: 2.25 cm thick. How many books in a stack that is 432 cm high?

7. Speed: 6.25 metres per second. Distance: 400 metres. How long to run 400 metres?

8. A kangaroo leap is about 12.5 metres. How many leaps in one kilometre?

9. Select the correct quotient from the three choices.

A $^{-}0.4 \div 0.2 = ?$ $(^{-}2, ^{-}0.2, ^{-}0.02)$ **D** $^{-}0.12 \div ^{-}0.6 = ?$ $(2, 0.2, 0.02)$

B $1.5 \div ^{-}3 = ?$ $(^{-}5, ^{-}0.5, ^{-}0.05)$ **E** $0.035 \div .7 = ?$ $(0.05, 0.5, 5)$

C $^{-}42 \div 0.06 = ?$ $(^{-}7, ^{-}70, ^{-}700)$ **F** $2.4 \div 0.08 = ?$ $(0.3, 3, 30)$

think

Since $3^3 = 27$, the last digit of the number 3^3 is 7.
What is the last digit of each of these numbers?

1. $(3^3)^3 = 27^3$ **2.** $((3^3)^3)^3 = (27^3)^3$ **3.** $(((3^3)^3)^3)^3$

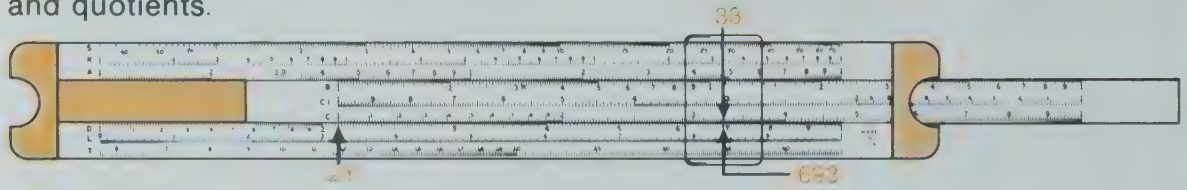
4. $(((3^3)^3)^3 \ldots)^3$, where there are 100 threes as exponents.

More practice, page S-18, Set 33

Estimating Products and Quotients

Investigating the Ideas

A **slide rule** can be used to find approximate products and quotients.



If you used a slide rule to find any of the six products below, you could read 693 for the product. You would then have to decide where to place the decimal point according to the factors involved.

Can you place the decimal point in 693 to show the product for each problem below?

A $2.1 \cdot 3.3$ **C** $0.021 \cdot 330$ **E** $0.21 \cdot 0.33$

B $0.21 \cdot 33$ **D** $21 \cdot 33$ **F** $2.1 \cdot 0.0033$

Discussing the Ideas

1. Explain how you decided to place the decimal point in each product.

2. It is often helpful to round each numeral to one significant digit in order to estimate answers.
   - **A** What is 2.1 rounded to one significant digit?
   - **B** What is 3.3 rounded to one significant digit?
   - **C** How can the product of the rounded factors help you place the decimal point in 693 correctly?

3. Explain how to locate the decimal point by estimation.

   A $7.42\overline{)818.93}$ (1104 . . .) **B** $0.0062\overline{)3.165}$ (5105 . . .) **C** $2.3\overline{)0.04352}$ (1892 . . .)

4. Scientific notation is helpful in estimation. For example, to estimate the product 587.3×79.54, round each number to one significant digit and write it in scientific notation as

   $$6 \times 10^2 \times 8 \times 10^1 = 48 \times 10^3.$$

   The product is about 48 000. Explain how to estimate the product 475.21×0.0397 in this way.

1. Round each number to one significant digit. Use scientific notation and estimate the products.

A 5846×6214 **D** 318.7×0.003187 **G** 0.06752×0.00225

B 691.3×87.65 **E** 0.0235×0.00829 **H** 41.689×0.000785

C 91.25×0.4863 **F** 0.00078×0.0192 **I** $0.00000692 \times 0.000406$

2. The correct digits are given in parentheses for each quotient. Using estimation, correctly locate the decimal point.

A $9.43 \div 4.1$ (23) **C** $8.0698 \div 2.57$ (314) **E** $0.153126 \div 0.047$ (3258)

B $16.544 \div 3.52$ (47) **D** $0.2856 \div 0.714$ (4) **F** $0.01517 \div 4.1$ (37)

3. Find the quotients.

A $10^3 \div 10^0$ **D** $(6 \times 10^2) \div 10^{-1}$ **G** $(8 \times 10^0) \div (2 \times 10^3)$

B $10^2 \div 10^{-2}$ **E** $(3 \times 10^4) \div 10^{-4}$ **H** $(6 \times 10^{-2}) \div (3 \times 10^2)$

C $10^{-3} \div 10^{-1}$ **F** $(4 \times 10^{-7}) \div 10^{-7}$ **I** $(8 \times 10^{-3}) \div (4 \times 10^{-5})$

4. Round each number to one significant digit. Use scientific notation and estimate the quotient.

A $597.2 \div 2.964$ **D** $0.6354 \div 215.83$ **G** $16.42 \div 0.000243$

B $79.62 \div 0.03982$ **E** $0.00396 \div 0.00000207$ **H** $0.08099 \div 0.03702$

C $0.8743 \div 0.0295$ **F** $0.000053 \div 987.65$ **I** $0.08909 \div 0.000251$

5. Study the example and then estimate the reciprocal of each number below.

EXAMPLE: Estimate the reciprocal of 809.4.

SOLUTION: The reciprocal is $\frac{1}{809.4} \approx \frac{1}{800} = \frac{1}{8 \times 10^2} = \frac{1}{8} \times 10^{-2}$

$= 0.125 \times 10^{-2} = 0.00125$

The reciprocal of 809.4 is about 0.0012

A 489.32 **B** 0.00197 **C** 0.07216 **D** 3.765

6. Use scientific notation to estimate each of the powers.

A $(0.00207)^3$ **C** $(402.6)^3$ **E** $(0.019832)^5$

B $(3012.876)^4$ **D** $(0.000178)^4$ **F** 2^{100} (HINT: $2^{10} = 1024$)

7. Use scientific notation, round each number to one significant digit, and estimate the answer. $\dfrac{(0.021)^2 \times 59.61 \times 0.000713}{694.35 \times (0.00001956)^3}$

More practice, page S-18, Set 34

Investigating the Ideas

When we use decimals to represent rational numbers, we often round to the nearest tenth, hundredth, thousandth, and so forth. If we divide numerator by denominator to get decimal approximations for the fraction $\frac{47}{82}$, we have

A

$$\begin{array}{r} 0.5 \\ 82\overline{)47.0} \\ 41\,0 \\ \hline 6\,0 \end{array}$$

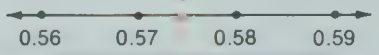
60 is more than half of 82. To the nearest **tenth**, $\frac{47}{82} \approx 0.6$.

Usual scale

B

$$\begin{array}{r} 0.57 \\ 82\overline{)47.00} \\ 41\,0 \\ \hline 6\,00 \\ 5\,74 \\ \hline 26 \end{array}$$

26 is less than half of 82. To the nearest **hundredth**, $\frac{47}{82} \approx 0.57$.

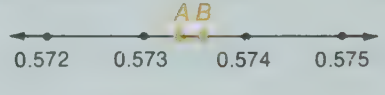


Magnified 10 times

Can you use division to find the decimal approximation of $\frac{47}{82}$ to the nearest **thousandth** and **ten thousandth**?

Discussing the Ideas

1. Which point on the number line, **A** or **B**, is nearer to the point for $\frac{47}{82}$?



2. **A** Which point, **C** or **D**, is nearer to the point for $\frac{47}{82}$?

B What is $\frac{47}{82}$ to the nearest ten thousandth?

3. What is the decimal for $\frac{1}{3}$ to the nearest

A tenth? **C** thousandth?

B hundredth? **D** ten thousandth?

4. The number 4.35 is midway between 4.3 and 4.4. In such cases, we agree always to "round up." We would round 4.35 to 4.4. Round each of the following numbers to the nearest hundredth.

A 4.735 **B** 3.655 **C** 7.955 **D** 15.455 **E** 0.809

1. Use the following division problems to round each of the numbers $\frac{10}{23}$, $\frac{23}{37}$, and $\frac{47}{31}$ to the nearest

A tenth:

$$\begin{array}{r} 0.434 \\ 23\overline{)10.000} \\ 92 \\ \hline 80 \\ 69 \\ \hline 110 \\ 92 \\ \hline 18 \end{array}$$

B hundredth:

$$\begin{array}{r} 0.621 \\ 37\overline{)23.000} \\ 222 \\ \hline 80 \\ 74 \\ \hline 60 \\ 37 \\ \hline 23 \end{array}$$

C thousandth:

$$\begin{array}{r} 1.516 \\ 31\overline{)47.000} \\ 31 \\ \hline 160 \\ 155 \\ \hline 50 \\ 31 \\ \hline 190 \\ 186 \\ \hline 4 \end{array}$$

2. Find a decimal approximation of each number to the nearest hundredth.

A $\frac{5}{6}$ **B** $\frac{13}{16}$ **C** $\frac{3}{7}$ **D** $\frac{11}{15}$ **E** $\frac{35}{64}$

3. Find a decimal approximation of each number to the nearest thousandth.

A $\frac{5}{10}$ **B** $\frac{7}{9}$ **C** $\frac{7}{12}$ **D** $\frac{65}{99}$ **E** $\frac{1455}{10\,000}$

4. Four league-leading teams have records as shown. Use decimals rounded to the nearest thousandth to find what part of the games played each team has won.

Team	Games won	Games played
A	34	56
B	35	60
C	32	55
D	36	57

5. **A** Which team in Exercise 4 has the best record?

B Which team has the worst record?

6. In football, a tie game is sometimes counted as $\frac{1}{2}$ game *won*. A team which has won one game, lost one game, and tied one game, has won $1\frac{1}{2}$ games out of three games. As a decimal, $1.5 \div 3 = 0.500$. Find the decimal (to the nearest thousandth) for each football team record shown in the table.

Team	Won	Lost	Tied	Decimal
A	6	2	0	
B	5	2	1	
C	3	3	2	
D	3	4	1	
E	2	4	2	

More practice, page S-19, Set 35

Discussing the Ideas

1. Since there is no ordinary decimal for the rational number $\frac{1}{3}$, we can represent the number by combining fraction and decimal notation.

$$\frac{1}{3} = \frac{10}{30} = \frac{9}{30} + \frac{1}{30} = 0.3 + \left(\frac{1}{3} \times \frac{1}{10}\right) = 0.3\frac{1}{3}$$

$$\frac{1}{3} = \frac{100}{300} = \frac{99}{300} + \frac{1}{300} = 0.33 + \left(\frac{1}{3} \times \frac{1}{100}\right) = 0.33\frac{1}{3}$$

$$\frac{1}{3} = \frac{1000}{3000} = \frac{999}{3000} + \frac{1}{3000} = 0.333 + \left(\frac{1}{3} \times \frac{1}{1000}\right) = 0.333\frac{1}{3}$$

Symbols like $0.3\frac{1}{3}$ are called **mixed decimal numerals**.
In the example above, explain why $\frac{9}{30} = 0.3$, and $\frac{99}{300} = 0.33$.

2. Give the correct numbers for ***a***, ***b***, ***c***, and ***d***.
Then give the mixed decimal numeral for ***n***.

A $\frac{2}{3} = \frac{a}{30} = \frac{b}{30} + \frac{2}{30} = \frac{c}{10} + \left(\frac{d}{3} \times \frac{1}{10}\right) = n$

B $\frac{2}{3} = \frac{a}{300} = \frac{b}{300} + \frac{2}{300} = \frac{c}{100} + \left(\frac{d}{3} \times \frac{1}{100}\right) = n$

C $\frac{2}{3} = \frac{a}{3000} = \frac{b}{3000} + \frac{2}{3000} = \frac{c}{1000} + \left(\frac{d}{3} \times \frac{1}{1000}\right) = n$

3. Read $0.3\frac{1}{3}$ as, "three and one third *tenths*," not "three and one third hundredths"
Read each mixed decimal numeral.

A $0.2\frac{1}{2}$ B $0.16\frac{2}{3}$ C $0.062\frac{1}{4}$ D $0.709\frac{1}{3}$ E $0.0020\frac{2}{7}$

4. Explain these two ways for finding a fraction for a mixed decimal numeral.

A $0.3\frac{1}{3} = \frac{3\frac{1}{3}}{10} = \frac{10}{3} \div 10 = \frac{1}{3}$

B $0.3\frac{1}{3} = \frac{3\frac{1}{3}}{10} = \frac{3\frac{1}{3} \times 3}{10 \times 3} = \frac{10}{30} = \frac{1}{3}$

5. Explain these two ways for finding a mixed decimal numeral.

A $\frac{1}{7} = \frac{100}{700} = \frac{14\frac{2}{7}}{100} = 0.14\frac{2}{7}$

B $\frac{1}{7} \rightarrow 7\overline{)1.00} \rightarrow 0.14\frac{2}{7}$

$$\begin{array}{r} 0.14 \\ 7\overline{)1.00} \\ \underline{7} \\ 30 \\ \underline{28} \\ 2 \end{array}$$

6. Study the division and then give the missing fraction in each mixed decimal numeral.

$$\begin{array}{r} 0.363 \\ 11\overline{)4.000} \\ \underline{33} \\ 70 \\ \underline{66} \\ 40 \\ \underline{33} \\ 7 \end{array}$$

tenths: $\frac{4}{11} = 0.3\frac{7}{11}$ ⟵ 70

hundredths: $= 0.36\frac{\text{IIIIIII}}{\text{IIIIIII}}$ ⟵ 40

thousandths: $= 0.363\frac{\text{IIIIIII}}{\text{IIIIIII}}$ ⟵ 7

7. What mixed decimal numeral is named in each part?

A One and two thirds **tenths** B Twelve and one half **hundredths**

C One hundred forty-two and six sevenths **thousandths**

1. Write a mixed decimal numeral for the number in each sentence.

A The sales tax rate in San Francisco is six and one fourth hundredths.

B Some credit card companies charge interest rates of one and one half hundredths interest on unpaid balances per month.

C About thirty-four and three tenths hundredths of the people in North America are under 18 years of age.

2. By dividing numerator by denominator, find the mixed decimal numeral in tenths, hundredths, and thousandths.

A $\frac{7}{12}$ **B** $\frac{5}{6}$ **C** $\frac{4}{3}$ **D** $\frac{15}{16}$ **E** $\frac{8}{37}$ **F** $\frac{5}{27}$

3. Find the fraction for f.

A $0.1\frac{1}{4} = \frac{1\frac{1}{4}}{10} = \frac{5}{4} \div 10 = f$

B $0.12\frac{1}{2} = \frac{12\frac{1}{2}}{100} = \frac{25}{2} \div 100 = f$

C $0.08\frac{1}{3} = \frac{8\frac{1}{3}}{100} = 8\frac{1}{3} \div 100 = f$

D $0.3\frac{3}{4} = \frac{3\frac{3}{4}}{10} = \frac{3\frac{3}{4} \times 4}{10 \times 4} = f$

E $0.62\frac{1}{2} = \frac{62\frac{1}{2}}{100} = \frac{62\frac{1}{2} \times 2}{100 \times 2} = f$

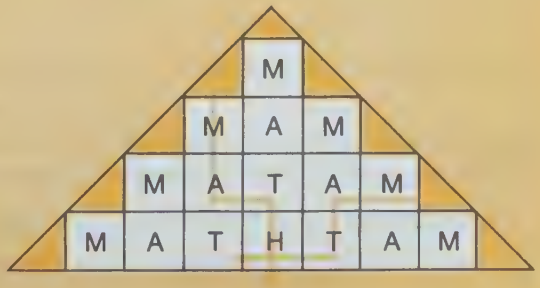
F $0.4\frac{1}{6} = \frac{4\frac{1}{6}}{10} = \frac{4\frac{1}{6} \times 6}{10 \times 6} = f$

4. The number $\frac{5}{32}$ can be written $0.1\frac{9}{16}$, $0.15\frac{5}{8}$, $0.156\frac{1}{4}$, $0.1562\frac{1}{2}$, and 0.15625. Give corresponding decimals for these.

A $\frac{15}{64}$ **B** $\frac{224}{625}$ **C** $\frac{3}{16}$ **D** $\frac{77}{125}$ **E** $\frac{39}{40}$ **F** $\frac{39}{80}$

think

The colored arrows show two paths you can follow in order to spell the word **MATH**. Find the **total** number of such paths.



Investigating the Ideas

Every rational number has a repeating decimal representation.
(The repeating part is sometimes all zeros.)

$\frac{2}{3} \rightarrow 3\overline{)2.000}$ $0.666\ldots$

$\begin{array}{r} 18 \\ \hline 20 \\ 18 \\ \hline 20 \\ 18 \\ \hline 2 \\ \vdots \end{array}$

$\frac{5}{8} \rightarrow 8\overline{)5.0000}$ $0.625000\ldots$

$\begin{array}{r} 48 \\ \hline 20 \\ 16 \\ \hline 40 \\ 40 \\ \hline 00 \\ 0 \\ \hline 0 \\ \vdots \end{array}$

$\frac{7}{11} \rightarrow 11\overline{)7.0000}$ $0.6363\ldots$

$\begin{array}{r} 66 \\ \hline 40 \\ 33 \\ \hline 70 \\ 66 \\ \hline 40 \\ 33 \\ \hline 7 \\ \vdots \end{array}$

The three dots " . . . " means that the pattern indicated continues indefinitely.

Can you find a repeating decimal for each of these?

A $\frac{5}{6}$ **B** $\frac{9}{25}$ **C** $\frac{5}{9}$ **D** $\frac{3}{7}$

Discussing the Ideas

Given number and its decimal	Repeating decimal notation
$\frac{2}{3} = 0.6666\ldots$	$= 0.\overline{6}$
$\frac{7}{11} = 0.636363\ldots$	$= 0.\overline{63}$
$\frac{5}{27} = 0.185185\ldots$	$=$
$\frac{123}{90} = 1.366666\ldots$	$=$

1. Instead of the dot notation, a bar is often used to indicate the repeating part of a decimal. What would be the bar notation for $\frac{5}{27}$ and $\frac{123}{90}$ in the table?

2. First select the correct decimal for n. Then explain why each of the other decimals is incorrect.
   A $0.\overline{17} = n$ (0.17777 . . . , 0.171111 . . . , 0.171717 . . .)
   B $0.\overline{254} = n$ (0.2545454 . . . , 0.254254 . . . , 0.254444 . . .)

3. Repeating decimals which repeats zeros, from some point on, are called **terminating decimals**.
   EXAMPLES: $\frac{7}{16} = 0.4375000\ldots = 0.4375\overline{0}$ is a terminating decimal.
   $\frac{11}{40} = 0.275000\ldots = 0.275\overline{0}$ is a terminating decimal.

   Can you decide, without dividing, whether or not a fraction will have a terminating decimal representation?

1. Write the repeating decimal notation for each decimal.

A 0.3333. . .	**D** 0.324324 . . .	**G** 0.024024 . . .	**J** 0.507507 . . .
B 0.7777 . . .	**E** 0.565656 . . .	**H** 0.37153715 . . .	**K** 0.626262 . . .
C 0.151515 . . .	**F** 0.9999 . . .	**I** 0.04260426 . . .	**L** 0.111111 . . .

2. For some rational numbers, the repeating part does not immediately follow the decimal.

EXAMPLE: $3.2465252\ldots = 3.246\overline{52}$

Write the repeating decimal notation for each decimal.

A 5.2636363 . . .	**D** 526.1444 . . .	**G** 56.661515 . . .
B 0.74444 . . .	**E** 73.4666 . . .	**H** 0.043212121 . . .
C 3.282828 . . .	**F** 77.77131313 . . .	**I** 0.762436243 . . .

3. Give the repeating decimal for each fraction.

A $\frac{1}{3}$	**C** $\frac{1}{9}$	**E** $\frac{8}{9}$	**G** $\frac{3}{11}$	**I** $\frac{1}{6}$	**K** $\frac{2}{99}$	**M** $\frac{1}{7}$
B $\frac{2}{3}$	**D** $\frac{3}{9}$	**F** $\frac{1}{11}$	**H** $\frac{9}{11}$	**J** $\frac{1}{99}$	**L** $\frac{71}{99}$	**N** $\frac{4}{7}$

4. The number of digits in the repeating block is called the **period** of a repeating decimal. The period of $1.42\overline{3172}$ is 4.
Give the period of each repeating decimal in Exercise 3.

5. Give the fraction or decimal for n and a.

A If $\frac{1}{9} = 0.\overline{1}$, then $\frac{2}{9} = n$.	**F** If $\frac{1}{999} = 0.\overline{001}$, then $\frac{285}{999} = n$.
B If $\frac{1}{9} = 0.\overline{1}$, then $n = 0.\overline{7}$.	**G** If $a = 0.\overline{12}$, then $2 \times a = n$.
C If $\frac{5}{11} = 0.\overline{45}$, then $\frac{1}{11} = n$.	**H** If $a = 0.\overline{04}$, then $n \times a = 0.\overline{36}$.
D If $\frac{5}{11} = 0.\overline{45}$, then $n = 0.\overline{90}$.	**I** If $a = 0.3\overline{12}$, then $5 \times a = n$.
E If $\frac{1}{99} = 0.\overline{01}$, then $\frac{43}{99} = n$.	**J** If $a = 0.\overline{473}$, then $n \times a = 0.\overline{946}$.

6. Using the facts that $\frac{1}{9} = 0.1111\ldots$, $\frac{1}{99} = 0.010101\ldots$, and $\frac{1}{999} = 0.001001\ldots$, you can show how every repeating decimal can be written as a fraction. Study the examples and then work the problems.

A $0.\overline{4} = 4 \times (0.111\ldots) = 4 \times \frac{1}{9} = \frac{4}{9}$

B $0.\overline{27} = 27 \times (0.0101\ldots) = 27 \times \frac{1}{99} = \frac{27}{99} = \frac{3}{11}$

C $0.3\overline{27} = .3 + 2.7 \times (0.0101\ldots) = \frac{3}{10} + \frac{2.7}{99} = \frac{3}{10} + \frac{27}{990} = \frac{324}{990} = \frac{18}{55}$

A $0.\overline{3} =$	**C** $0.\overline{28} =$	**E** $0.\overline{813} =$	**G** $0.70\overline{34} =$
B $0.\overline{71} =$	**D** $0.9\overline{4} =$	**F** $0.64\overline{1} =$	**H** $0.30\overline{6} =$

COMPUTING WITH REPEATING DECIMALS

By working from left to right, instead of the usual way, you can do some computing using repeating decimals.

1. Find the sums. Carry out each sum until you get a repeating decimal.

A
$$\begin{array}{r} 0.1111\ldots \\ +0.5555\ldots \\ \hline \end{array}$$

B
$$\begin{array}{r} 0.2323\ldots \\ +0.6464\ldots \\ \hline \end{array}$$

C
$$\begin{array}{r} 1.010101\ldots \\ +0.892892\ldots \\ \hline \end{array}$$

2. Find the differences.

A
$$\begin{array}{r} 0.7777\ldots \\ -0.3333\ldots \\ \hline \end{array}$$

B
$$\begin{array}{r} 0.6262\ldots \\ -0.1313\ldots \\ \hline \end{array}$$

C
$$\begin{array}{r} 0.101010\ldots \\ -0.090909\ldots \\ \hline \end{array}$$

3. Find the products.

A
$$\begin{array}{r} 0.333\ldots \\ \times 2 \\ \hline \end{array}$$

B
$$\begin{array}{r} 0.161616\ldots \\ \times 2 \\ \hline \end{array}$$

C
$$\begin{array}{r} 0.666\ldots \\ \times 2 \\ \hline \end{array}$$

★ **4.** In the example, the remainders are given in color. The division has not yet yielded the repeating decimal for $\frac{1}{17}$. Explain why the period of the repeating decimal cannot be as great as 20.

EXAMPLE:
$$\begin{array}{r} 0.05882 \\ 17\overline{)1.00000\ldots} \\ 85 \\ \hline 150 \\ 136 \\ \hline 140 \\ 136 \\ \hline 40 \\ 34 \\ \hline 60 \\ \vdots \end{array}$$

★ **5.** Each example illustrates an interesting fact about repeating decimals. Copy each example and complete the computation by adding the decimals in Example A and by multiplying the decimals in Example B.

A
$$\begin{array}{r} \frac{2}{3} = 0.6666\ldots \\ +\frac{1}{3} = 0.3333\ldots \\ \hline 1 = 0.\blacksquare \end{array}$$

B
$$\begin{array}{r} \frac{1}{9} = 0.11111\ldots \\ \times 9 = 9 \\ \hline 1 = 0.\blacksquare \end{array}$$

Strange as it may seem, these examples show that the repeating decimal $0.999\ldots$ must be another name for 1.

think

The example shows how to multiply $0.\overline{2}$ by $0.\overline{3}$. Copy and continue the multiplication until you are sure that the product is a repeating decimal.

$0.\overline{2} = \frac{2}{9}$ and $0.\overline{3} = \frac{1}{3}$ $\quad \frac{2}{9} \cdot \frac{1}{3} = \frac{2}{27}$.

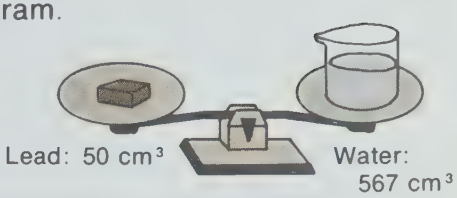
Find the repeating decimal for $\frac{2}{27}$ and check your multiplication.

$0.\overline{2}$	$0.22222\ldots$
$0.\overline{3}$	$\times 0.33333\ldots$
$0.3 \times 0.\overline{2}$	$0.06666\ldots$
$0.03 \times 0.\overline{2}$	$+0.00666\ldots$
	$0.07333\ldots$
$0.003 \times 0.\overline{2}$	$+0.00066\ldots$
	$0.07399\ldots$
$\vdots$	$\vdots$

Specific Gravity

The specific gravity of a substance is found by dividing its mass by the mass of an equal volume of water. The mass of one cubic centimetre (cm^3) of water is one gram. Thus, one litre of water has a mass of one kilogram. Lead has a specific gravity of 11.34.

Thus, the mass of 1 cm^3 of lead is 11.34 grams. The mass of 50 cm^3 of lead is $50 \cdot 11.34$ grams $= 567.0$ grams.



Element	Symbol	Specific gravity
Aluminum	Al	2.70
Carbon	C	3.52
Copper	Cu	8.92
Gold	Au	19.3
Helium	He	0.00018
Hydrogen	H	0.00009
Iron	Fe	7.86
Lead	Pb	11.34

Element	Symbol	Specific gravity
Mercury	Hg	13.55
Nitrogen	N	0.00125
Oxygen	O	0.001429
Platinum	Pt	21.37
Potassium	K	0.87
Silver	Ag	10.5
Tin	Sn	7.3
Uranium	U	18.7

1. Find the mass of the given volume of each element.

- **A** 1000 cm^3 of Au
- **B** 10 cm^3 of C
- **C** 76 cm^3 of Ag
- **D** 10 cm^3 of Fe
- **E** 23 cm^3 of Hg
- **F** 100 000 litres of H
- **G** 500 000 litres of He
- **H** 800 litres of N

2. Air is composed mostly of nitrogen and oxygen. It has a specific gravity of about 0.0013.

- **A** What is the mass of one litre of air?
- **B** What is the mass of one cubic metre of air?

3. A classroom has a volume of 384 cubic metres (m^3). What is the mass of the air in the classroom?

4. How many litres of water does it take to have a mass equal to the mass of one litre of mercury?

5. If you found a nugget of pure gold with a mass of one kilogram, what would be the volume of the nugget to the nearest tenth of a cubic centimetre?

REVIEWING THE IDEAS

1. Express each numeral as a power of ten.

A 10 000 **B** $\frac{1}{10}$ **C** $\frac{1}{1000}$

2. Give the usual numeral for $8 \cdot 10^3 + 4 \cdot 10^2 + 5 \cdot 10^{-1} + 6 \cdot 10^{-2}$.

3. Express 563.081 in expanded notation.

4. In the decimal 495.876,

A the 4 tells how many ? .

B the 7 tells how many ? .

5. Find the sums.

A $12.1 + 5.73 + 8.15$

B $160 + 0.376 + 13.5$

C $0.15 + 0.015 + 0.0015$

6. Find the differences.

A $8.59 - 4.87$ **C** $75 - 69.53$

B $6.1 - 1.39$ **D** $100 - 3.079$

7. Find the sum.

$5.73 + {}^-6.87 + 10.9 + {}^-0.355$

8. Find the products.

A 0.01×875 **D** 4.3×0.005

B 1000×4.08 **E** 0.12×1.3

C 0.5×19.43 **F** 95.1×2.1

9. Give the usual numeral for each.

A 9×10^5 **C** 5.3×10^{-2}

B 6.3×10^3 **D** 3.88×10^{-6}

10. Express each number in scientific notation.

A 2500 **C** 0.059

B 45 000 **D** 0.00031

11. Round 385 700 to two significant digits.

12. What is the greatest possible error in the measurement 3.79×10^5 km?

13. Use the fact $224 \div 16 = 14$ to give your answer.

A $2.24 \div 16$ **D** $0.0224 \div 160$

B $2.24 \div 0.16$ **E** $0.00224 \div 0.016$

C $22.4 \div 1.6$ **F** $2240 \div 0.016$

14. Find the quotients.

A $18\overline{)4.5}$ **C** $0.013\overline{)7.28}$

B $0.6\overline{)0.75}$ **D** $5.3\overline{)^-48.23}$

15. Round each number to one significant digit. Then estimate each product and quotient.

A 5795×78 **C** $63\,955 \times 795$

B 32×0.069 **D** 2005×0.019

16. Find a decimal approximation for each number to the nearest thousandth.

A $\frac{5}{16}$ **B** $\frac{4}{15}$ **C** $\frac{8}{9}$

17. Find a mixed decimal numeral for $\frac{7}{11}$ to the nearest hundredth.

18. Give the repeating decimal for each.

A 0.757575 . . . **B** 0.304304 . . .

19. Give the repeating decimal for each fraction.

A $\frac{5}{6}$ **B** $\frac{5}{9}$ **C** $\frac{2}{7}$

TEST YOURSELF

1. Find the exponent n in each equation
   - A $100 = 10^n$
   - B $10\,000 = 10^n$
   - C $\frac{1}{10} = 10^n$
   - D $0.01 = 10^n$

2. In the decimal 2.7134,
   - A the 7 tells the number of ?.
   - B the 1 tells the number of ?.
   - C the ? means 3×10^{-3}.

3. Find the sum or difference.
   - A $6.24 + 18.1 + 11.729$
   - B $126.4 - 92.48$

4. Find the products.
   - A 1000×0.0628
   - B 0.01×346.4
   - C 19.43×0.08

5. Round 0.007439 to two significant digits and express in scientific notation.

6. The average distance from Earth to the moon is 3.83×10^5 km. What is the greatest possible error in this measurement?

7. Find the quotients.
   - A $0.18\overline{)0.702}$
   - B $168\overline{)0.7896}$

8. Find a decimal approximation for each number to the nearest thousandth.
   - A $\frac{7}{16}$
   - B $\frac{29}{48}$

9. Find a repeating decimal for each rational number.
   - A $\frac{5}{9}$
   - B $\frac{7}{12}$
   - C $\frac{1}{2}$

10. Which number is the best estimate for the quotient $12.49 \div 0.00618$?
    - A 2
    - B 20
    - C 200
    - D 2000

RESEARCH PROJECTS

A Simon Stevin (1548–1620) is credited with inventing decimal notation for rational numbers. Stevin's notation was somewhat different than our modern notation. Find out about Stevin's decimal symbols and report on your research to the class. (See *A History of Mathematics* by Carl B. Boyer; New York: John Wiley and Sons, 1968, pp. 347–350)

B Find out how to use a slide rule to multiply and divide. Demonstrate and explain to your classmates how it is used.

C Many physical constants that occur in scientific work are either very large or very small. Therefore, they are often expressed in scientific notation. Find some reference books in chemistry or physics and make a list of some of these constants. For example, find the speed of light, the length of a light year, Avogadro's number, the mass of the Earth, the size of a molecule or an atom, and so on. Make a display showing these magnitudes expressed in scientific notation.

A Numeration System with Base $\frac{1}{3}$

Suppose we agree to write numerals with base $\frac{1}{3}$.
The basic symbols that we shall agree to use
(as we must do in any base) for base $\frac{1}{3}$ are {0, 1, 2}.

EXAMPLES: $11_{(\frac{1}{3})}$ means $1 \cdot \left(\frac{1}{3}\right)^1 + 1 \cdot \left(\frac{1}{3}\right)^0 = \frac{2}{3} + 1 = 1\frac{2}{3} = 1.66\overline{6}_{(ten)}$

$212_{(\frac{1}{3})}$ means $2 \cdot \left(\frac{1}{3}\right)^2 + 1 \cdot \left(\frac{1}{3}\right)^1 + 2 \cdot \left(\frac{1}{3}\right)^0$

$= \frac{2}{9} + \frac{1}{3} + 2 = 2\frac{5}{9} = 2.\overline{5}_{(ten)}$

$20_{(\frac{1}{3})}$ means $2 \cdot \left(\frac{1}{3}\right)^{-1} = 2 \cdot 3^1 = 6_{(ten)}$

$212_{(\frac{1}{3})}$ means $1 \cdot \left(\frac{1}{3}\right)^{-1} + 2 \cdot \left(\frac{1}{3}\right)^{-2} = 1 \cdot (3) + 2 \cdot (3^2) = 3 + 18 = 21_{(ten)}$

$1001_{(\frac{1}{3})}$ means $1 \cdot \left(\frac{1}{3}\right)^{-1} + 0 \cdot \left(\frac{1}{3}\right)^{-2} + 1 \cdot \left(\frac{1}{3}\right)^{-3} = 1 \cdot (3) + 0 \cdot (3^2) + 1 \cdot (3^3)$

$= 3 + 0 + 27 = 30_{(ten)}$

Counting with base-$\frac{1}{3}$ numerals is interesting but quite awkward. The table gives the first few counting numbers in base ten and base $\frac{1}{3}$. Try these activities.

Base ten	Base $\frac{1}{3}$
1	1
2	2
3	0.1
4	1.1
5	2.1
6	0.2
7	1.2

1. Continue the base-$\frac{1}{3}$ numerals through $12_{(ten)}$.

2. Write your age in years as a numeral in base $\frac{1}{3}$.

3. Write some addition equations using base-$\frac{1}{3}$ numerals.
   Check them using base-ten numerals.
   base-$\frac{1}{3}$ → 0.1 + 1.1 = 1.2
   base ten → 3 + 4 = 7

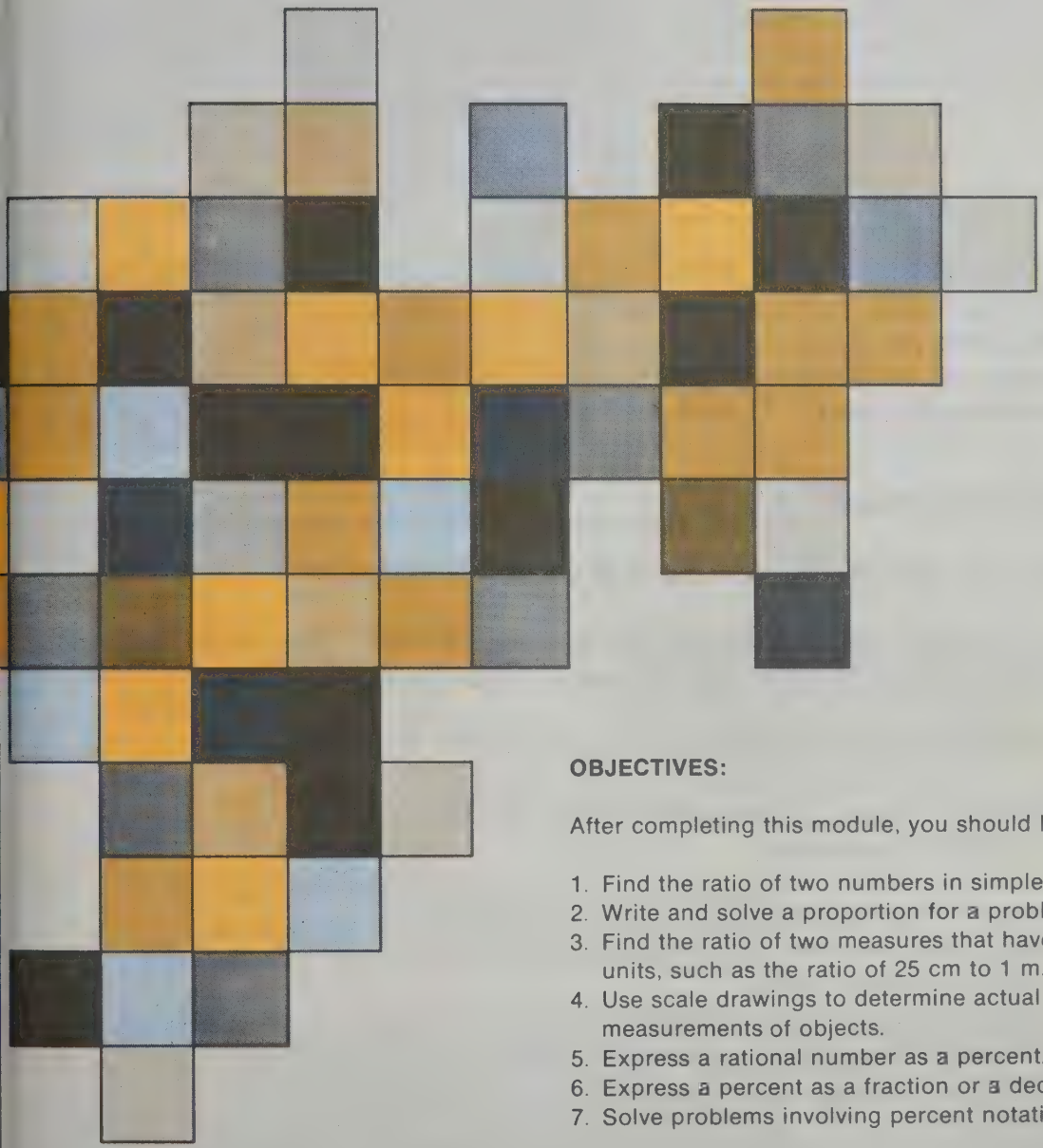
4. Is the base-$\frac{1}{3}$ multiplication correct? $2 \times 0.1 = 0.2$
   Check it using base-ten numerals.

TEST YOURSELF Answers

1. A 2; B 4; C $^{-}1$; D $^{-}2$ 2. A tenths; B ten thousandths; C 3
3. A 36.069; B 33.92 4. A 62.8; B 3.464; C 1.5544 5. 7.4×10^{-3}
6. 0.005×10^5 km or 500 km 7. A 3.9; B 0.0047 8. A 0.438;
   B 0.604 9. A $0.\overline{5}$; B $0.583\overline{3}$; C $0.5\overline{0}$ 10. D

UNIT F: *Rational Numbers*

MODULE 3: *Ratio and Percent*



OBJECTIVES:

After completing this module, you should be able to:

1. Find the ratio of two numbers in simplest form.
2. Write and solve a proportion for a problem.
3. Find the ratio of two measures that have different units, such as the ratio of 25 cm to 1 m.
4. Use scale drawings to determine actual measurements of objects.
5. Express a rational number as a percent.
6. Express a percent as a fraction or a decimal.
7. Solve problems involving percent notation.

Investigating the Ideas

Suppose you are chairman of the Food Committee for a class party. The committee has decided to buy the food shown on the list at the right.

Grocery list:

1 pizza for every 6 students
3 bottles of pop for every 2 students
5 doughnuts for every 3 students
2 bags of potatochips for every 10 students

Can you figure out how many of each item you would have to buy to feed all the students in your mathematics class?

Discussing the Ideas

1. Explain how you decided how much of each item to buy.

2. Suppose a class had 30 students. Explain how to find how many of each item you would need.

3. The **ratio** of pizzas to students is 1 to 6. This ratio can be denoted by the symbols 1 : 6 or $\frac{1}{6}$.
   Give the ratio of each of the other items on the list to the number of students.

4. Just as the fraction symbols $\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots$ are called equivalent fractions, the ratios 1 : 4, 2 : 8, 3 : 12, . . . are called **equivalent ratios**.
   What are three equivalent ratios that would compare
   - **A** the number of bottles of pop to the number of students?
   - **B** the number of doughnuts to the number of students?

5. The ratio of bags of potato chips to students is 2 : 10 or $\frac{2}{10}$. Since the lowest-terms fraction for $\frac{2}{10}$ is $\frac{1}{5}$, we say that the **simplest-terms ratio** for 2 : 10 is 1 : 5.

   What is the simplest-terms ratio for each of the following?

   A 6 : 8 **B** 15 to 12 **C** $\frac{6}{15}$ **D** 8 : 24

1. Give three more ratios for each set of equivalent ratios.

A {1 to 2, 2 to 4, 3 to 6, . . .}

B {3:2, 6:4, 9:6, . . .}

C {3 to 4, 6 to 8, 9 to 12, . . .}

D $\left\{\frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \ldots\right\}$

E {2:5, 4:10, 6:15, . . .}

F $\left\{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \ldots\right\}$

2. Find the number for n so that each ratio is equivalent to 3:5.

A n:10 **C** 12 to n **E** $\frac{30}{n}$ **G** n:25 **I** $\frac{36}{n}$

B 9:n **D** n to 15 **F** $\frac{n}{35}$ **H** 6 to n **J** n:30

3. Give the simplest-terms ratio for each of the following.

A 4:8 **B** 15:12 **C** 63:36 **D** 42:51 **E** 14:91

4. We can express the ratio of two numbers even though they are not whole numbers. Thus, the ratio $\frac{1}{2}$ to $\frac{3}{4}$ can be written as a ratio involving whole numbers:

$$\frac{1}{2} \text{ to } \frac{3}{4} = \frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \cdot \frac{4}{3} = \frac{4}{6} = \frac{2}{3}$$

Write each ratio as a ratio of whole numbers.

A $\frac{1}{2}$ to $\frac{2}{3}$ **B** $\frac{1}{9}$ to $\frac{1}{2}$ **C** $\frac{3}{10}$ to $\frac{1}{3}$ **D** $1\frac{1}{2}$ to $\frac{7}{4}$ **E** $2\frac{1}{4}$:5

5. Approximate each ratio with a decimal rounded to the nearest hundredth.

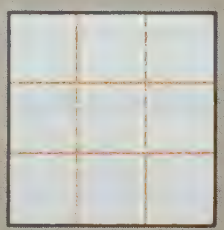
EXAMPLE: $\frac{7}{12}$ $\frac{7}{12}$

A $\frac{9}{14}$ **B** 15:23 **C** $\frac{7}{24}$ **D** 18:14 **E** 5 to 23

★ **6.** In all the basketball games for a season, Jack made 3 out of every 5 free throws that he tried. He missed a total of 26 free throws. How many did he make?

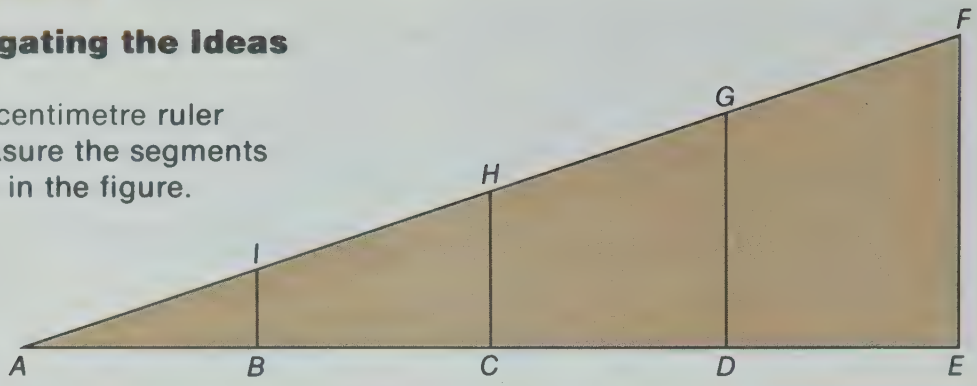
think

Trace the square. Shade the interior and cut it out. Now cut along the dashed lines. Fold along the lines that are not dashed to make a cube that has all faces shaded.



Investigating the Ideas

Use a centimetre ruler to measure the segments shown in the figure.



How many pairs of segments can you find in which the ratio of their lengths, in simplest terms, is 1:3?

Discussing the Ideas

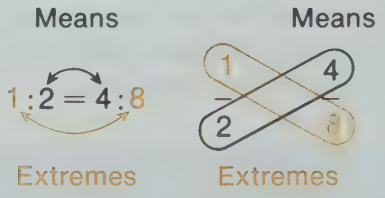
1. A What is the length of $\overline{EF}$? of $\overline{AE}$?
   B What is the ratio of the length $\overline{EF}$ to $\overline{AE}$?
   C What is the ratio in simplest terms?

2. An equation stating that two ratios are equivalent is called a **proportion**. Each of these is a proportion:

   $\frac{1}{3} = \frac{4}{12}$ $\qquad$ $1:3 = 4:12$ $\qquad$ 1 to 3 = 4 to 12

   What are some other ratios that could form a proportion when $\frac{1}{3}$ is one of the ratios?

3. In the proportion at the right, the "end" numbers (1 and 8) are called the **extremes**. The "middle" numbers (2 and 4) are called the **means**. What do you think is the relation between the means and the extremes in a proportion?



4. You can use the fact that

   In a proportion, the product of the means equals the product of the extremes.

   to solve problems involving proportions. Study and complete the example.

EXAMPLE:

$$\frac{2}{3} = \frac{8}{x}$$

$$2 \cdot x = 3 \cdot 8$$

$$x = 24 \div 2 =$$

1. In each proportion, show that the product of the means is equal to the product of the extremes.

A $1:3 = 3:9$ **C** $4:6 = 10:15$ **E** $3:5 = 60:100$

B $\frac{3}{5} = \frac{12}{20}$ **D** $\frac{7}{3} = \frac{28}{12}$ **F** $30:90 = 27:81$

2. Solve the proportions.

A $\frac{5}{6} = \frac{15}{s}$ **D** $4:6 = y:15$ **G** 10 to 20 = 5 to b **J** $\frac{14}{16} = \frac{x}{8}$

B $\frac{t}{40} = \frac{6}{8}$ **E** $d:12 = 12:16$ **H** c to 6 = 5 to 30 **K** $\frac{15}{r} = \frac{5}{8}$

C $\frac{4}{10} = \frac{w}{40}$ **F** $10:12 = n:6$ **I** 7 to 10 = h to 10 **L** $\frac{14}{16} = \frac{a}{24}$

3. Write and solve a proportion for each problem.

EXAMPLE: Two of every 5 students in class are boys. There are 14 boys in the class. How many students are in the class?

SOLUTION: Let x represent the number of students in the class.

$$\text{Boys} \rightarrow \frac{5}{\text{Students} \rightarrow x} = \frac{14}{x}$$

$$5 \cdot x = 5 \cdot 14$$

$$x = 35$$

A Oranges sell at 5 for 39 cents. How much do 15 oranges cost?

B In a group, the ratio of girls to boys is 3 to 4. If there are 15 girls, how many boys are there?

C On a map, 2 cm represents 50 km. What distance is represented by 7 cm?

D Two of every 100 people at a concert are over 80 years old. If there are 15 people over 80 years old, how many people are at the concert?

4. On a map, 0.7 cm represents 14 km. What distance would be represented by 1.3 cm?

★ **5.** Solve the proportions.

A $\frac{x}{1\frac{1}{2}} = \frac{3}{4}$ **B** $\frac{1}{x} = \frac{2\frac{1}{4}}{7}$ **C** $\frac{2}{3} = \frac{x}{7}$ **D** $\frac{2\frac{1}{2}}{3\frac{1}{4}} = \frac{6}{x}$

★ **6.** Solve the proportions.

A $\frac{2}{3} = \frac{x}{x+12}$ **B** $\frac{3}{4} = \frac{n}{28-n}$ **C** $\frac{s+9}{s} = \frac{5}{2}$ **D** $\frac{24-b}{b} = \frac{3}{5}$

More practice, page S-19, Set 36

Ratio and Units of Measure

It is often necessary to convert measurements to different units. For example, a speed in kilometres per hour (km/h) must be changed to metres per minute or, perhaps, a measurement in millilitres must be changed to litres. Ideas of ratio and proportion are helpful in such calculations.

Highest speed by man: 39 889 km/h by the Apollo X astronauts

Thus, $\frac{48\text{ km}}{1\text{ h}} = \frac{48\text{ km}}{60\text{ min}} = \frac{48\,000\text{ m}}{60\text{ min}} = \frac{800\text{ m}}{1\text{ min}} = 800$ metres per minute

and $780\text{ ml} = 780\text{ ml} \cdot \frac{1\,\ell}{1000\text{ ml}} = 0.780\,\ell$

1. Find the correct number for each ▒.

A 7 m = ▒ cm
B 7 g = ▒ mg
C 19 m = ▒ mm
D 180 min = ▒ h
E 48 h = ▒ day
F 3000 ℓ = ▒ kl
G ▒ cm = 32 mm
H 36 mo = ▒ yr
I ▒ mg = 20 g
J ▒ kg = 4000 g
K 2 m = ▒ cm
L ▒ m = 450 cm
M ▒ m = 0.9 km
N 0.3 m = ▒ cm
O ▒ mm = 30 cm
P 600 g = ▒ kg
Q ▒ cm = 2 m
R 7 h = ▒ s

2. Copy the equations and give the numbers for ***a*** and ***b***.

A $\frac{120\text{ km}}{1\text{ h}} = \frac{120\text{ km}}{a\text{ min}} = \frac{b\text{ km}}{1\text{ min}} = b$ km/min

B $\frac{1400\text{ km}}{1\text{ wk}} = \frac{1400\text{ km}}{a\text{ d}} = \frac{b\text{ km}}{1\text{ d}} = b$ km/d

C $\frac{3000\,\ell}{1\text{ min}} = \frac{3000\,\ell}{a\text{ s}} = \frac{b\,\ell}{1\text{ s}} = b\,\ell$/s

3. Sometimes it is helpful to give the ratio of two measurements without giving the units. Both measurements must first be expressed with a common unit.

Thus, $\frac{20\text{ cm}}{3\text{ m}} = \frac{20\text{ cm}}{300\text{ cm}} = \frac{20}{300} = \frac{1}{15}$,

and 12 min : 1 h 20 min = 12 min : 80 min = 12 : 80 = 3 : 20.
Express these ratios without units.

A $\frac{8\text{ mm}}{4\text{ cm}}$ **B** $\frac{25\text{ min}}{1\text{ h}}$ **C** $\frac{1\text{ cm}}{2\text{ km}}$ **D** $\frac{40\text{ s}}{1\text{ min }40\text{ s}}$ **E** $\frac{680\,\ell}{2\text{ kl}}$

SPORTS RATIOS

1. Jim Plunkett completed 158 passes in 328 attempts during one football season.
   - **A** What is the ratio of completed passes to attempts in simplest form?
   - **B** Find a decimal to the nearest thousandth for that ratio.

2. Billie Jean King won the Wimbledon Tennis Championship five times in eight years. Express this ratio as a decimal to the nearest thousandth.

3. George Blanda once completed 37 passes in 68 attempts in a single football game. Find a 3-place decimal to express the ratio of completed passes to attempts.

4. Phil Esposito's NHL scoring record: 152 points, 78 games. What is the ratio of points to games?

5. In 1972, Ken Dryden played 54 hockey games, conceding 119 goals. Express his goals-against average as a decimal of three significant digits.

6. In a certain year the Los Angeles Lakers lost only 13 games out of 82 games that they played. What is the ratio of the games won to the games played? the games lost to the games played?

7. The leading scorer in the NBA in a certain year was Kareem Abdul-Jabbar. In that year, he made 1159 field goals in 2019 attempts. Find the ratio of field goals to attempts as a decimal to the nearest thousandth.

8. Steve Carlton was the leading pitcher in the National League one year when he won 27 games and lost 10. What part of the games did he win?

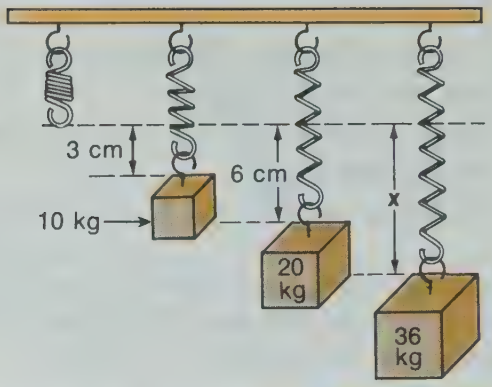
9. Rod Carew led the American League in hitting with 170 hits in 535 times at bat. Find the ratio of hits to times at bat as a decimal to the nearest thousandth.

10. The Toronto Maple Leafs hockey team had a record of 33 wins, 31 losses, and 14 ties for a season. What part of their games did they win? lose? tie?

Investigating the Ideas

Suppose that a 10-kilogram mass stretches a spring 3 centimetres. Then a 20-kilogram mass will stretch the spring twice as far, or 6 centimetres. The stretching of the spring is **proportional** to the masses:

that is, $\frac{3\text{ cm}}{6\text{ cm}} = \frac{10\text{ kg}}{20\text{ kg}}$.

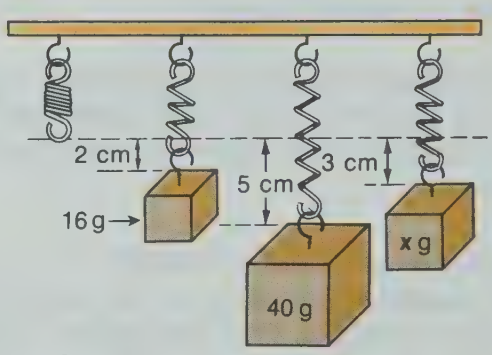


Can you find how far a 36-kilogram mass would stretch the same spring?

Discussing the Ideas

1. **A** What proportion involving distance **x** above can be written for the Investigation question?
   B Explain how the proportion can be solved for **x**.

2. **A** Is the ratio $\frac{2\text{ cm}}{5\text{ cm}}$ the same as the ratio $\frac{16\text{ g}}{40\text{ g}}$?



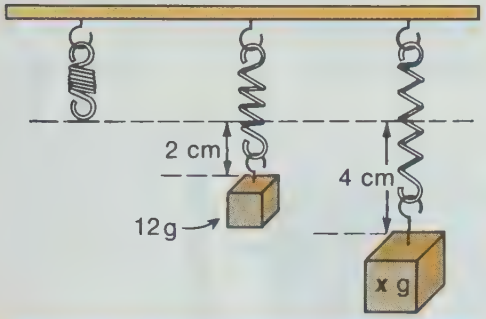
   B What proportion can you write to find the mass **x**?
   C Show how you would solve the proportion.

3. How is a **spring scale** like the one shown at the right related to the problems above?

4. If you were conducting an experiment with a spring and some masses, would you need to be careful about how large the masses are? Explain.

For Exercises 1 through 6, find the missing measure.

1.



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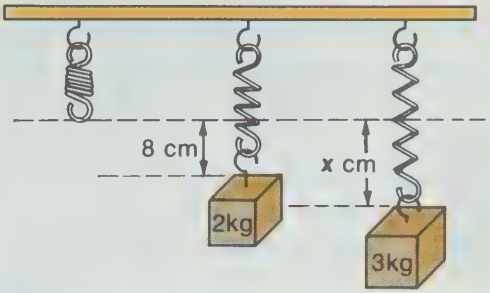
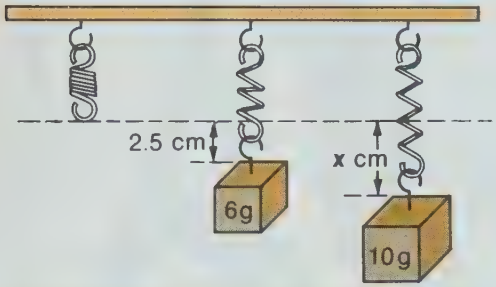


3.



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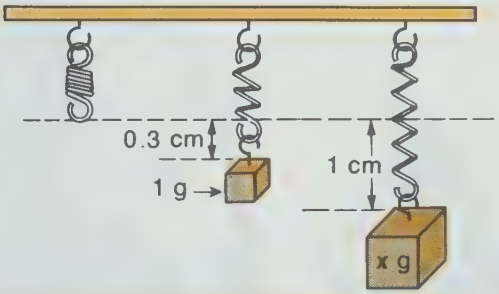
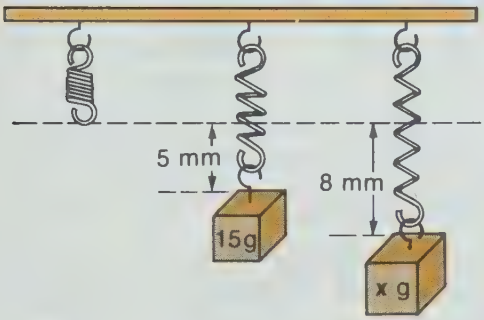
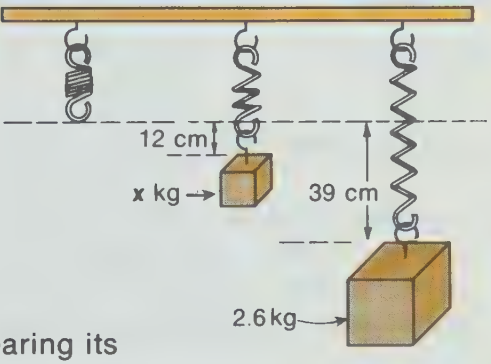


5.



6.

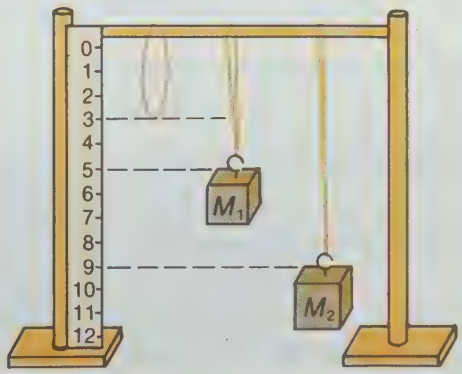


7. A certain spring will stretch 12 cm before nearing its breaking point. If a 2-kg mass stretches the spring 5 mm, how much will the spring be stretched by

   A 10 kg? C 40 kg?

   B 1 kg? D 12 kg?

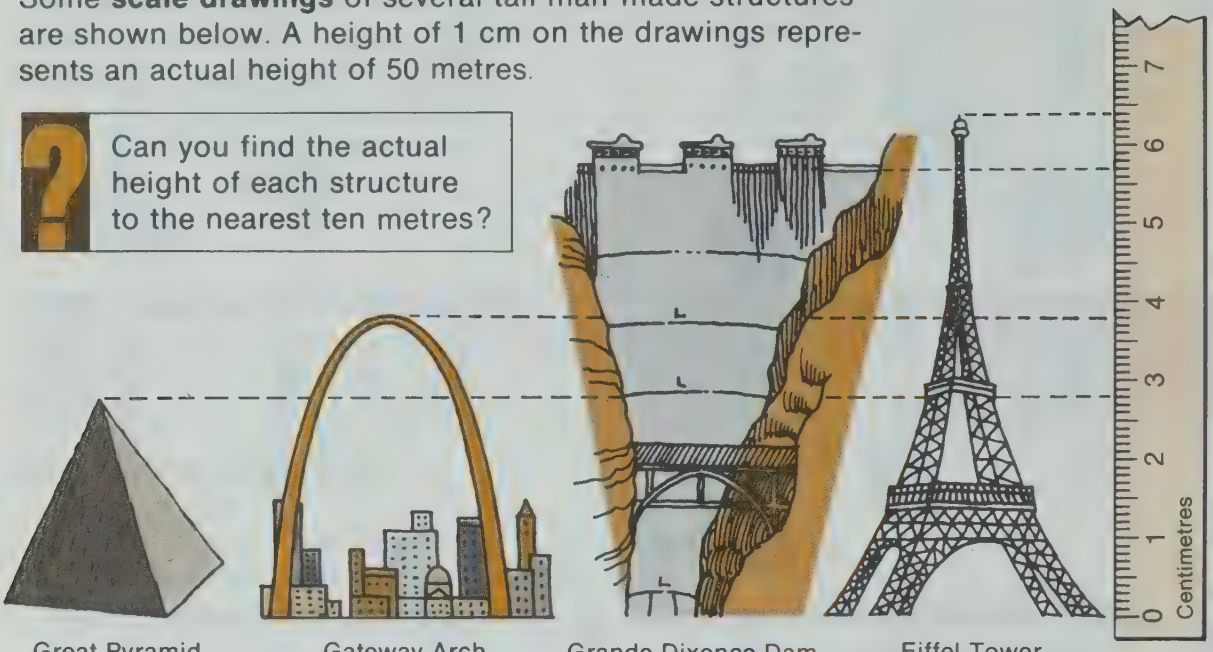
★ 8. Two masses M_1 and M_2 were suspended on the same rubber band as shown in the diagram. How many times greater is the mass M_2 than M_1?



Investigating the Ideas

Some **scale drawings** of several tall man-made structures are shown below. A height of 1 cm on the drawings represents an actual height of 50 metres.

Can you find the actual height of each structure to the nearest ten metres?



Great Pyramid of Egypt

Gateway Arch St. Louis

Grande Dixence Dam Switzerland

Eiffel Tower Paris

Discussing the Ideas

1. Pictures of large or small objects are often drawn to scale. The scale of a drawing is the ratio of the distance measured on the drawing to the corresponding actual distance. What is the scale for the drawings in the Investigation?

2. A scale for a drawing might be indicated in several ways. Explain how each way expresses the same scale.

   A 1:5000 **B** 1 cm = 50 metres **C**

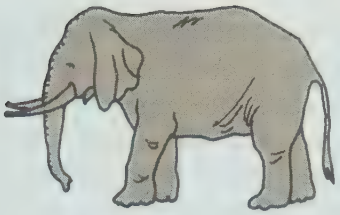
Metres

3. The drawing of the ant is four times as large as an actual ant. Is the scale for the drawing 1:4 or 4:1?

4. The drawing of the ant is about 24 mm long. Explain how to find the actual length of the ant.

5. A scale on a map is 1 cm = 80 km. What is the actual distance between cities whose distance apart on the map measures 2.8 cm?

1. Measure the height of the elephant drawing with a centimetre ruler. What is the actual height of the elephant?



SCALE: 1 cm = 1.2 m

2. Measure the length of the elephant from the tip of his tusks to his tail. Find the actual length of the elephant in metres

3. Use the scale drawing to estimate the Statue of Liberty's height in metres. The scale is 1 : 1000.

Statue of Liberty

4. A Estimate the diameter of a white blood cell in millimetres.

B Estimate the diameter of a red blood cell in millimetres.

SCALE: 1000 to 1

5. On a map, the distance between Paris and Rome measures 4.4 centimetres. The airline distance between these two cities is about 1100 kilometres. What is the scale for this map?

6. Suppose you drew a map of your province using the scale 1 cm = 100 km.

A About how long would the north-south distance be on your drawing?

B About how long would the east-west distance be?

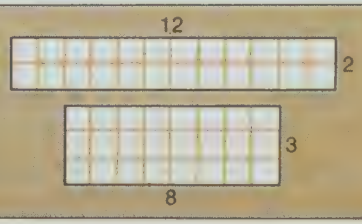
7. A swimming pool is 50 metres long and 20 metres wide.

A Make a scale drawing using the scale 1 : 500.

B Make a scale drawing using the scale 1 cm = 10 m.

Think

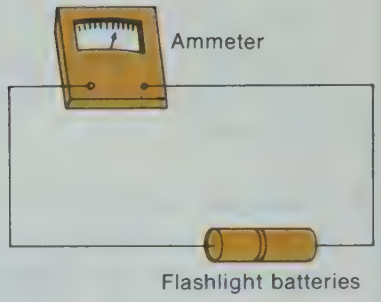
Make a copy of each rectangle on graph paper. Then show how to divide the 3 by 8 rectangle into two identical parts that will exactly cover the 2 by 12 rectangle.



More practice, page S-20, Set 37

Ratios and Electricity

An ordinary dry cell (flashlight) battery produces an electromotive force E of about 1.5 volts. Two such cells connected as in the figure will produce 3 volts. An **ammeter** measures the rate of flow of current, I, in **amperes** (A). If an electromotive force of 3 volts produces a current of 5 A in a circuit, then an electromotive force of 6 volts in the same circuit will produce 10 A, 9 volts will produce 15 A, and so on. Notice that we have the proportions:



$$\frac{3}{6} = \frac{5}{10}, \quad \frac{3}{9} = \frac{5}{15} \quad \frac{E_1}{E_2} = \frac{I_1}{I_2}$$

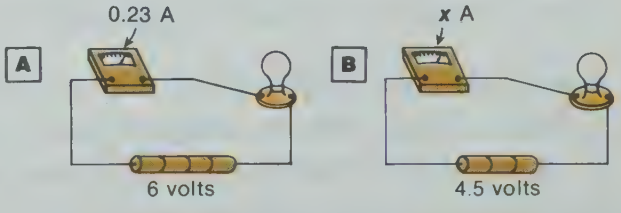
1. Hooking up one dry cell (1.5 volts) to a certain electrical circuit induces a current of 1.2 A. What will be the current using three dry cells (4.5 volts)?

2. An electromotive force of 10 volts in a certain electrical circuit induces a current of 3.2 A. Give the current induced in the same circuit by an electromotive force of

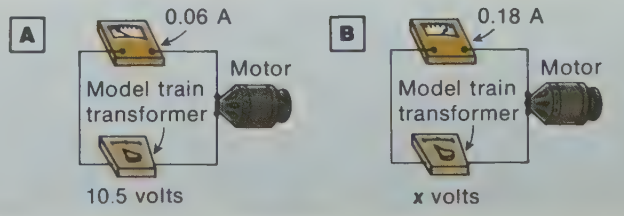
   A 5 volts **B** 20 volts **C** 16 volts **D** 4.5 volts

3. For a certain electrical circuit, when $E_1 = 3.6$, then $I_1 = 5.4$. Use the proportion $\frac{E_1}{E_2} = \frac{I_1}{I_2}$ to find the amperage I_2 induced by an electromotive force $E_2 = 9.6$ volts in the circuit.

4. Figures **A** and **B** show the same circuit but with different electromotive forces. Find the amperage **x** for figure **B**.



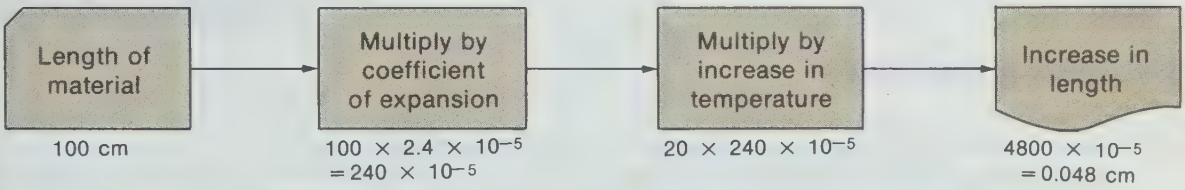
5. The model train transformer can vary the electromotive force which changes the current in the circuit shown. Write and solve a proportion to find the electromotive force **x**.



Expansion of Metals

It has been found in experiments that a given material expands the same amount for each degree of increase in temperature. For example, a rod of aluminum will increase about 2.4×10^{-5} cm (0.000024 cm) for each centimetre of its length when the temperature is raised 1° C. The number 2.4×10^{-5} is called the **coefficient of linear expansion**.

EXAMPLE: Find the increase in length of an aluminum rod 100 cm long when the temperature is increased 20° C. The flow chart below gives instructions for finding the increase in length.


Length of material
100 cm
Multiply by coefficient of expansion
100 × 2.4 × 10⁻⁵ = 240 × 10⁻⁵
Multiply by increase in temperature
20 × 240 × 10⁻⁵
Increase in length
4800 × 10⁻⁵ = 0.048 cm

1. A copper wire is 100 metres long at 0° C. How much does the length increase when the wire is heated to 40° C? Express your answer in centimetres.

2. At 0° C an aluminum bar is 65 cm long. How much will its length increase if it is placed in boiling water (100° C)?

3. A steel beam is 30 metres in length at 0° C. How much longer will it be (in cm) when the temperature rises to 20° C?

Material	Coefficient of linear expansion
Aluminum	2.4×10^{-5}
Copper	1.4×10^{-5}
Gold	1.3×10^{-5}
Steel	1.1×10^{-5}
Silver	1.7×10^{-5}
Tin	2.3×10^{-5}
Tungsten	4.2×10^{-6}

4. Tungsten does not expand readily. What would be the increase in length of a tungsten wire 10 cm long at 0° C if the temperature of the wire is raised to 500° C?

5. The tallest television tower in the world is in Fargo, North Dakota. It is about 628.8 metres tall. If the tower is made of steel, how many centimetres will it expand when the temperature rises from 0° C to 30° C?

Investigating the Ideas

Each card below contains a symbol for a rational number.

A	8 to 400	B	0.76	C	$\frac{3}{5}$
D	$0.3\frac{1}{3}$	E	7:20	F	$\frac{11}{8}$

Can you write a fraction with a denominator of 100 for each rational number shown on the cards?

Discussing the Ideas

1. Explain how you found the fraction with denominator 100 for each numeral shown on the cards.

2. Rational numbers may be represented by fractions or decimals. Another way of representing rational numbers is by **percents** (%). When we use the language of percent we are thinking of **hundredths**.

   For example, $0.25 = \frac{25}{100} = 25\%$; $2.37 = \frac{237}{100} = 237\%$; $0.33\frac{1}{3} = \frac{33\frac{1}{3}}{100} = 33\frac{1}{3}\%$

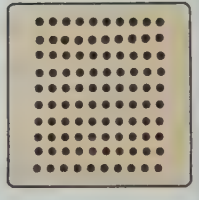
   What percent can be written for each of the fractions you found in the Investigation?

3. What percent can be written for each decimal?

   A 0.43 B 1.25 C $0.66\frac{2}{3}$ D 0.5

4. When we say, "The number of objects in set A is seven percent (7%) of the number of objects in set B," we mean that there are 7 objects in A for every 100 objects in B.

   $A =$

   $B =$ 

   A If there are 7% as many objects in set A as in set B and there are 500 objects in set B, how many objects are in set A?

   B 28 out of 400 students are absent from school. How many are absent from each 100? What percent were absent?

5. Explain the steps for representing the fraction as a percent.

   $\frac{1}{6} = \frac{100}{600} = \frac{100 \div 6}{600 \div 6} = \frac{16\frac{2}{3}}{100} = 0.16\frac{2}{3} = 16\frac{2}{3}\%$

1. Write each ratio as a percent.

A 19 to 100	**C** 16 to 400	**E** $\frac{1}{2}$ to 100	**G** 13 to 20
B 13 to 50	**D** 150 to 100	**F** 50 to 50	**H** 240 to 1000

2. Express each fraction as a percent. EXAMPLE: $\frac{22}{300} = \frac{22 \div 3}{300 \div 3} = \frac{7\frac{1}{3}}{100} = 7\frac{1}{3}\%$

A $\frac{15}{300}$	**C** $\frac{5}{200}$	**E** $\frac{42}{1200}$	**G** $\frac{23}{1000}$	**I** $\frac{9}{1800}$
B $\frac{24}{600}$	**D** $\frac{25}{800}$	**F** $\frac{17}{600}$	**H** $\frac{371}{1000}$	**J** $\frac{33}{4400}$

3. Copy each exercise, replacing ***a*** and ***b*** by the correct number.

A $\frac{4}{30} = \frac{a}{300} = \frac{b}{100} = b\%$

B $\frac{2}{3} = \frac{a}{300} = \frac{b}{100} = b\%$

C $\frac{5}{12} = \frac{a}{1200} = \frac{b}{100} = b\%$

D $\frac{1}{8} = \frac{a}{800} = \frac{b}{100} = b\%$

4. Carry out each indicated division to two decimal places and then express your answer to the nearest whole percent.

EXAMPLE: $\frac{31}{47} \rightarrow 31 \div 47 = 0.65\frac{45}{47} \rightarrow 65\frac{45}{47}\% \rightarrow 66\%$

A $\frac{7}{12}$	**C** $\frac{5}{6}$	**E** $\frac{24}{35}$	**G** $\frac{131}{85}$	**I** $\frac{4}{7}$
B $\frac{13}{18}$	**D** $\frac{18}{24}$	**F** $\frac{7}{111}$	**H** $\frac{7}{8}$	**J** $\frac{22}{23}$

5. Each quotient is given to four decimal places. Express each fraction to the nearest tenth of a percent.

A $\frac{7}{12} = 0.5833\ldots$	**C** $\frac{45}{187} = 0.2406\ldots$	**E** $\frac{37\,000}{155\,000} = 0.2386\ldots$
B $\frac{5}{64} = 0.0781\ldots$	**D** $\frac{257}{71} = 3.6197\ldots$	**F** $\frac{4256}{654\,000} = 0.0065\ldots$

6. Carla was absent 9 days out of 180 school days during a certain year. What percent of the days was she absent?

7. In a free-throw shooting contest, Dick made 29 out of 40 free throws. Dick said, "If I make the next free throw, I will have made 75% of my shots."

A Show that Dick's statement is incorrect.

B What is the fewest number of free throws that Dick must make in a row after his first 40 shots in order to make at least 75% of his shots?

think

Arrange the eight digits below so that a numeral will be formed in which the 1's are separated by one digit, the 2's by two digits, the 3's by three digits, and the 4's by four digits.

1 1 2 2 3 3 4 4

Discussing the Ideas

Percent language is useful in presenting information. For computing purposes, however, it is usually easier to use decimals or fractions for percents. You should be able to express decimals and fractions as percents, and percents as decimals or fractions.

1. Study the examples for changing percents to decimals.

EXAMPLES: [A] $17\% = 0.17$ [B] $23\% = 0.23$ [C] $3.8\% = 0.038$

Change each percent to a decimal.

A 5% B 44% C 125% D 16.4% E 1%

2. Give the lowest-terms fraction for each percent.

EXAMPLES: [A] $48\% = \frac{48}{100} = \frac{12}{25}$ [B] $37.5\% = \frac{37.5}{100} = \frac{375}{1000} = \frac{3}{8}$

A 20% B 4% C 12.5% D 0.1% E 1000%

3. You should be able to find the lowest-terms fraction for each of these percents mentally. Can you do it?

A 10% C 50% E 90% G 100% I 75%

B 25% D 30% F 40% H 80% J $33\frac{1}{3}\%$

4. The example shows how to find a percent for a fraction to the nearest whole percent. What percent is $\frac{11}{23}$ to the nearest whole percent?

EXAMPLE:

$$\frac{17}{26} \rightarrow 26\overline{)17.00} \rightarrow \frac{17}{26} = 0.65\frac{10}{26}$$

$$\begin{array}{r} 0.65 \\ 26\overline{)17.00} \\ 15\,6 \\ \hline 1\,40 \\ 1\,30 \\ \hline 10 \end{array}$$

5. Round each decimal to the nearest hundredth. Then express the rounded decimal as a percent.

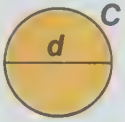
A 0.2199 . . . B 0.36248 . . . C 2.144 D 0.009617

More practice, page S-20, Set 38

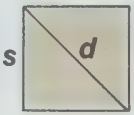
1. About 73.5% of the people in the United States live on 1.5% of the land. Express each percent as a decimal.

2. Nearly 87% of Quebec's population speak French. Express this percent as a fraction and as a decimal.

3. The diameter ***d*** of any circle is 0.3183 . . . as long as the circumference ***C*** of the circle. Express this decimal as a percent to the nearest whole percent.



4. The side ***s*** of any square is about 0.7071 as long as the diagonal ***d*** of the square. What percent of the length of the diagonal is the side of the square?



5. A baseball player's batting average is reported as 0.329. What percent of the times at bat are hits according to this decimal?

6. In some places you must pay 4¢ sales tax on each dollar of sales. What is the sales tax rate as a percent?

7. A picture of an animal in a book is $\frac{1}{50}$ the actual size of the animal. What percent of the actual size is the picture?

8. If 74 out of every 140 registered voters voted in an election, what percent of the registered voters voted, to the nearest whole percent?

9. A carton of milk holds 0.946 litres. What percent of a litre is the carton, to the nearest whole percent?

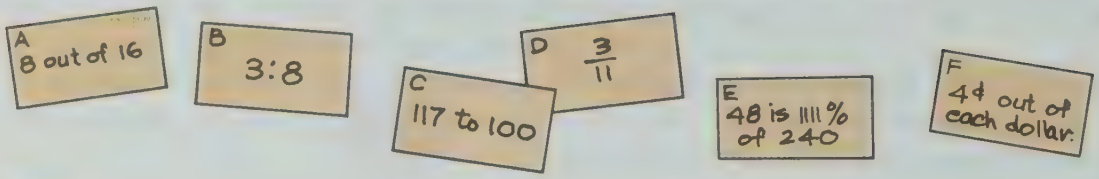
10. A student answered 39 out of 48 questions correctly. What percent were answered correctly, to the nearest whole percent?

11. The inflation rate in the United States in 1973 was 8.5%. Express this percent as a decimal.

12. The total amount of water on Earth is estimated to be 1.3×10^9 km^3. About 2.9×10^7 km^3 of the water is in ice caps and glaciers. About what percent of the Earth's water is in ice caps and glaciers?

Investigating the Ideas

Study the cards below.



Can you give a percent for the number suggested on each card?	First write the number as a fraction with denominator 100.

Discussing the Ideas

1. Explain your method for finding some of the percents.

2. Each card suggests the problem of finding what percent one number is of another number. For example, for card D, $\frac{3}{11}$ suggests finding what percent 3 is of 11.

$$\text{If } \frac{3}{11} = x\%, \text{ then } \frac{3}{11} = \frac{x}{100} \text{ (Why?)}$$
$$\text{and } 11x = 300 \text{ (Why?)}$$
$$\text{and } x = 27\tfrac{3}{11}$$
$$\text{Therefore, } \tfrac{3}{11} = 27\tfrac{3}{11}\%$$

Use this method to find the percent for card B.

3. Notice in the example above that $x\%$ is written $\frac{x}{100}$.
We can also represent $x\%$ as $x \cdot \frac{1}{100}$ or $x \cdot 0.01$.
Which numerals below represent the same number as 8%?

A $8 \cdot \frac{1}{100}$	C $\frac{80}{100}$	E $8 \cdot 0.01$	G $\frac{8}{100}$
B 0.08	D $\frac{8}{10}$	F $\frac{2}{25}$	H 0.8

4. Card E suggests the equations shown at the right.
   - A Explain each step and then solve the equation for x.
   - B Solve: 8 is what percent of 40?

$$48 = x\% \text{ of } 240$$
$$48 = x\% \cdot 240$$
$$\frac{48}{240} = x\%$$
$$\frac{48}{240} = \frac{x}{100}$$

1. Math test: 20 problems.
Had 18 correct.
What percent correct?

2. Class: 30 students.
Today: 24 present.
What percent present?

3. Played 10 games.
Lost 4 games.
What percent lost?

4. True-False quiz: 24 questions.
Had 6 incorrect.
What percent correct?

5. Regular price: $40.
Sale price: $10 off.
What percent off?

6. Made 12 out of 16 free throws.
What percent were made?

7. Earnings: $120.
Tax: $12.
What percent tax?

8. Rent: $150 per month.
Salary: $600 per month.
What percent for rent?

9. Attendance: $\frac{4}{5}$ of students.
What percent attended?

10. If 16 out of 24 students voted for Denise, what percent is this?

11. Played 24 games. Won 10 games.
Lost what percent?

12. Test: 50 questions. Had 43 correct.
What percent incorrect?

★ 13. 60 students. 12 wear glasses.
$\frac{1}{3}$ of those with glasses have red hair.

A What percent wear glasses?

B What percent wearing glasses have red hair?

C What percent of the students wear glasses and have red hair?

think

A school club composed of 99 students planned a drawing for three prizes. Each student held a different number from 1 through 99.

Each of the three students whose numbers were drawn would receive as a prize as many free movie passes as the sum of the digits of his number. For example, if number 17 were drawn, the holder would get 8 passes.

John was absent during the drawing and the next day he asked the other members who had won. The club secretary wouldn't tell him.

"Well, then, tell me the total number of free passes you gave out," John said.

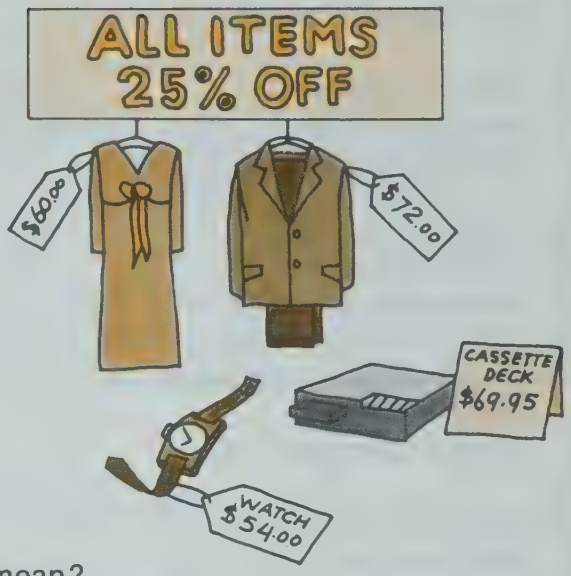
"If I do, I'll give away the numbers drawn," the secretary replied.

Then John said, "Well, never mind. You've already told me."

What were the winning numbers?

Investigating the Ideas

Suppose a store advertises "25% off" the marked price on each article pictured. The sales tax rate is 6%.



Which of the articles shown could you buy if you had $50?

Discussing the Ideas

1. What does "25% off" the marked price mean?

2. The percent "off" is called the **discount rate**. The amount that is "marked off" is the **discount**. How can you find the sale price of this item?

Marked price →	$60
Discount rate →	× 0.25
	300
	120
Discount →	$15.00

3. Since $25\% = \frac{1}{4}$, how could you use this fraction to compute the amount of the discount?

4. Should the original price or the sale price be used in computing the sales tax?

A

Original price →	$60
Tax rate →	× 0.06
Sales tax →	$3.60

B

Sale price →	$45
Tax rate →	× 0.06
Sales tax →	$2.70

5. Can you make a chart like this and complete it?

Article	Original price	Discount	Sale price	Sales tax (6%)	Total amount
Dress					
Suit					
Cassette deck					
Watch					

1. An $18 sweater is on sale. It is marked "25% off."
   - A What is the amount of discount?
   - B What is the sale price of the sweater?

2. Dick bought for half price a shirt which was originally priced at $11.00. He had to pay a 5% sales tax.
   - A What was the sale price of the shirt?
   - B What was the total cost of the shirt including sales tax?

3. Maria saw a pair of shoes marked $21.95. They were on sale for 10% off the marked price. Maria had only $20 with her.

   - A Did she have enough money to pay for the shoes?
   - B If the sales tax rate was 4%, would Maria have enough money?

4. A 10-speed bicycle regularly priced $125 is marked down 15% for a sale. Find the total amount needed to purchase the bicycle on sale including a sales tax of 6%.

5. Which is less expensive?
   - A A Super 174XC turntable on sale for $69.95.
   - B A Super 174XC turntable regularly priced $79.95 but on sale at 15% off of the regular price.

6. Nicole saw the same brand and style of sweater on sale in two different stores. At Erton's Department Store the price was $15 with 20% off. At Sampson's the price was $18 with $\frac{1}{3}$ off. What is the sale price of the sweater at each store?

7. The price of a $180 lounge chair was reduced 20%. Later the sale price was reduced by 10%. What was the final sale price?

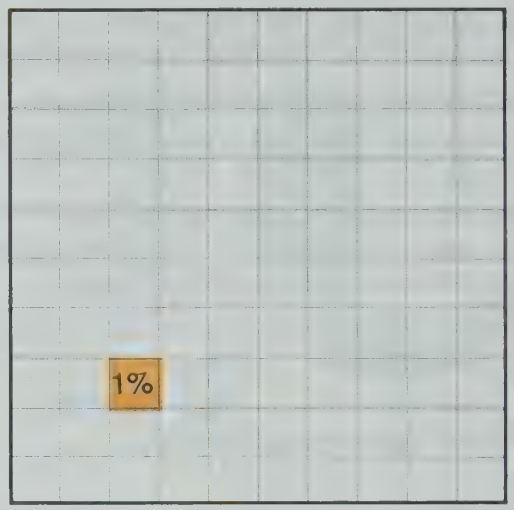
8. Suppose the price of the lounge chair originally marked $180 was reduced 30%.
   - A What is the sale price of the lounge chair?
   - B How much more or less is this price than the final sale price of the lounge chair in Exercise 7?

More practice, page S-21, Set 39

Investigating the Ideas

The large square contains 100 small squares. Each small square is 1% of the large square.

Draw a large square region containing 100 small squares on a piece of graph paper.



Can you shade and label three regions within your square so that each one is less than 1% of the large region?

Discussing the Ideas

1. A What part of the small square region is shaded?
   B What fractional percent of the large region is this?

2. Study the example. Then explain how to find the amount for each part.

> EXAMPLE: Find $\frac{1}{2}$% of \$1000.
> SOLUTION: $\frac{1}{2}\% = 0.005$
> $\$1000 \times 0.005 = \$5.00$

   A $\frac{1}{2}$% of \$840 B $\frac{1}{4}$% of \$70 000 C 0.1% of \$10 000

3. If you wanted to cut out a region from a sheet of graph paper that was 250% the size of the large square region shown in the Investigation, how many of the small squares should your new region contain?

4. You can write decimals for percents greater than 100%.

$$250\% = \frac{250}{100} = 2.50$$

   What decimal represents each of these percents?
   A 300% B 120% C 1000% D 105%

1. Write a decimal for each percent.

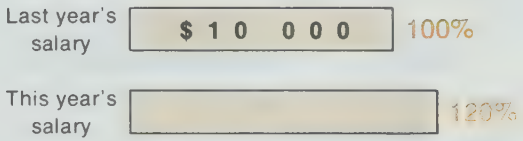
A $\frac{1}{2}\%$	**D** 800%	**G** $\frac{1}{100}\%$	**J** 0.2%
B 274%	**E** 0.7%	**H** 210%	**K** 0.05%
C $\frac{3}{10}\%$	**F** $\frac{1}{4}\%$	**I** 350%	**L** $\frac{1}{8}\%$

2. A piece of material is guaranteed not to shrink more than $\frac{7}{10}\%$ when washed. What is the greatest amount of shrinkage expected from a piece one metre long?

3. An aluminum rod 500 centimetres long is found to expand 0.12% of its length when heated. How many centimetres does the rod expand?

4. Fredrick read that the cost of living today is 160% of what it was ten years ago. Is this *160 times*, *16 times*, or *1.6 times* the cost of living of ten years ago?

5. Joan earned $10 000 last year. This year Joan earned 120% of last year's salary. How much did she earn this year?



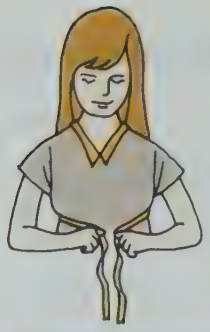
6. The population of Sunnyvale in 1960 was 53 000. In 1970, it was 180% of that. What was the population in 1970?

7. Ralph borrowed $1200 to buy an automobile. He had to pay back 109% of what he had borrowed. How much did he have to pay back?

8. Sometimes you may hear this expression: "There is only one chance in a million!" Express this ratio as a percent.

More practice, page S-21, Set 40

Investigating the Ideas



Carry out this measurement activity:

1. Exhale. Then find your chest measurement in centimetres.
2. Inhale as much as possible. Measure your chest again.

About how much does your chest measurement increase when you inhale?

Discussing the Ideas

1. When a measurement increases, it is often convenient to compare the increase to the original measurement by means of a percent. This is called the **percent of increase** in the measurement.

   A What was your chest measurement when you exhaled?

   B How many centimetres did it increase when you inhaled?

   C Find the ratio: $\frac{\text{Increase}}{\text{Exhaled measurement}}$

   D Express the ratio as a percent to the nearest tenth.

Exhaled measurement: 80 cm
Inhaled measurement: 88 cm
Increase: 8 cm

$\frac{8}{80} = \frac{1}{10} =$

2. If a measurement decreases, you can find the **percent of decrease** as follows:

   A How much did your chest measurement decrease when you exhaled?

   B Find the ratio: $\frac{\text{Decrease}}{\text{Inhaled measurement}}$

   C Express the ratio as a percent to the nearest tenth.

Inhaled measurement: 88 cm
Exhaled measurement: 80 cm
Decrease: 8 cm

Percent of decrease: $\frac{8}{88} = \frac{1}{11} \approx$

3. Can you explain why the percent of increase (Exercise 1**D**) is slightly different from the percent of decrease (Exercise 2**C**)?

1. Copy and complete the table.

	Chest Measurement (cm) Exhaled	Inhaled	Increase	$\frac{\text{Increase}}{\text{Exhaled measurement}}$	Percent of increase (nearest tenth)
	89	97	8	$\frac{8}{89}$	$\frac{8}{89} \approx 9\%$
A	84	90			
B	78	86			
C	80	89			
D	70	75			

2. Margaret was 140 centimetres tall on her 13th birthday. On her 14th birthday, she was 146 centimetres tall.

- **A** What was her increase in height?
- **B** What was her percent of increase in height to the nearest tenth of a percent?

3. On July 12, Mr. Marsh weighed 142 kilograms. Then he went on a diet and on December 15, he weighed 110 kilograms.

- **A** How much had his weight decreased?
- **B** What percent had his weight decreased to the nearest tenth?

4. Linda earned $122 per week. She got a raise and now earns $130 per week. What percent raise did she get?

5. In 1960, the population of a B.C. town was 12 230. By 1970, the population had increased to 19 366. What was the percent of increase in population to the nearest whole percent?

6. In 1960, Bisbee, Arizona, had a population of 9914. By 1970, the population had decreased to 8328. What was the percent of decrease in the population?

REVIEWING THE IDEAS

1. Give three ratios equivalent to 2:5.

2. Give the simplest-terms ratio for each.
   A 7:28 B $1\frac{1}{2}$ to 4 C $\frac{1}{2}$ to $\frac{3}{4}$

3. What numbers are the means in the proportion 6:16 = 9:24?

4. Write a proportion which has 4 and 9 as the means and 3 and 12 as the extremes.

5. Complete: In any proportion, the product of the means equals ? ? ? ? ?.

6. Solve each proportion for x.
   A $\frac{4}{6} = \frac{x}{12}$ D $\frac{9}{15} = \frac{15}{x}$
   B $\frac{x}{3} = \frac{10}{6}$ E $\frac{6}{75} = \frac{x}{100}$
   C $\frac{4}{9} = \frac{x}{36}$ F $\frac{x}{30} = \frac{7}{35}$

7. A distance of 2 cm on a map represents 120 km. What distance is represented by 1 cm on the map?

8. An enlargement of a photograph of an insect shows the insect 100 times its actual size. If an eye of the insect in the photograph measures 9 mm, what is the actual size of the eye?

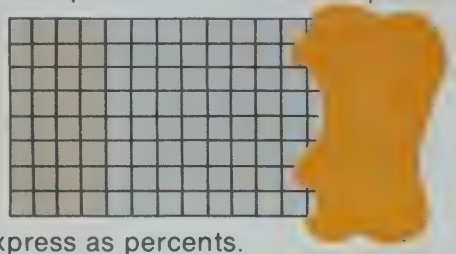
9. A mass of 400 grams stretches a spring 3.2 cm. How far would a mass of 500 grams stretch the spring?

10. Which is the ratio of 36 mm to 10 cm?
    A $\frac{36}{10}$ B $\frac{10}{36}$ C $\frac{9}{25}$ D $\frac{18}{5}$

11. Express 18 km/h in terms of metres per second.

12. Express each ratio as a percent.
    A 11 to 100 C 3:25
    B 4:50 D 8 to 200

13. None of the covered squares in the figure is shaded. Tell what percent of the squares are shaded if each row contains
    A 100 squares. C 20 squares.
    B 50 squares. D 800 squares.



14. Express as percents.
    A $\frac{3}{5}$ C $\frac{2}{3}$ E 0.002
    B 0.17 D 1.25 F $\frac{9}{4}$

15. Barbara got 17 out of 20 problems correct. What percent of her problems were correct?

16. A sport coat regularly priced $55 is on sale for 20% off. What is the sale price?

17. A province has a sales tax rate of 5%. How much tax must be paid on a purchase of $19.98?

18. 1960 population: 4 million.
    1970 population: 5 million.
    What is the percent of increase?

19. If you bought a car for $800 and later sold it for 120% of what you paid for it, what was the selling price?

20. In a school of 400 students only 12 were absent one day. What percent were present?

TEST YOURSELF

1. What ratio compares the number of elements in set A with the number of elements in set B?

   A: {□ □ □ □ . . .}

   B: {○○ ○○ ○○ ○○ . . .}

2. What is the simplest-terms ratio for 12:20?

3. Solve the proportions.

   A $\frac{3}{8} = \frac{12}{x}$ B $\frac{x}{9} = \frac{4}{12}$

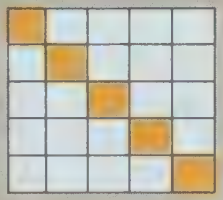
4. How many kilometres per hour is 5 m/s?

5. Which is the ratio of 1 mm to 1 dm?

   A $\frac{1}{10}$ B $\frac{1}{100}$ C $\frac{1}{1000}$ D $\frac{1}{10\,000}$

6. A scale drawing for a room has the scale 1 cm = 2.5 m. The length of the room shown in the drawing is 3.2 cm. What is the actual length of the room?

7. What percent of the squares are shaded?



8. Express each symbol as a percent.

   A 6:10 B 0.37 C $\frac{5}{4}$

9. A Express $12\frac{1}{2}\%$ as a fraction.

   B Express 0.3% as a decimal.

10. Solve these short story problems.

    A Regular price: $85.
    Sale price: 20% off.
    What is the sale price?

    B 1965 population: 450 000.
    1970 population: 1.5% less.
    What is the 1970 population?

RESEARCH PROJECTS

A Look up the topic "The Golden Section" in a mathematics reference book. Illustrate this famous ratio with posters or drawings. (See *Mathematical Snapshots* by H. Steinhaus; New York: Oxford University Press, 1969, pp. 45–51.)

B Follow the Dow Jones Industrial Average in a newspaper for a week, noting its increases and decreases. Make a chart showing each day's change. Express the increase or decrease in terms of percent for each day.

C Many people have possessed extraordinary powers of mental calculation. Find out about some of the calculating prodigies and write a report on them. (See *Mathematical Recreations and Essays* by W. Rouse Ball; New York; Macmillan, 1960, pp. 350–378.)

CUMULATIVE REVIEW

1. What part of the rectangle is shaded?

2. Name five fractions equivalent to $\frac{5}{6}$.

3. **A** What is the opposite of $-\frac{2}{3}$?

B What is the opposite of $\frac{5}{4}$?

4. Give the lowest-terms fraction.

A $\frac{63}{81}$ **B** $\frac{2^3}{2^4}$ **C** $\frac{110}{345}$

5. Solve the equations.

A $\frac{3}{n} = \frac{18}{12}$ **C** $\frac{9}{10} = \frac{m}{90}$

B $\frac{r}{48} = \frac{3}{16}$ **D** $\frac{1}{8} = \frac{8}{v}$

6. Copy each problem and give the correct symbol ($<$, $>$, or $=$) for each ◍.

A $\frac{3}{4}$ ◍ $\frac{7}{8}$ **C** $\frac{5}{2}$ ◍ $\frac{8}{3}$

B $\frac{6}{48}$ ◍ $\frac{7}{56}$ **D** $\frac{11}{16}$ ◍ $\frac{11}{17}$

7. Compute.

A $8\frac{1}{3} + 6\frac{2}{5}$ **C** $2\frac{1}{7} + 3\frac{3}{4}$

B $8\frac{1}{3} - 6\frac{2}{5}$ **D** $\frac{7}{8} - \frac{1}{9}$

8. Give the principle illustrated in each problem.

A $\frac{2}{3} + \frac{4}{7} = \frac{4}{7} + \frac{2}{3}$

B $\left(\frac{2}{3} + \frac{4}{7}\right) + \frac{3}{4} = \frac{3}{4} + \left(\frac{2}{3} + \frac{4}{7}\right)$

C $\frac{3}{5} \cdot \left(\frac{2}{3} + \frac{1}{3}\right) = \frac{3}{5}$

D $\frac{4}{5} \cdot \left(\frac{5}{9} + \frac{10}{11}\right) = \left(\frac{4}{5} \cdot \frac{5}{9}\right) + \left(\frac{4}{5} \cdot \frac{10}{11}\right)$

9. Compute.

A $\frac{7}{9} \cdot \frac{3}{8}$ **B** $\frac{7}{9} \div \frac{3}{8}$ **C** $\frac{16}{47} \div \frac{16}{47}$

10. Give the reciprocal of each number.

A $\frac{2}{3}$ **B** 4 **C** $\frac{-1}{9}$

D Does zero have a reciprocal? Explain.

11. Find the sum or difference.

A $\frac{3}{4} + \frac{-2}{3}$ **C** $1 - \frac{-2}{3}$

B $\frac{-1}{8} + \frac{-1}{2}$ **D** $\frac{-4}{5} - \frac{1}{8}$

12. Find the product or quotient.

A $\frac{1}{3} \cdot \frac{-3}{4}$ **C** $\frac{-6}{7} \div \frac{-14}{15}$

B $\frac{-1}{5} \cdot \frac{-3}{2}$ **D** $\frac{-3}{8} \div \frac{11}{16}$

13. A music store is selling some LP records at $\frac{1}{3}$ off the regular price. What is the sale price of a $4.98 record?

14. Mr. Wood bought a house costing $26 000. He paid $\frac{1}{5}$ of the cost as a down payment. How much was the down payment?

15. Express each fraction as a decimal.

A $\frac{47}{100}$ **C** $\frac{2}{5}$ **E** $\frac{1}{6}$

B $\frac{8}{25}$ **D** $\frac{3}{8}$ **F** $\frac{2}{7}$

16. Express each decimal as a fraction in simplest terms.

A 0.08 **C** 0.33 **E** 16.11

B 0.3 **D** 2.4 **F** 8.35

17. Find n.

A $0.2 + 73.7 + 4 + 0.006 = n$

B $35 + 0.421 + 0.3 + 8.1 = n$

18. Write each number in scientific notation.

A 0.0073

B 58 000

C 0.000082

D 98 000 000 000

19. Solve the equations.

A $4.76 - 3.128 = d$ **C** $3.2 + a = 9.6$

B $3 - 0.1962 = v$ **D** $s + 2.2 = 6.1$

20. Find the products.

A $10 \cdot 7.1$ D $0.1 \cdot 2.17$

B $100 \cdot 0.082$ E $0.01 \cdot 639$

C $100 \cdot 2.16$ F $0.001 \cdot 55.4$

21. Find the products.

A $6.2 \cdot 14$ C $3.4 \cdot 6.17$

B $0.014 \cdot 395$ D $3 \cdot 0.0072$

22. Find the quotients.

A $81.4 \div 10$ C $96 \div 0.01$

B $6.28 \div 0.1$ D $2.83 \div 100$

23. Find the quotients.

A $38.46 \div 0.002$ C $10.26 \div 0.38$

B $783.123 \div 0.3$ D $3.9545 \div 55$

24. Round each number to one significant digit. Then estimate the product.

A $3.9 \cdot 7.2$ C $624 \cdot 0.1934$

B $0.301 \cdot 5.88$ D $88\,947 \cdot 0.29$

25. Round each number to one significant digit. Then estimate the quotient.

A $0.0842 \div 0.0217$

B $17.92 \div 3.14$

C $5.36 \div 0.00984$

26. What is the simplest ratio that compares the number of circles to the number of triangles?

27. Solve each proportion.

A $\frac{7}{x} = \frac{14}{32}$ C $\frac{x}{11} = \frac{28}{44}$ E $\frac{6}{x} = \frac{24}{32}$

B $\frac{2}{3} = \frac{x}{27}$ D $\frac{1}{8} = \frac{8}{x}$ F $\frac{3}{x} = \frac{15}{45}$

28. Which is the larger of the two ratios, 3 to 9 or 5 to 16?

29. Express each number as a percent.

A $\frac{3}{4}$ C $\frac{33}{100}$ E $\frac{11}{10}$

B $\frac{7}{10}$ D 0.53 F $\frac{4}{25}$

30. Ellen got 21 problems correct on a test of 25 problems. What percent did she have correct?

31. Virginia was 150 cm tall. During the summer, she grew 3 cm. By what percent did her height increase?

32. A dress was originally marked \$14.70. If there was a discount of $33\frac{1}{3}\%$, what was the sale price of the dress?

33. A province has a 6% sales tax rate.

A How much tax would be charged on a suit costing \$75.95?

B How much would the suit cost including the sales tax?

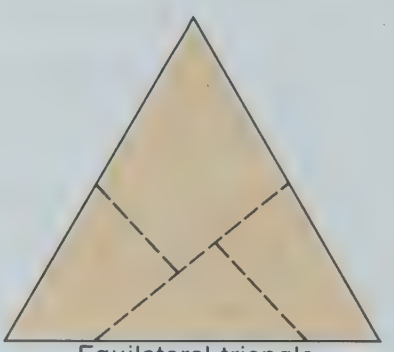
34. What percent of 98 is 7 to the nearest tenth of a percent?

35. Sam sold a used car for \$600. He made a profit of 20%. How much did he originally pay for the car?

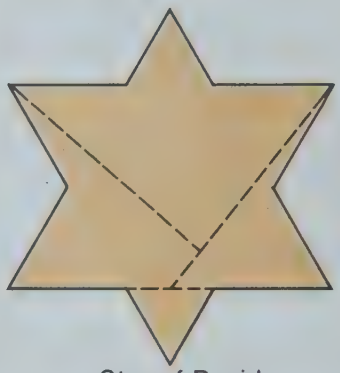
36. What is 0.4% of 15?

DISSECTION PUZZLES

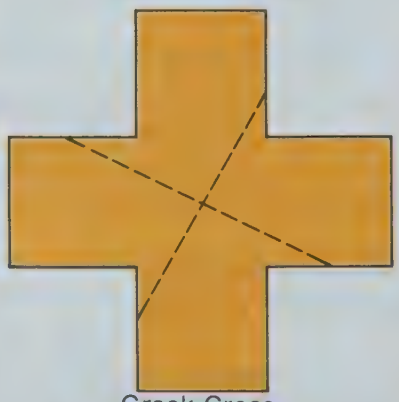
The pieces in each region can be arranged to make a square region which has the same area as the original region. Trace each region, cut it out, and cut it into the pieces along the dashed lines. Can you make a square from each set of pieces?



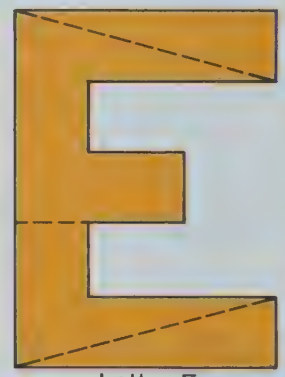
Equilateral triangle



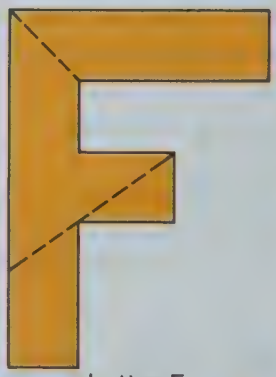
Star of David



Greek Cross



Letter E



Letter F

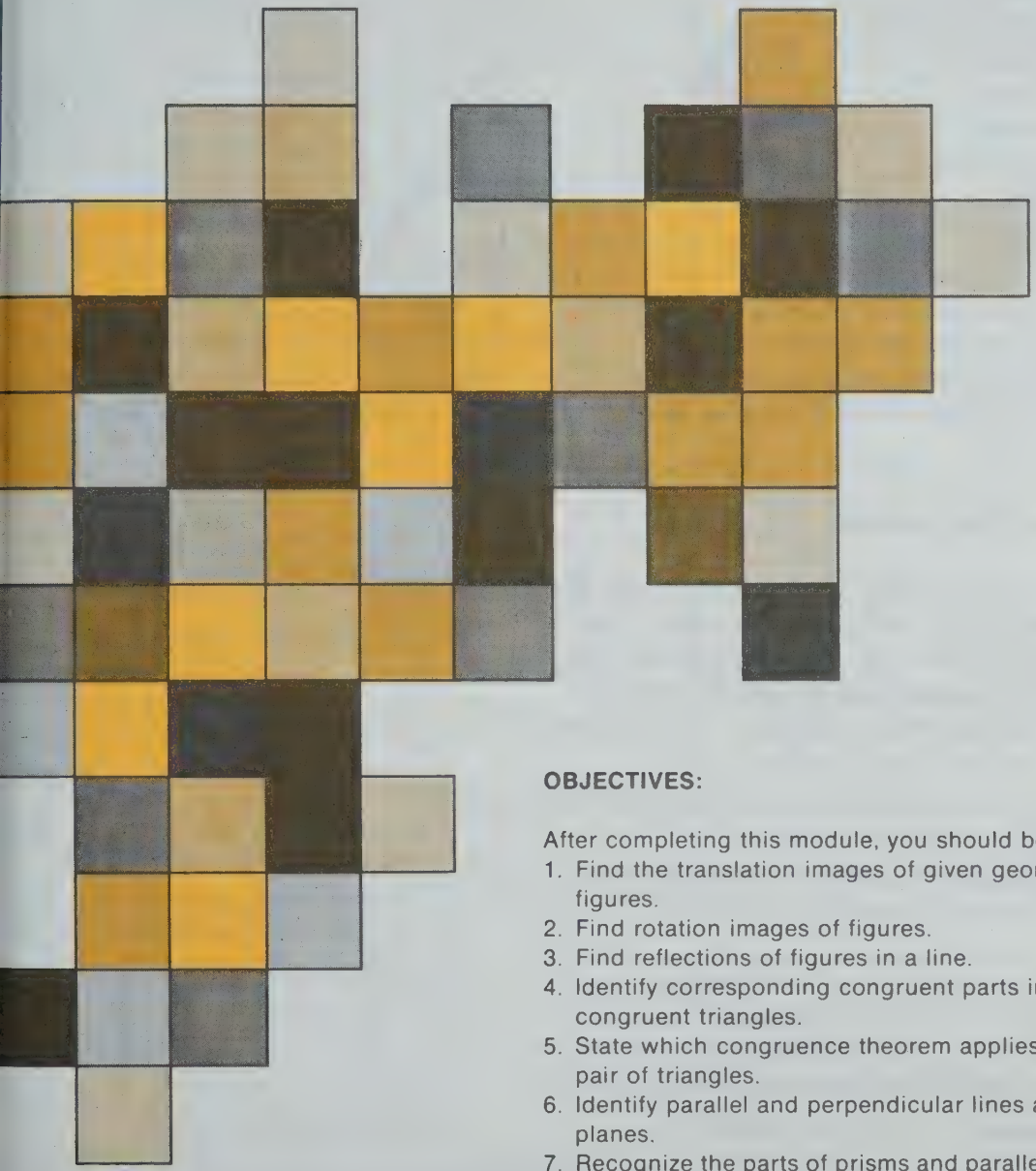
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. 1 to 2 **2.** 3:5 **3.** **A** 32; **B** 3 **4.** 18 km/h
5. **B** **6.** 8 m **7.** 20% **8.** **A** 60%; **B** 37%; **C** 125%
9. **A** $\frac{1}{8}$; **B** 0.003 **10.** **A** $68; **B** 443 250

UNIT G: *Geometry*

MODULE 1: *Rigid Motions and Congruence*



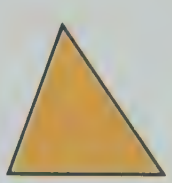
OBJECTIVES:

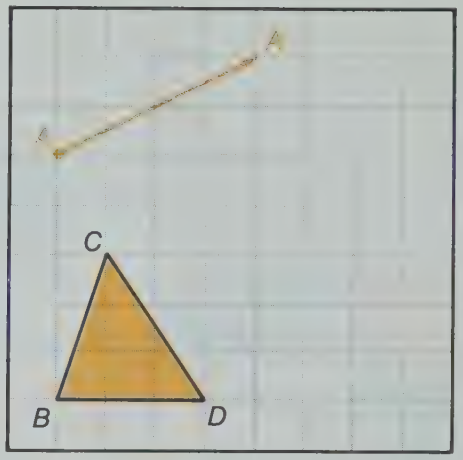
After completing this module, you should be able to:

1. Find the translation images of given geometric figures.
2. Find rotation images of figures.
3. Find reflections of figures in a line.
4. Identify corresponding congruent parts in pairs of congruent triangles.
5. State which congruence theorem applies for a pair of triangles.
6. Identify parallel and perpendicular lines and planes.
7. Recognize the parts of prisms and parallelepipeds.

Investigating the Ideas

Make a drawing of $\triangle BCD$ on graph paper. Also, cut out a triangular region that is the same size and shape as $\triangle BCD$.



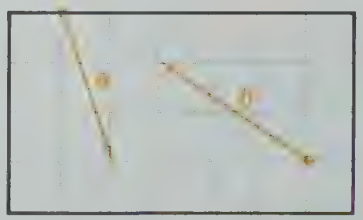


Can you use the triangular cut-out to show the new location of $\triangle BCD$ if all the points of the plane are slid the same distance and in the same direction as the arrow from A to A'?

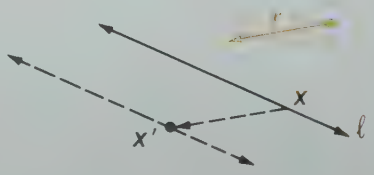
Discussing the Ideas

1. A *motion* which slides all points of the plane the same distance and in the same direction is called a **translation** or simply a **slide**. In the figure above, point A' is called the **translation image** of point A.
   We say that the translation "maps A onto A'" and denote this by writing: $A \rightarrow A'$
   - **A** Label the translation image of $\triangle BCD$. What points do B, C, and D map onto?
   - **B** What relations do you see between $\triangle BCD$ and its translation image?

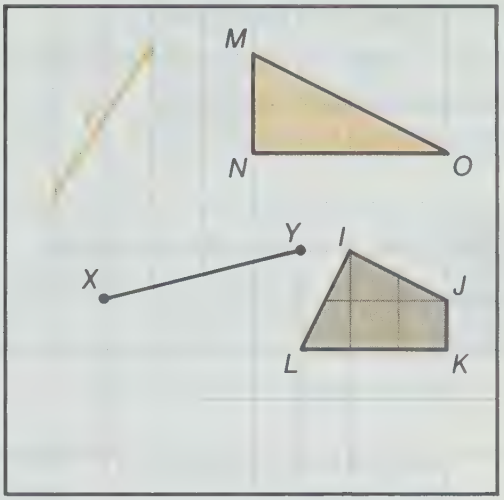
2. Arrows such as a and b show the distance and direction of translations. Draw a geometric figure on graph paper and show how to find its image for translations by
   - **A** arrow ***a***.
   - **B** arrow ***b***.



3. - **A** Draw a line ℓ and an arrow for a translation. Show the translation image of the line.
   - **B** How are the line and its translation image related?
   - **C** Can you describe a translation in which the image of line ℓ coincides with the original line?



1. Draw the figures shown on graph paper. Then show the translation image of each figure after the translation given by arrow ***t***.



2. Using the results of Exercise 1, complete each sentence.

A The translation image of $\triangle MNO$ is a ? that has the same size and shape as $\triangle MNO$.

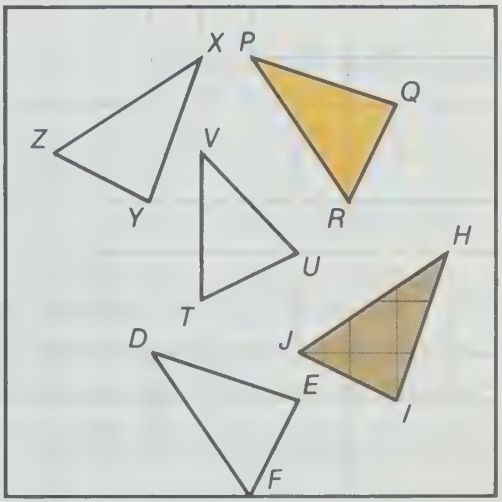
B The translation image of $\overline{XY}$ is a ? just as long as $\overline{XY}$.

C The translation image of quadrilateral *IJKL* is a ? with the same size and shape as *IJKL*.

3. **A** In the figure at the right, which triangle is a translation image of $\triangle PQR$?

B Complete this mapping for the translation:

$P \rightarrow$ ▒, $Q \rightarrow$ ▒, $R \rightarrow$ ▒



4. **A** Using the same grid as for Exercise 3, which triangle is a translation image of $\triangle HIJ$?

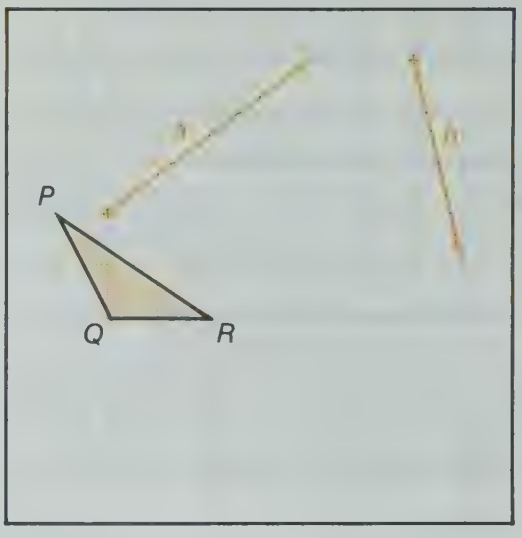
B Complete this mapping for the translation:

$H \rightarrow$ ▒, $I \rightarrow$ ▒, $J \rightarrow$ ▒

5. The arrow ***a*** shows a translation so that $X \rightarrow Y$. Is there a translation so that $Y \rightarrow X$? (That is, X is the image of Y.) How is this translation related to the translation given by arrow ***a***?

Investigating the Ideas

Draw $\triangle PQR$ on graph paper and cut out a triangular region that is the same size and shape as $\triangle PQR$.



Can you use the triangular region to show the following translation images?

1. The translation image of $\triangle PQR$, using arrow ***a***.

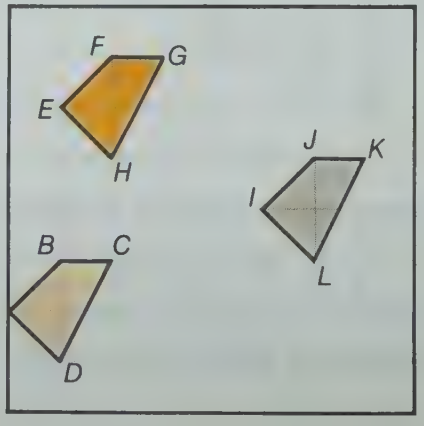
2. The translation image of the image found in step 1, using arrow ***b***.

Discussing the Ideas

1. The translation for arrow ***a*** could be described as "right 4, up 3." How could the translation for arrow ***b*** be described?

2. The combination of translations in the Investigation is a translation by "***a* followed by *b***."
   - **A** Is there a single translation that would produce the same final image as "***a*** followed by ***b***?"
   - **B** How would you describe the translation?

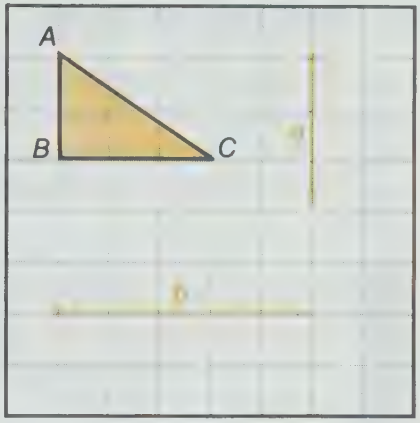
3. **A** Describe the translation that maps *ABCD* onto *EFGH*; that is, $A \rightarrow E$, $B \rightarrow F$, $C \rightarrow G$, and $D \rightarrow H$.
   - **B** Describe the translation that maps *EFGH* onto *IJKL*.
   - **C** The combination of the translations for parts **A** and **B** will map *ABCD* onto *IJKL*. Describe the single translation that will give the same final image as the combination of the translations for parts **A** and **B**.



1. **A** Draw $\triangle ABC$ on graph paper. Show the image of $\triangle ABC$ by translation ***a***.

B Show the image of $\triangle ABC$ for the combination "***a*** followed by ***b***."

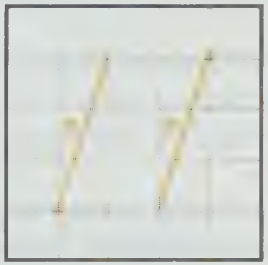
C Show an arrow that gives the same translation as "***a*** followed by ***b***."



2. Draw a square on graph paper. Show the final translation image of the square for the combination of translations "***a*** followed by ***b***" where ***a*** is "right 5, down 2" and ***b*** is "left 4, up 3."

3. **A** Draw a figure of your choice on graph paper.

B Show the translation image of the figure after the combination of translations "left 3, up 5" followed by "right 3, down 5."



4. Two translations are **inverses** of each other if the combination of the two translations leaves all points in their original position. Are the two translations in Exercise 3 inverses of each other?

5. The translations "right 2, up 1" and "left 2, down 1" are inverse translations. Give the inverse translation of each of the following:

A left 1, up 1

B right 4, down 5

C right 2, up 7

D left 4, down 4

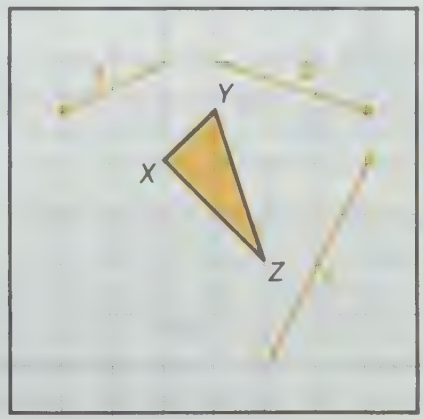
E up 3

F left 2

6. **A** Draw a triangle like $\triangle XYZ$ on graph paper.

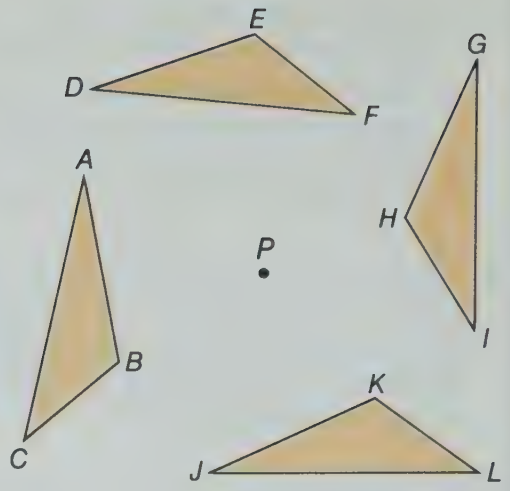
B Show the final translation image of $\triangle XYZ$ after the combination of "***a*** followed by ***b*** followed by **c**."

C Try translations ***a***, ***b***, and **c** again but in a different order, such as "**c** followed by ***a*** followed by ***b***." Is the final translation image in a position that is different from that for part **B**?



Investigating the Ideas

Place a piece of tracing paper over these figures. Trace $\triangle ABC$ and point P. Place your pencil point at P and turn the tracing paper.



Which one of the other three triangles will match the tracing of $\triangle ABC$ when the paper is turned around point P?

Discussing the Ideas

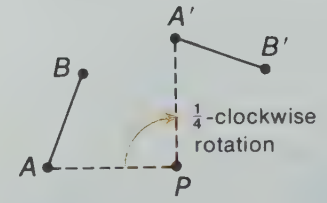
1. **A** Which point of any of the other triangles matched point A when you turned the paper around point P?
   B Which other points matched each other?

2. The motion that matched the two triangles is called **a rotation with centre at *P***, or **a rotation about point *P***. You can think of each point in the plane, except P, rotating through the same size angle with point P as the vertex of the angle. Draw a figure of your choice and then show a rotation of this figure about some point.

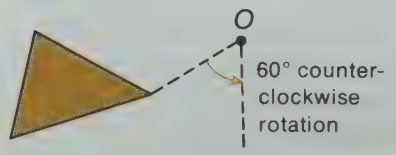
3. A rotation is determined by
   [1] its *centre,*
   [2] a *direction* (clockwise or counterclockwise), and
   [3] *amount of rotation* (usually a fractional part of a complete rotation or a degree measure).

   The figure at the right shows a **$\frac{1}{4}$-clockwise rotation with centre *P***. The rotation image of point A is A', and we write $A \rightarrow A'$.

   What is the rotation image of point B?

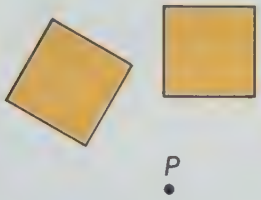


4. Draw a triangle and a point O. Explain how to find the rotation image of the triangle for "a 60° counterclockwise rotation with point O as the centre."

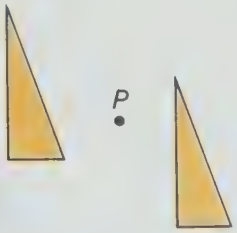


1. Using tracing paper, trace one of the figures and point P. Then tell whether the other figure is a rotation image of the first figure with centre at P.

A

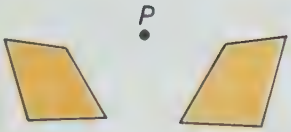


B

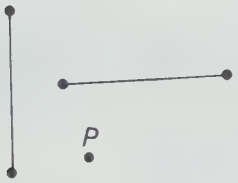


C

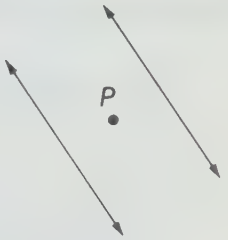
D



E

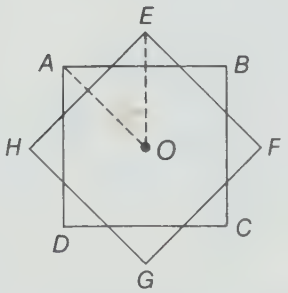


F



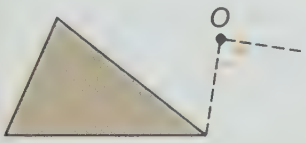
2. A rotation with centre O maps point A of square $ABCD$ onto point E.

- **A** What is the rotation image of point B? C? D?
- **B** Give the amount and direction for this rotation.

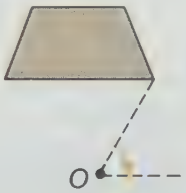


3. Use tracing paper to copy each figure. Then find the rotation image using the amount and direction of rotation shown. Point O denotes the centre of rotation for each part.

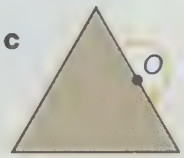
A



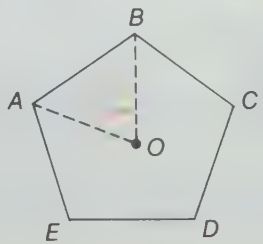
B



C

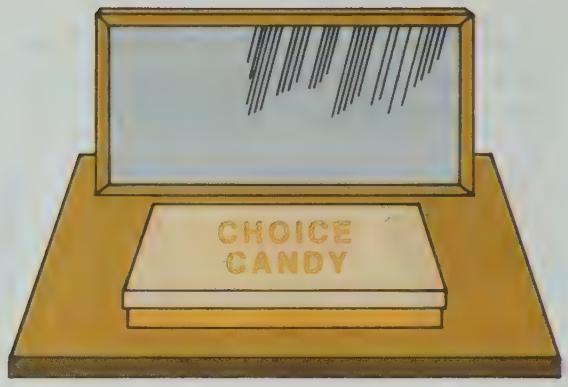


★ **4.** For the rotation with centre O shown, $A \rightarrow B$, $B \rightarrow C$, $C \rightarrow D$, $D \rightarrow E$, and $E \rightarrow A$. Consider another rotation with centre at point O so that $A \rightarrow C$. What would be the rotation image for B? C? D? E?



Investigating the Ideas

Suppose you saw this box of candy lying in front of a mirror.

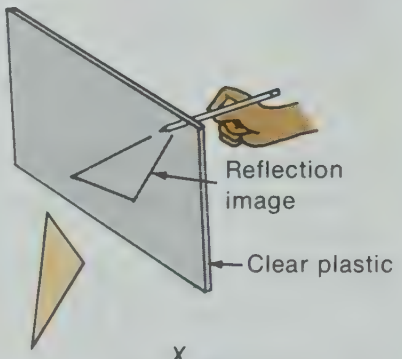


Can you draw the image that you would see in the mirror?

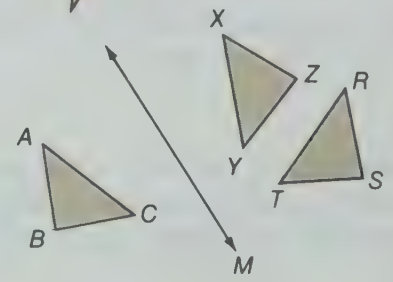
Discussing the Ideas

1. Which word on the box of candy looks the same when you see its reflection in the mirror?

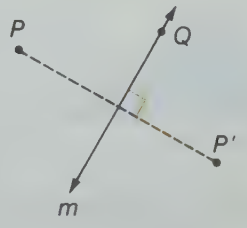
2. You can use a rectangular piece of clear plastic as a mirror to help you draw reflections of figures. Hold the piece of plastic upright on your paper. Look through the plastic to see the **reflection image**. Draw some figures and then use this method to find their reflection images.



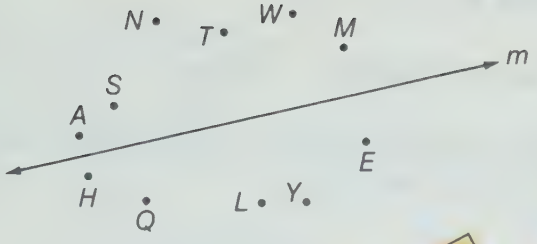
3. Suppose line m represents a mirror.
   - **A** Which figure is the reflection image of $\triangle ABC$ in line m?
   - **B** Which point is the reflection image of A? B? C?



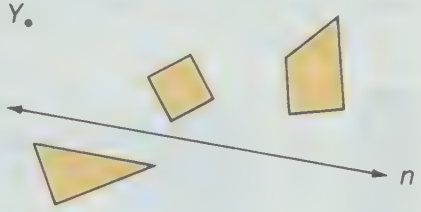
4. A reflection of the points of the plane in a line m can be described as follows:
   - [1] If P is not on m, then $P \rightarrow P'$ so that m is the perpendicular bisector of PP'.
   - [2] If Q is on m, then Q is its own image, $Q \rightarrow Q'$.
   - **A** Draw some points on one side of a line. Use a clear plastic "mirror" to find their reflection images.
   - **B** Connect some points to their images. Is the line the perpendicular bisector of these segments?



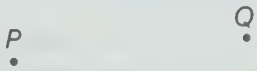
1. Which pairs of points are reflection images of each other in line m?



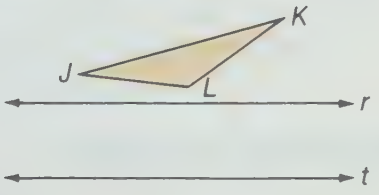
2. Draw or trace figures like the three shown here. Use a clear plastic "mirror" to draw the reflection image of each figure in line n.



3. Mark two points P and Q on your paper. Suppose P and Q are reflection images of each other. Use a plastic mirror to find and draw the reflecting line.



4. Draw two parallel lines r and t and $\triangle JKL$ as shown.
   - **A** Find the reflection image of $\triangle JKL$ in line r. Call it $\triangle MNO$.
   - **B** Find the reflection image of $\triangle MNO$ in line t. Call it $\triangle PQR$.
   - **C** Is there a single reflection in a line so that $\triangle PQR$ is the reflection image of $\triangle JKL$?
   - **D** Is $\triangle PQR$ a translation image of $\triangle JKL$?



5. Line ℓ is a **line of symmetry** of the square because the reflection images of all the points of the square in line ℓ will be on the square.
   - **A** How many different lines of symmetry does a square have?
   - **B** How many lines of symmetry does a rectangle which is not a square have?

6. How many lines of symmetry does each figure have?

A

Equilateral triangle

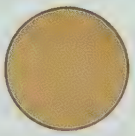
B

The letter H

C

Parallelogram

D

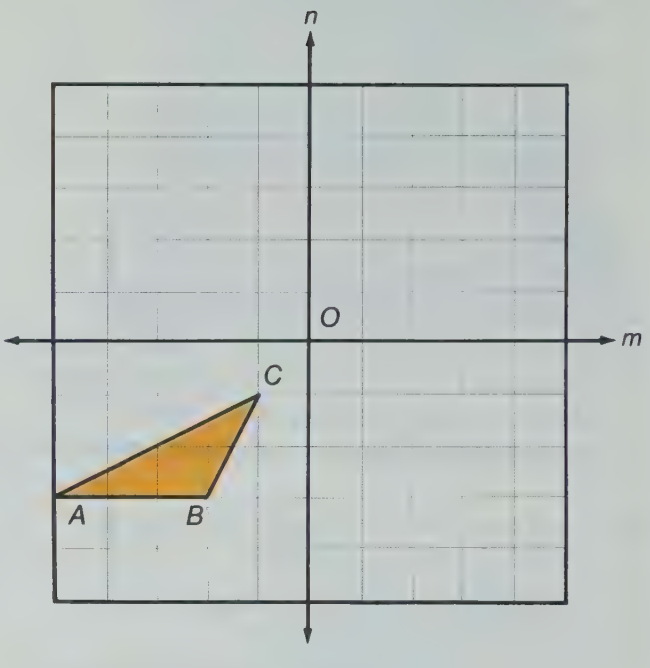


Circle

Investigating the Ideas

Draw $\triangle ABC$ and perpendicular lines m and n on a sheet of graph paper.

Can you find the following reflection images?



A Find the reflection image of $\triangle ABC$ in line m. Call it $\triangle A'B'C'$.

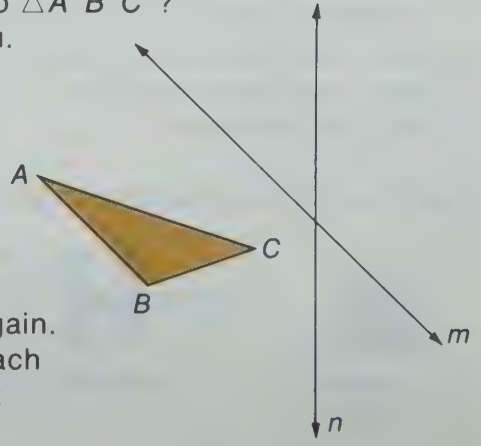
B Find the reflection image of $\triangle A'B'C'$ in line n. Call it $\triangle A''B''C''$.

Discussing the Ideas

1. $\triangle A''B''C''$ is said to be the reflection image of $\triangle ABC$ after a reflection in line m followed by a reflection in line n.
   - **A** Is there a single reflection that will map $\triangle ABC$ onto $\triangle A''B''C''$? Use a clear plastic mirror to help you.
   - **B** Is there a single translation that will map $\triangle ABC$ onto $\triangle A''B''C''$?
   - **C** Is there a rotation that maps $\triangle ABC$ onto $\triangle A''B''C''$? Use a piece of tracing paper to help you.

2. Try the Investigation again with lines m and n intersecting but not perpendicular. What conclusions can you draw about this combination of reflections?

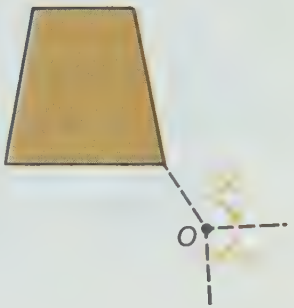
3. Try the same combination of reflections again. This time, have lines m and n parallel to each other. How is this result different from that found in Exercise 2?



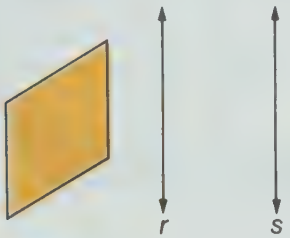
For each exercise, draw or trace the figures given. Then perform the combination of motions given and show the final image. Use a plastic mirror, graph paper, or tracing paper to help you find the final image.

1. Translation: ***a*** followed by ***b***.

2. Clockwise rotations: 120° followed by 90° with centre *O*.

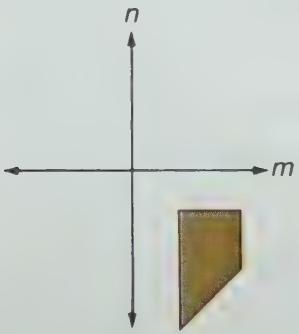


3. Reflection in line *r* followed by a reflection in line *s*.

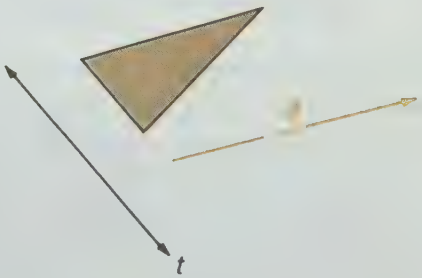


4. 90° clockwise rotation followed by translation **c**.

5. Reflection in line *n* followed by a reflection in line *m*.



6. Reflection in line *t* followed by translation ***d***.

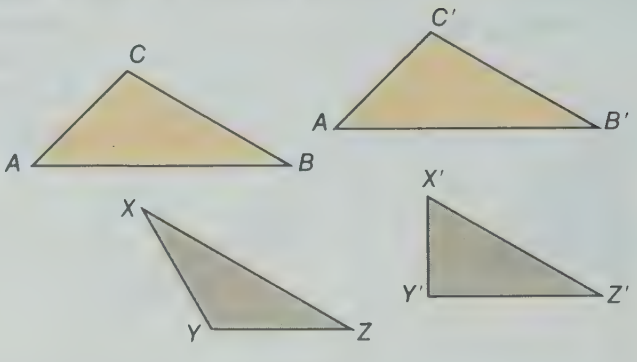


Discussing the Ideas

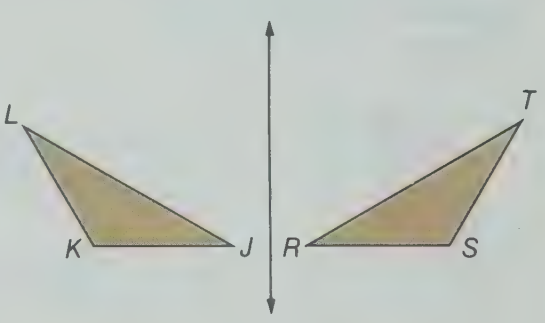
1. Each of the motions you have studied (translations, reflections, and rotations) is called a **rigid motion** because the image of a given figure after one of the motions is the same size and shape as the original figure.

   A Which rigid motion is illustrated if $\triangle A'B'C'$ is the image of $\triangle ABC$?

   B Could $\triangle X'Y'Z'$ be the image of $\triangle XYZ$ for one of the three rigid motions? Explain your answer.



2. The image of any geometric figure for any rigid motion or combination of rigid motions is a figure which is **congruent** to the original figure. Thus, if $\triangle RST$ is the image of $\triangle JKL$ for some rigid motion, then we say that $\triangle JKL$ is congruent to $\triangle RST$. We write this as $\triangle JKL \cong \triangle RST$. The parts of each figure that match each other are called **corresponding parts** and are **congruent to each other**.



   For example, in the figure,

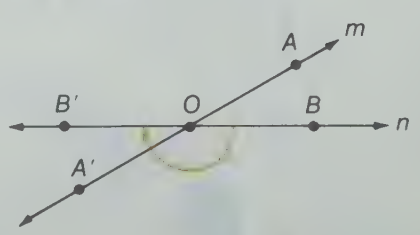
   $\overline{KJ} \cong \overline{SR}$ because $K \leftrightarrow S$ and $J \leftrightarrow R$

   $\angle LJK \cong \angle TRS$ because $L \leftrightarrow T$, $J \leftrightarrow R$, and $K \leftrightarrow S$.

   A What other pairs of corresponding parts can you find?

   B What statements of congruence can you write for them?

3. Draw two intersecting lines m and n and mark points A and B as shown.



   A Find the images of points A and B after a $\frac{1}{2}$-clockwise rotation about point O. Call the images A' and B'.

   B Is $\overline{OB} \cong \overline{OB'}$? Is $\overline{OA} \cong \overline{OA'}$? Why?

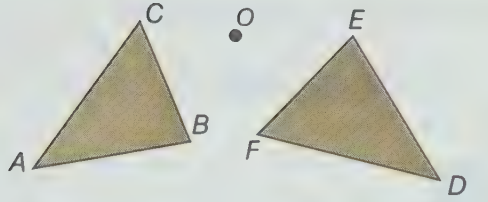
   C Since $\angle A'OB'$ is a rotation image of $\angle AOB$, is $\angle AOB \cong \angle A'OB'$?

1. $\triangle ABC$ is a rotation image of $\triangle DEF$ with point O as the centre of rotation. Complete each statement:

A $\overline{AB} \cong$ ▒▒▒ **D** $\angle A \cong$ ▒▒▒

B $\overline{BC} \cong$ ▒▒▒ **E** $\angle B \cong$ ▒▒▒

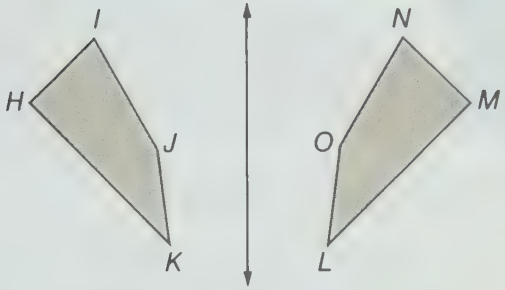
C $\overline{AC} \cong$ ▒▒▒ **F** $\angle C \cong$ ▒▒▒



2. Suppose the two quadrilaterals are reflection images of each other in line t.

A Write statements of congruence for the corresponding pairs of segments.

B Write statements that show which pairs of angles are congruent.



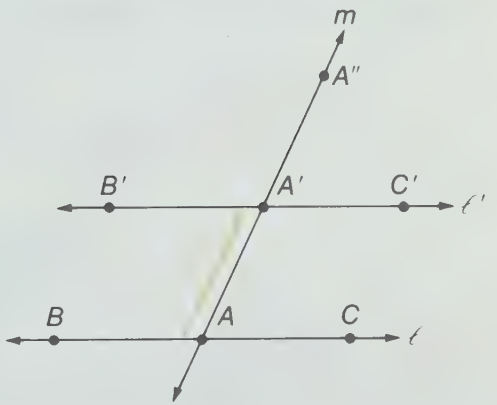
3. Lines ℓ and m intersect at point A. Consider a translation of the points in the plane by the arrow from A to A' The translation maps $B \rightarrow B'$, $C \rightarrow C'$, $A \rightarrow A'$, and $A' \rightarrow A''$.

A What is the image of line ℓ?

B $\angle A''A'C'$ is the translation image of $\angle$? .

C Is $\angle A''A'C' \cong \angle A'AC$?

D List other pairs of congruent angles in this figure.



Use the figures on the grid for Exercises 4 and 5.

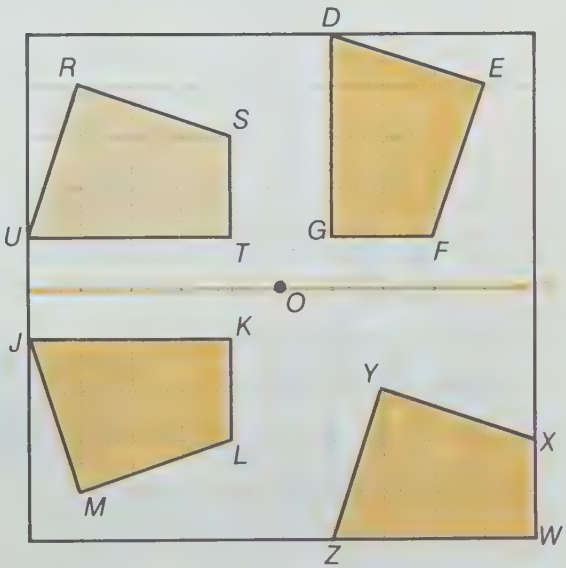
4. Find two quadrilaterals which are

A rotation images of each other.

B reflection images of each other.

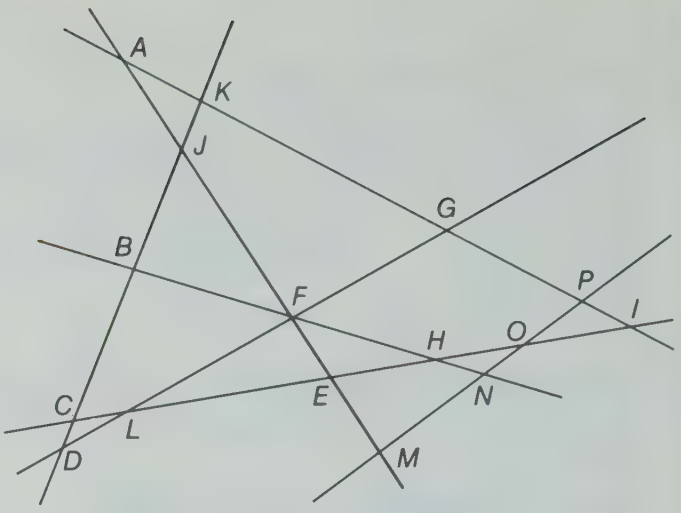
C translation images of each other.

5. Write a statement of congruence for each pair of quadrilaterals named in Exercise 4**A**, **B**, and **C**.



Investigating the Ideas

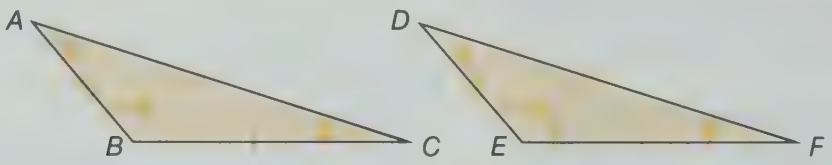
Study the drawing at the right.



Can you locate a pair of congruent triangles in the drawing? Check by tracing the triangles.

Discussing the Ideas

1. The figure below shows how corresponding segments and angles of congruent triangles can be marked.



$\triangle ABC \cong \triangle DEF$ $\Longleftrightarrow$ $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$ and $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$

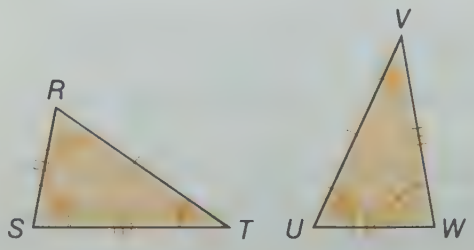
What statements can you make about the corresponding parts of the congruent triangles in the Investigation?

2. The statement $\triangle JKL \cong \triangle XYZ$ means that the vertices of the two congruent triangles are matched this way:

$J \leftrightarrow X$, $K \leftrightarrow Y$, and $L \leftrightarrow Z$

- **A** The correspondence above means that $\overline{JK} \cong \overline{XY}$. What other pairs of segments must be congruent?
- **B** The correspondence above also means that $\angle J \cong \angle X$. What other pairs of angles are congruent?

3.
- **A** Name the pairs of sides that are congruent.
- **B** Name the pairs of angles that are congruent.
- **C** What statement of congruence can be written for the two triangles?

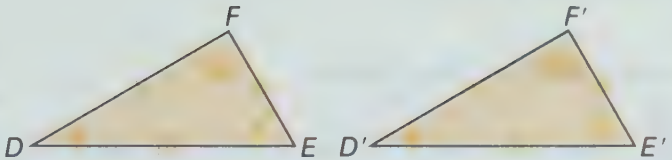


1. In the figure, $\triangle DEF \cong \triangle D'E'F'$.

A $\overline{DE} \cong$ ▒▒▒ **D** $\angle D \cong$ ▒▒▒

B $\overline{FE} \cong$ ▒▒▒ **E** $\angle E \cong$ ▒▒▒

C $\overline{DF} \cong$ ▒▒▒ **F** $\angle F \cong$ ▒▒▒



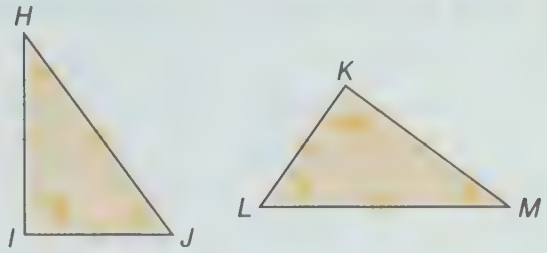
2. Complete each part.

A $\overline{HI} \cong$ ▒▒▒ **E** $\angle I \cong$ ▒▒▒

B $\overline{IJ} \cong$ ▒▒▒ **F** $\angle J \cong$ ▒▒▒

C $\overline{HJ} \cong$ ▒▒▒ **G** $\triangle HIJ \cong$ ▒▒▒

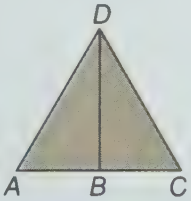
D $\angle H \cong$ ▒▒▒



3. $\triangle RST \cong \triangle XWY$. Write statements of congruence for the pairs of congruent segments and pairs of congruent angles.

4. In each figure, a pair of congruent triangles is shown. Decide which vertices correspond, then write a statement of congruence about the triangles.

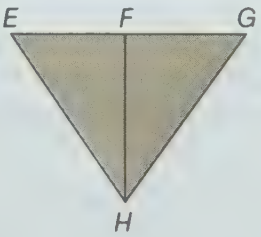
A



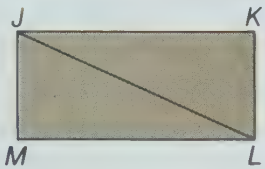
EXAMPLE:

$\triangle ABD \cong \triangle CBD$

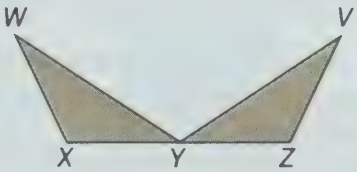
B



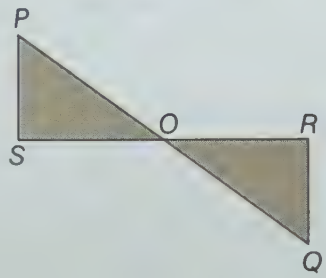
C



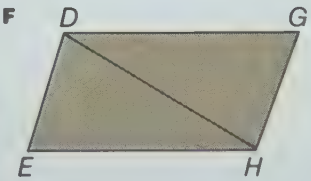
D



E

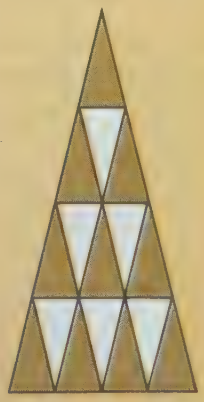


F



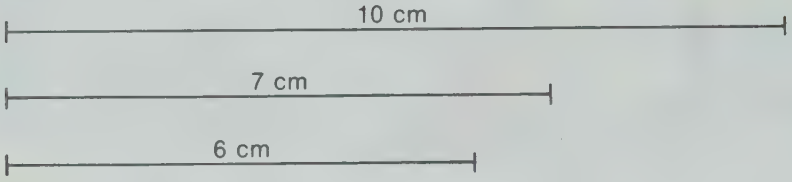
think

How many different triangles are in this figure?



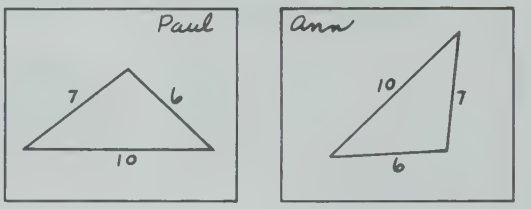
Investigating the Ideas

Can you make a drawing of a triangle whose sides are as long as the segments below?



Discussing the Ideas

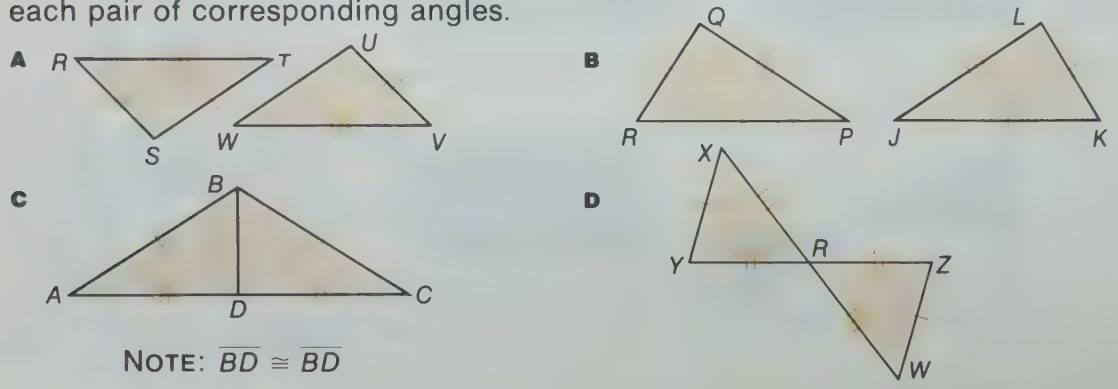
1. Compare the triangle that you drew with a triangle drawn by one of your classmates. You may have to rotate, translate, or reflect one of the triangles. Do the two triangles seem to be congruent?



2. From the result of the Investigation, you may be able to see that the following statement or **theorem** of geometry is true.

> The ide- ide- ide Theorem ( )
>
> If the sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.

Each pair of triangles below are congruent because of the SSS Theorem. Write a statement of congruence for each pair of corresponding angles.



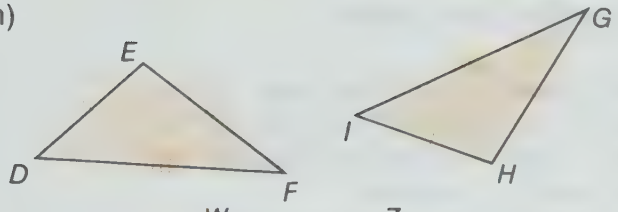
NOTE: $\overline{BD} \cong \overline{BD}$

1. **A** $\triangle ? \cong \triangle ?$. (SSS Theorem)

B $\angle D \cong \angle ?$

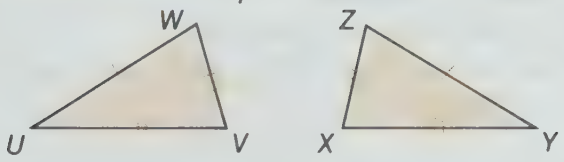
C $\angle E \cong \angle ?$

D $\angle F \cong \angle ?$



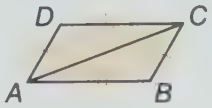
2. **A** $\triangle ? \cong \triangle ?$.

B Which pairs of angles are congruent?



3. The two triangles in each figure are congruent because of the SSS Theorem. You will also notice that the two triangles in each figure have a *common side*. Name the common side and then write statements of congruence for corresponding sides of the triangles.

EXAMPLE:

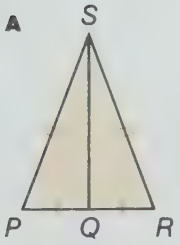


SOLUTION: *Common side* $\overline{AC}$

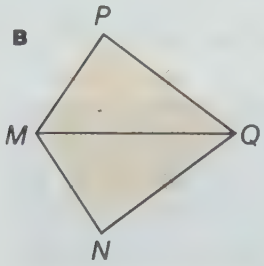
$\overline{AC} \cong \overline{AC}$, $\overline{AD} \cong \overline{CB}$, $\overline{AB} \cong \overline{CD}$

$\triangle ABC \cong \triangle CDA$

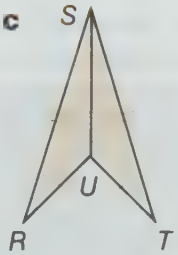
A



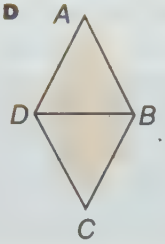
B



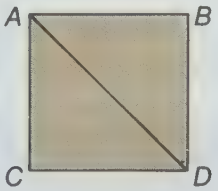
C



D



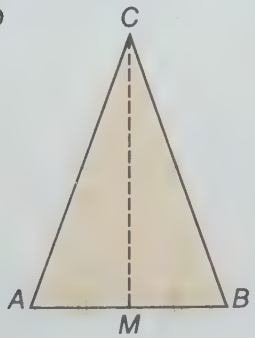
4. *ABCD* is a square. Show that $\triangle ABD \cong \triangle ACD$ by the SSS Theorem.



5. An **isosceles triangle** has a pair of congruent sides. Suppose $\triangle ABC$ is isosceles and $\overline{AC} \cong \overline{BC}$. Let M be the midpoint of $\overline{AB}$ so that $\overline{AM} \cong \overline{MB}$.

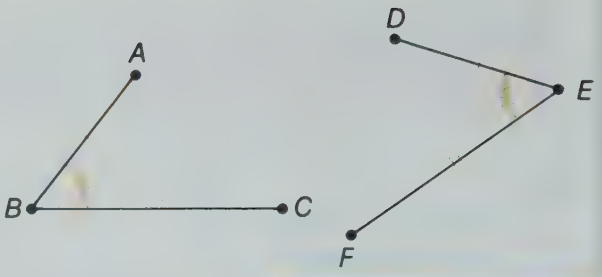
A Is $\triangle AMC \cong \triangle BMC$? Why?

B Is $\angle A \cong \angle B$?



Investigating the Ideas

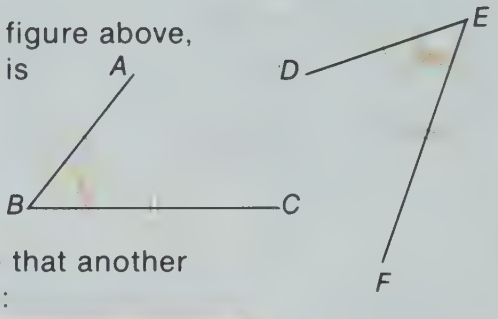
Suppose a rigid motion maps one figure onto the other so that the pairs of corresponding parts are congruent as marked.
Do you think the two triangles with vertices A, B, C and D, E, F will be congruent?



Can you trace the figures and complete the triangles, to check your guess?

Discussing the Ideas

1. From the parts marked congruent in the figure above, the correspondence for the rigid motion is $A \leftrightarrow D$, $B \leftrightarrow E$, and $C \leftrightarrow F$.



   A Is $\overline{AC} \cong \overline{DF}$?

   B Is $\triangle ABC \cong \triangle DEF$? Why?

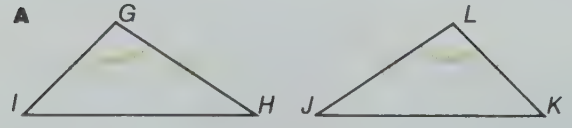
2. From the discussion above, you may see that another congruence theorem for triangles is true:

> The ide- ngle- ide Theorem ( )
>
> If two sides and the included angle of one triangle are respectively congruent to the corresponding parts of another triangle, then the triangles are congruent.

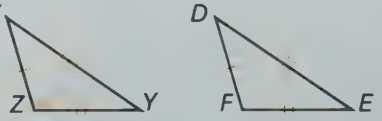
   In the Investigation picture, $\angle B$ is the "included angle" for sides $\overline{AB}$ and $\overline{BC}$. What sides form the included angle E?

3. Each pair of triangles below are congruent. Which congruence theorem, SSS or SAS, applies to each pair?

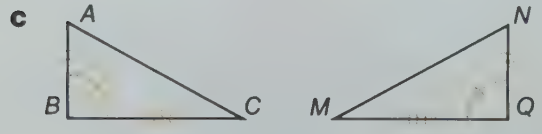
A



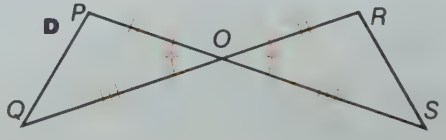
B



C



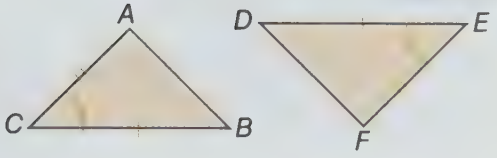
D



1. **A** $\triangle ? \cong \triangle ?$.

B Which congruence theorem is illustrated?

C $\overline{AB} \cong ?$; $\angle A \cong ?$; $\angle B \cong ?$

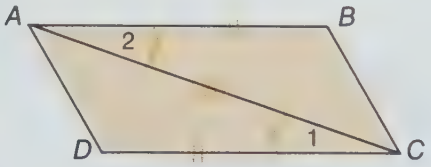


2. Complete each statement.

A $\overline{AB} \cong ?$ **B** $\overline{AC} \cong ?$

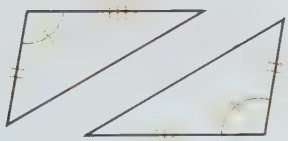
C $\angle 1 \cong ?$

D $\triangle ACD \simeq \triangle ?$ because of the ? Theorem.

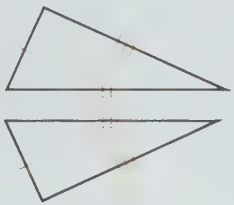


3. In each pair of triangles, pairs of congruent sides or angles are marked. Tell whether each pair of triangles illustrates the SSS Theorem or the SAS Theorem.

A

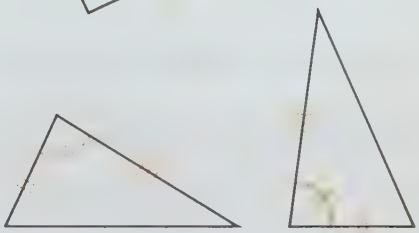


B



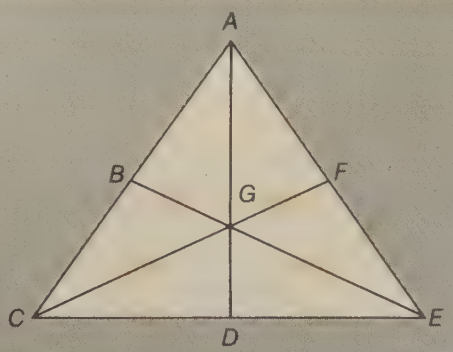
C

D



think

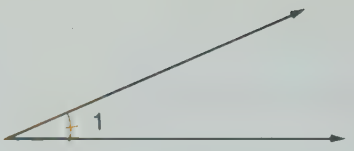
How many pairs of congruent triangles can you find and name for this figure?

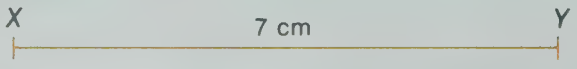


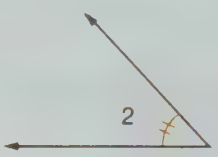
The Angle-Side-Angle Theorem

Investigating the Ideas

Make a drawing or tracing of a triangle, $\triangle ABC$, in which $\angle A \cong \angle 1$, $\angle B \cong \angle 2$, and $\overline{AB} \cong \overline{XY}$.



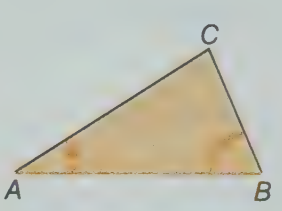




Can you compare your triangle with a classmate's triangle to see whether they are congruent?

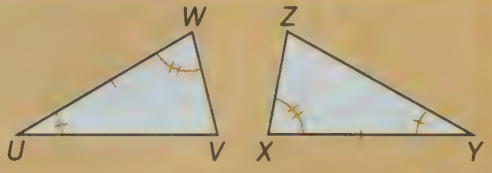
Discussing the Ideas

1. In $\triangle ABC$ of the Investigation, $\overline{AB}$ is said to be the "included side" with respect to angles A and B. Does the investigation suggest another theorem about congruent triangles? Can you state it?



2. The Investigation suggests this congruence theorem:

> The ngle- ide- ngle Theorem ( )
>
> If two angles and the included side of one triangle are congruent to the corresponding parts of another triangle, then the triangles are congruent.



   A What is the "included side" for $\triangle UVW$? for $\triangle XYZ$?
   B Which side is congruent to $\overline{WV}$? to $\overline{UV}$?
   C Which angle is congruent to $\angle V$?

3. Each pair of triangles are congruent. Which congruence theorem (SSS, SAS, or ASA) applies to each pair?

A

B

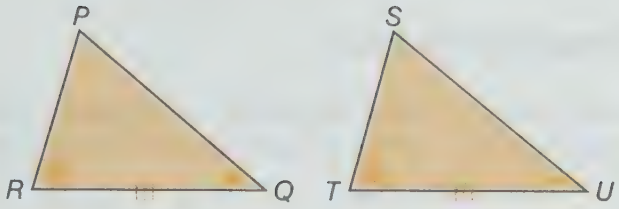
C

1. Complete each statement.

A $\triangle PQR \cong \triangle$? because of the ? Theorem.

B $\angle P \cong \angle$?

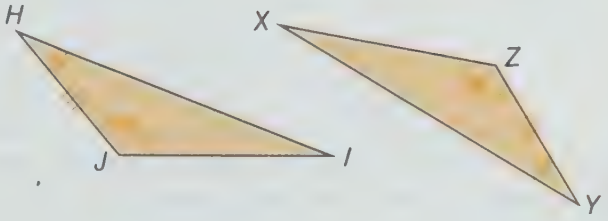
C $\overline{RP} \cong$? and $\overline{PQ} \cong$?



2. Complete each statement.

A $\triangle XYZ \cong \triangle$?

B $\overline{JI} \cong$?, $\overline{HI} \cong$?, and $\angle X \cong \angle$?



3. Each part illustrates either the SSS, SAS, or ASA congruence theorem. Give the correct theorem.

A

B

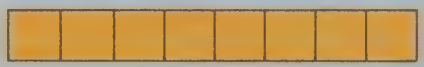
C

D

4. Tell which single rigid motion could be used to map one of the triangles in each part of Exercise 3 onto the other triangle.

think

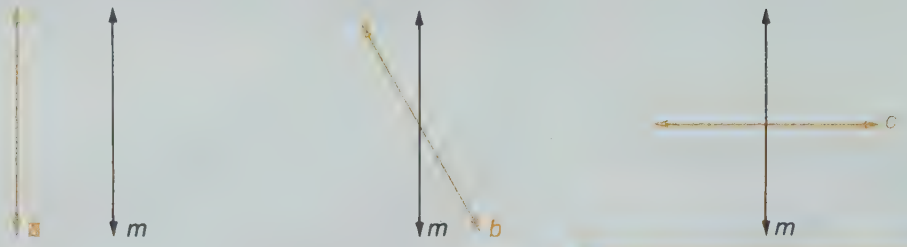
Make a strip of squares, eight squares long by one square wide.



1. Fold the strip into a model of a cube. (Some squares will be hidden.)
2. Can you make a cube using a strip only seven squares long?

Investigating the Ideas

Trace or copy these figures on your paper. Then use a clear plastic mirror to complete this Investigation.



Can you find the reflection image in line m for each of the lines a, b, and c?

Discussing the Ideas

1. If lines a and m do not intersect, they are **parallel lines** and we write $a \parallel m$. Suppose the reflection image of a is a'. Are lines a and a' parallel?

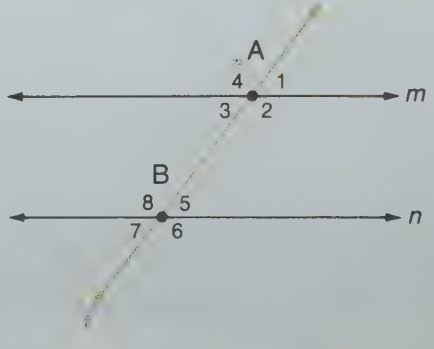
2. What kind of lines are b and m, *parallel* or *intersecting*? What can you say about line b and its reflection image?

3. How are line c and its reflection image related?

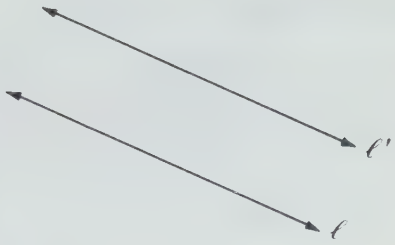
4. Since c is reflected onto itself by m, we say that c and m are **perpendicular** lines and write this as $c \perp m$. What kind of angle do perpendicular lines form?

5. Suppose a line ℓ is translated by some arrow $\boldsymbol{a}$. What is the relation between line ℓ and its translation image?

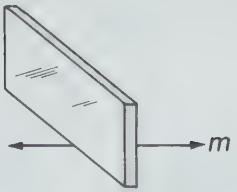
6. Lines m and n are parallel. Line t is a **transversal** of m and n.
   - **A** Is $\angle 1 \cong \angle 3$? (HINT: Think about a $\frac{1}{2}$-clockwise rotation with centre A.)
   - **B** Is $\angle 1 \cong \angle 5$? (HINT: Think about a translation from A to B.)



1. Draw a line ℓ on your paper. Use a translation to show a line ℓ' which is parallel to ℓ.



2. Draw a line m on your paper. Use a clear plastic mirror to help locate a line n which is perpendicular to m.



3. Perpendicular lines form **right angles** (SYMBOL:). Are two right angles congruent to each other?

4. Make a drawing of lines ℓ, m, and t such that $\ell \parallel m$ and $\ell \perp t$. What relationship is there between lines m and t?

5. Make a sketch of lines r, s, and t for these conditions:

$r \perp s$ at point A
$r \perp t$ at point B and $A \neq B$

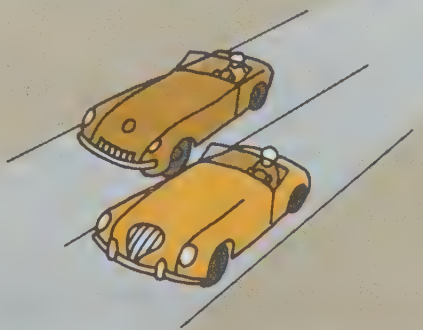
How are lines s and t related?

6. Make a sketch of lines a, b, and c such that $a \parallel b$ and $b \parallel c$. How are lines a and c related?

think

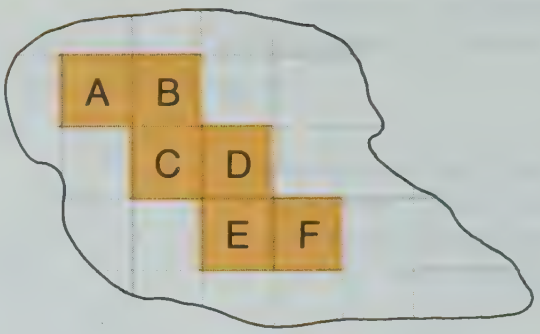
Two race cars sped two laps around a track. The first car averaged 200 km/h on the first lap and 240 km/h on the second lap. The second car averaged 220 km/h for both laps.

Which car won the race, or was it a tie?



Investigating the Ideas

Suppose you cut out this shape from graph paper and fold it along the dotted segments to form a cube.



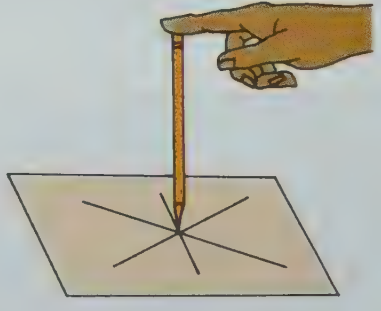
Can you decide which pairs of lettered faces will be opposite each other on the cube?

Discussing the Ideas

1. The pairs of faces of the cube that are opposite each other lie in **parallel planes**. How would you define parallel planes?

2. Two of the faces that intersect, such as A and B, are in **perpendicular planes**. Which other pairs of faces of the cube are in perpendicular planes?

3. Draw three lines that intersect at one point on your paper.
   - **A** Can you hold your pencil so that it is perpendicular to exactly one of the lines?
   - **B** Can you hold your pencil so that it is perpendicular to two of the lines but not the third line?

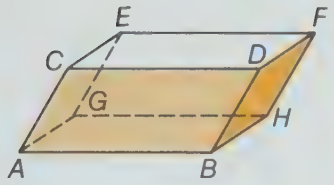


4. What objects in your room suggest parallel planes?

5. What objects in your room suggest perpendicular planes?

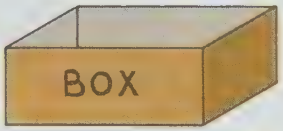
1. The figure shows a **parallelepiped**. The opposite faces are congruent parallelograms.

- **A** How many faces does it have?
- **B** Which face is opposite *DFHB*?
- **C** Are the opposite faces parallel?



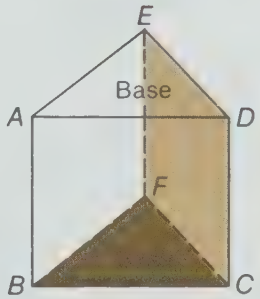
2. A "box" is a *rectangular parallelepiped*.

- **A** Are the intersecting faces perpendicular to each other?
- **B** What shape is each face?



3. **Prisms** have bases which are congruent polygons in parallel planes and **lateral faces** which are parallelograms.

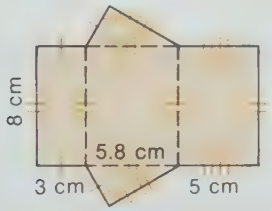
- **A** Name the bases in this figure.
- **B** Quadrilateral *ABCD* is a lateral face. Name the other lateral faces.



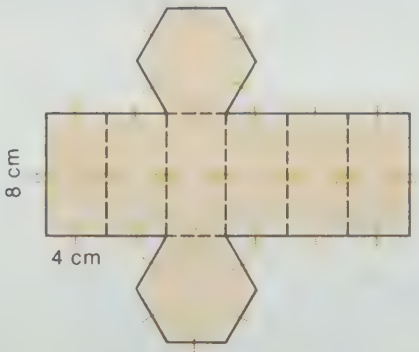
Triangular Prism

4. Prisms are named according to their kind of base. What kind of prism would these patterns make?

A



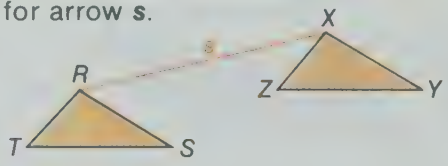
B



★ **5.** Make the models shown in Exercise 4.

REVIEWING THE IDEAS

1. $\triangle XYZ$ is the translation image of $\triangle RST$ for arrow **s**.



A What is the image of point T?

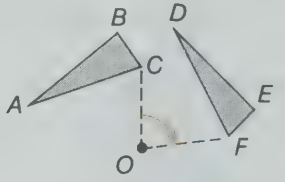
B What is the image of $\overline{RS}$?

2. **A** Draw a figure on graph paper.

B Show a "right 2, down 3" translation of the figure.

C Translate the image of part **B** "left 4, up 1."

3. The figure shows a rotation with centre O.

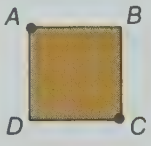


A Which direction is the rotation?

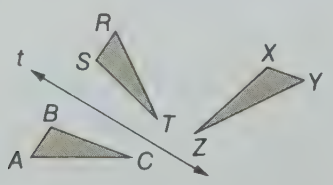
B What is the image of point B?

C What is the image of $\angle BAC$?

4. Draw a square $ABCD$. Show the rotation image of $ABCD$ after a $\frac{3}{4}$-counterclockwise rotation about point C.



5. Which triangle is the reflection image of $\triangle ABC$ in line t?



6. How many lines of symmetry does a rectangle have?

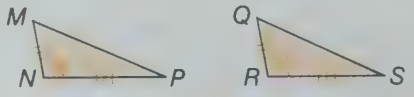
7. Draw an equilateral triangle. Show all its lines of symmetry.

8. True or false?

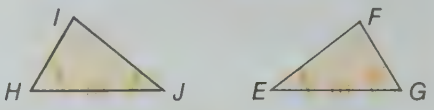
The image of any geometric figure for any rigid motion is a figure congruent to the original figure.

9. Write a statement of congruence for the triangles in each part and tell whether the SSS, SAS, or the ASA congruence theorem applies.

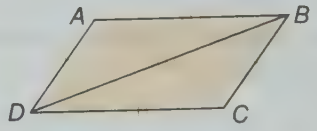
A



B

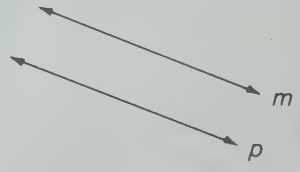


C

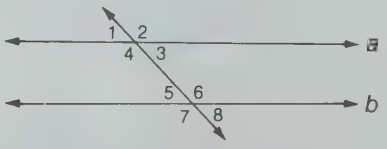


10. Line p is a translation image of line m.

What is the relation of m to p?



11. Lines a and b are parallel.

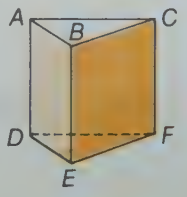


A Name all the angles that are congruent to $\angle 3$.

B Name all the angles that are congruent to $\angle 2$.

12. **A** Name the bases of the prism.

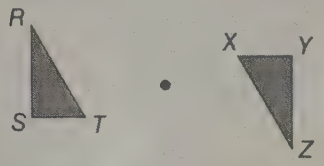
B What shape are the lateral faces of the prism?



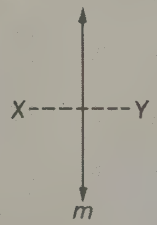
TEST YOURSELF

1. Which rigid motion can be thought of as a motion that moves all points of the plane the same distance and in the same direction?

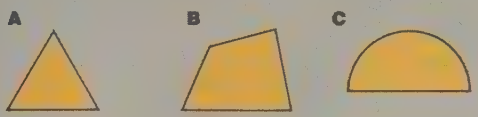
2. In the figure, $\triangle RST \cong \triangle XYZ$. What single rigid motion would map $\triangle RST$ onto $\triangle XYZ$?



3. Point X is the reflection image of Y in line m. How is $\overline{XY}$ related to line m?



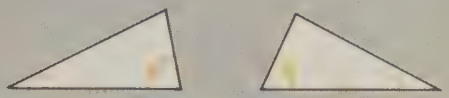
4. Which figure does not have a line of symmetry?

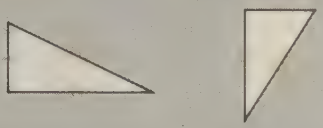


5. How many lines of symmetry does a square have?

6. Suppose $\triangle GHI \cong \triangle RST$.
   - A $\overline{HI} \cong$?
   - B $\angle G \cong$?

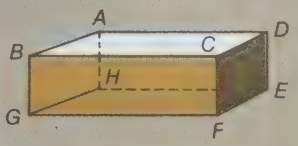
In Exercises 7, 8, and 9, some pairs of congruent angles and segments are marked. Tell which congruence theorem (SSS, SAS, or ASA) can be used to show the pair of triangles are congruent.

7. 

8. 

9.

10. This figure is a rectangular parallelepiped.



   - A Faces $ABCD$ and ? are parallel to each other.
   - B Faces $BGFC$ and $CDEF$ are ? to each other.

RESEARCH PROJECTS

A Investigate the topic of mazes. Try constructing some mazes and see if your classmates can find their way into or out of your maze. (See *The Second Scientific American Book of Puzzles and Diversions* by Martin Gardner; New York: Simon and Schuster, 1960, pp. 112–118.)

B Mathematicians think that every map in a plane can be colored with no more than four colors regardless of the number of countries on the map. However, no one has been able to prove this. Obtain some outline maps and color the states, provinces, or counties on the map with four or fewer colors. Try making your own maps and coloring them.

HEXOMINOS

A **hexomino** is a pattern of six connected squares.

You can cut out hexominos from graph paper.

The hexomino in **A** can be folded along the segments to form a cube.

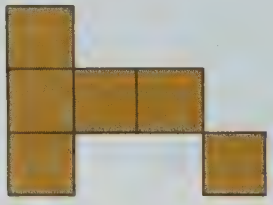
A

Hexomino **B** cannot be folded to form a cube.

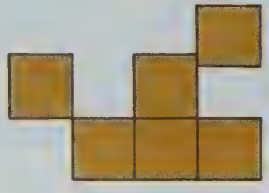
B

Regions like **C** and **D** are not hexominos.

C



D



1. How many different hexominos can you show on graph paper?

2. How many hexominos can be folded to form cubes?

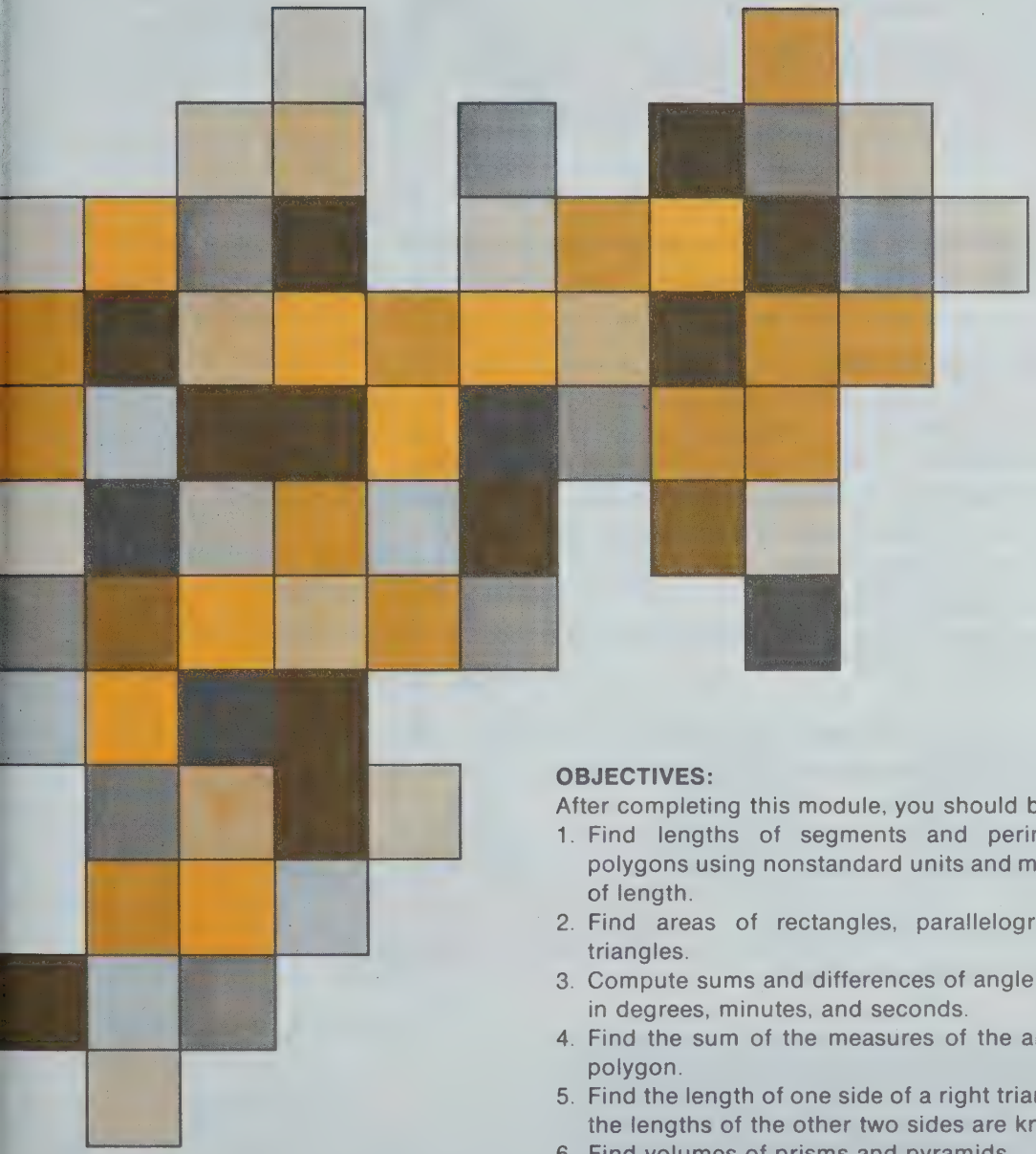
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. translation **2.** rotation **3.** $m \perp$ bisector of $\overline{XY}$
4. **B** **5.** 4 **6.** **A** $\overline{ST}$; **B** $\angle R$ **7.** ASA **8.** SSS
9. SAS **10.** **A** *HGFE*; **B** perpendicular

UNIT G: *Geometry*

MODULE 2: *Measurement*



OBJECTIVES:

After completing this module, you should be able to:

1. Find lengths of segments and perimeters of polygons using nonstandard units and metric units of length.
2. Find areas of rectangles, parallelograms, and triangles.
3. Compute sums and differences of angle measures in degrees, minutes, and seconds.
4. Find the sum of the measures of the angles of a polygon.
5. Find the length of one side of a right triangle when the lengths of the other two sides are known.
6. Find volumes of prisms and pyramids.
7. Find the capacity of rectangular prisms using metric units.

Investigating the Ideas

Suppose $\overline{AB}$ represents one of your steps (or paces) when you walk. About how many steps would you take to walk from X to Y?

Estimate how many of your actual steps it would take to walk across your classroom. Then check your guess.

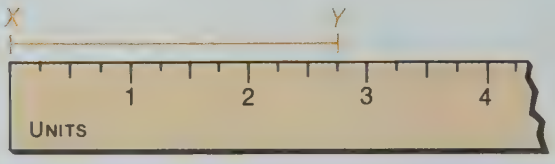
Discussing the Ideas

1. To find the **length** of a segment, we first choose a *unit segment*. Then we count the number of times the unit segment is contained in the segment to be measured. If $\overline{AB}$ is the unit segment in the Investigation, what is the length of $\overline{XY}$?

2. If one of your steps is the unit, what is the length of your classroom?

3. Why might two students get different numbers for the length of the classroom using their steps as units?

4. The length of $\overline{XY}$, **to the nearest unit**, is 3 units. We write: $\ell(\overline{XY}) = 3$ units. How can you find the length of $\overline{XY}$ **to the nearest half unit**?



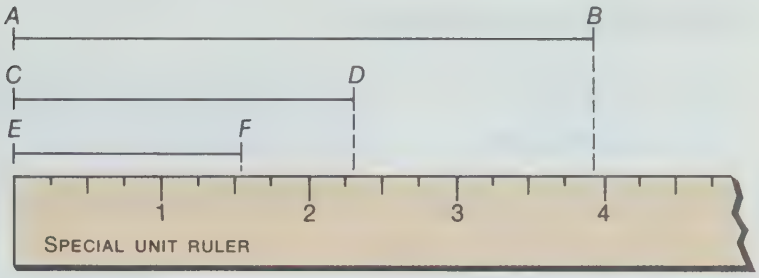
5. A Use a segment as long as $\overline{MN}$ as the unit. Measure the distance across your desk top.

   B Divide $\overline{MN}$ into ten smaller units. Measure the distance across your desk top to the nearest tenth of a unit.

1. **A** Give the length of each segment to the nearest unit.

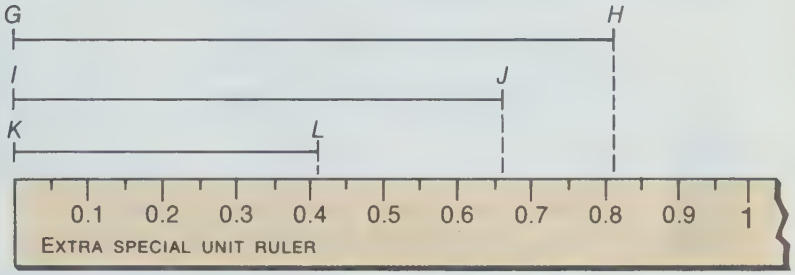
B Give the length of each segment to the nearest half unit.

C Find the sum of the lengths for part **B**.

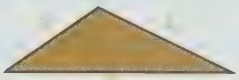


2. **A** What is the length of each segment to the nearest tenth of a unit?

B Find the sum of the lengths for part **A**.



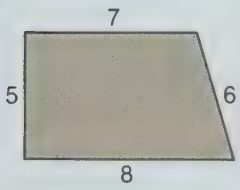
3. The **perimeter** of a region bounded by segments is the sum of the lengths of the segments bounding the region.



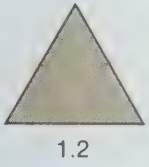
Perimeter = 3 + 6 + 4 = 13 units

Find the perimeter of each region.

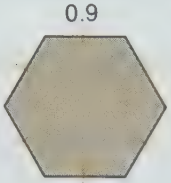
A



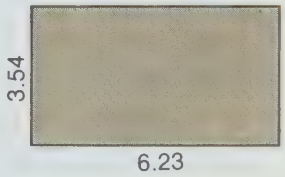
B



C



D



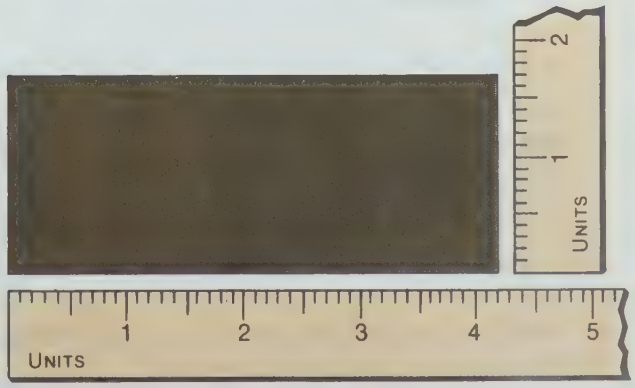
4. Give the length and width of the rectangle

A to the nearest unit.

B to the nearest half unit.

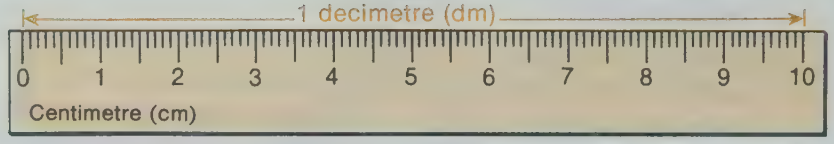
C to the nearest tenth unit.

5. What is the perimeter of the rectangle using the measurements of Exercise 4, part **A**? **B**? **C**?



Investigating the Ideas

The **metre** (m) is the standard unit for measuring length. Smaller or larger units are simply related to the metre by powers of ten.



Thus,

1 decimetre is 0.1 metre.	10 dm = 1 m
1 centimetre is 0.01 metre.	100 cm = 1 m
1 millimetre is 0.001 metre.	1000 mm = 1 m

Can you carry out one or more of the measurement activities below and record your results?

A Find the width of each finger on one of your hands in millimetres.

B Find your height to the nearest decimetre.

C Find the length or width of your classroom to the nearest metre.

Discussing the Ideas

1. The prefixes in the metric system tell the part or multiple of the basic unit. Give the missing number for each ▮.

   A 1 centimetre (cm) = ▮ metre (m)

   B 1 hectometre (hm) = ▮ m

   C 1 decametre (dam) = ▮ m

   D 1 kilometre (km) = ▮ m

Prefix	Meaning
milli-	one thousandth
centi-	one hundredth
deci-	one tenth
deca-	ten
hecto-	one hundred
kilo-	one thousand

2. The most frequently used units of length are the metre, kilometre, centimetre, and millimetre. Which unit or units would most likely be used to measure each of the following:

   A The length of a hockey field.

   B Airline distance between cities.

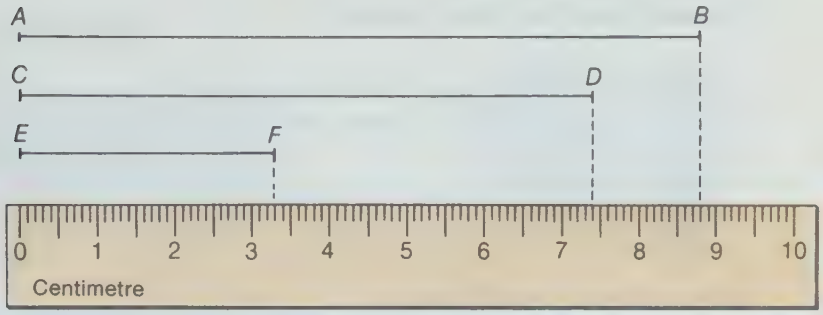
   C The thickness of a coin.

   D A person's waist.

3. What would be the width of your middle finger in centimetres if it were 19 millimetres wide?

1. Give the length of each segment to the nearest

- **A** centimetre.
- **B** half centimetre.
- **C** millimetre.



2. Express each measure in metres.

A 500 cm	**D** 2 km	**G** 47 dam	**J** 25 cm
B 30 dm	**E** 1.6 km	**H** 15 hm	**K** 0.001 km
C 1200 mm	**F** 227 cm	**I** 358 dm	**L** 0.1 dam

3. Express the measurement given in each statement in metres.

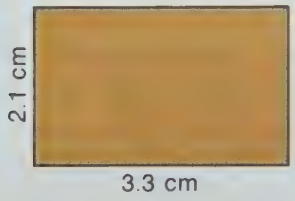
- **A** Some Australian earthworms are 152 cm long.
- **B** A giraffe may be 5480 mm in height.
- **C** A large blue whale was 0.0276 km long.
- **D** A large African elephant may be 366 cm tall.
- **E** A large polar bear may be 34 dm in length.

4. Each measurement is given in metres. Express it in millimetres.

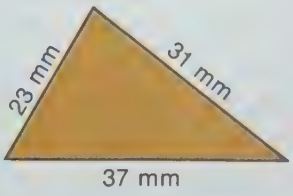
- **A** The world's smallest antelope is only about 0.25 m tall.
- **B** The white-tooth shrew is about 0.038 m long.
- **C** The eye of a giant squid is about 0.3 m in diameter.
- **D** A small hummingbird has a wingspan of about 0.005 m.

5. Find the perimeter of each region.

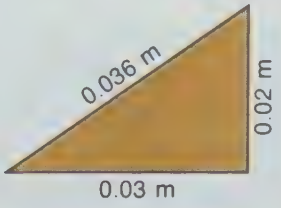
A



B



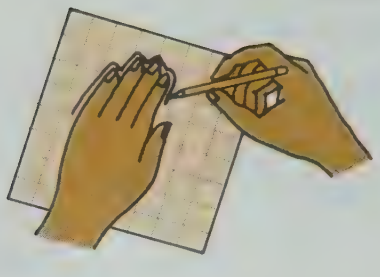
C



More practice, page S-22, Set 41

Investigating the Ideas

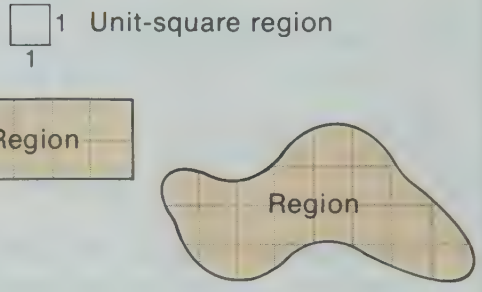
Place your hand on a sheet of graph paper and trace around it.



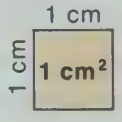
About how many squares does your hand cover?

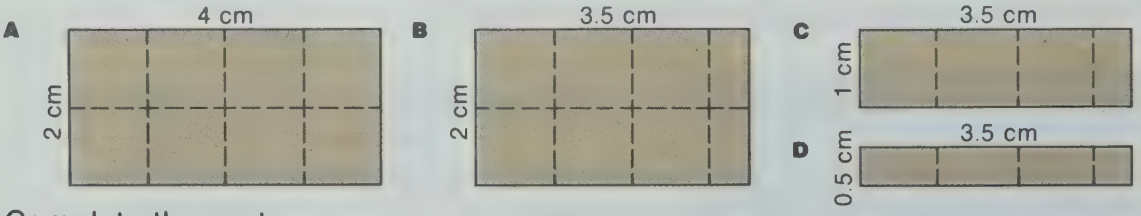
Discussing the Ideas

1. To find the **area** of a region, we think of dividing it into *unit-square regions* and then counting the number of units. Even if the borders are curved, we still "think" of counting unit-square regions to find the area. What is the area of your hand using the square region of the graph paper as the unit?



2. If the unit square is a square that is 1 cm on each edge, then the area is expressed in *square centimetres* (cm^2). Explain how to find the area of each rectangular region.

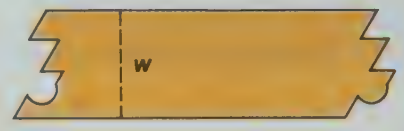




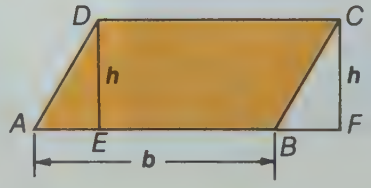
3. Complete the sentence.

The formula for the area A of a rectangle with length ℓ and width w is ? .

4. "Cutting apart" and "gluing together" can sometimes help you to find area. Use this figure to explain this idea.



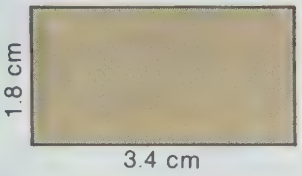
5. Parallelogram *ABCD* has a **base** of length b and **height** of length h.
   - A What is the length and width of rectangle *DEFC*?
   - B Using the letters b and h, give a formula for the area of the parallelogram.
   - C What is the area if $b = 3.7$ cm and $h = 1.1$ cm?



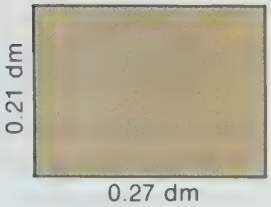
1. Find the area of each rectangular region.

A

B



C

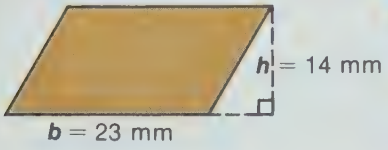


2. A rectangular room is 4.7 metres long and 4.4 metres wide. What is the area of the room?

3. What would it cost to carpet the floor of the room in Exercise 2 if carpeting costs $11.98 per square metre?

4. What is the area of each parallelogram?

A



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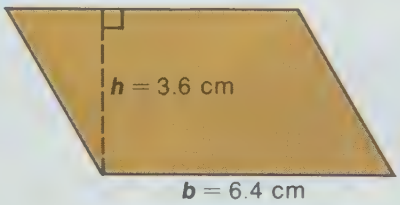
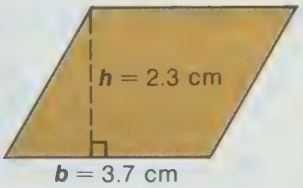
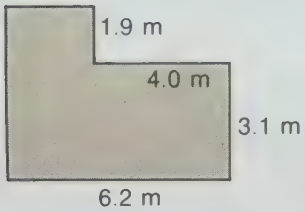


C

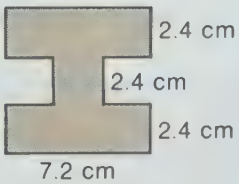


5. Find the area of each region.

A



B

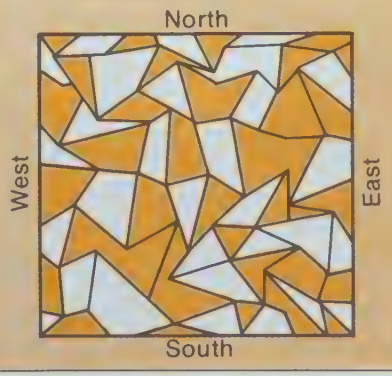


6. A house has a rectangular base that is 22.4 metres long and 18.9 metres wide. The house is located on a rectangular lot 53.6 metres long by 32.6 metres wide.

- **A** What is the area of the base of the house?
- **B** What is the area of the lot?
- **C** What is the area of the part of the lot not occupied by the house?

think

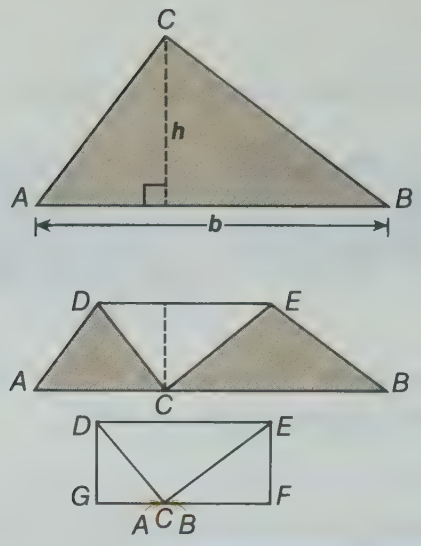
What is the smallest number of "fences" you can cross in going from the west side to the east side of the square if you cannot cross at any of the points of intersection?



More practice, page S-22, Set 42

Investigating the Ideas

Draw a large triangular region on a sheet of paper. Let $\angle C$ be the largest angle of the triangle. Let h be the height of the triangle to the base b.
Cut out the region.
Fold vertex C to base b so that the fold line $\overline{DE}$ is parallel to $\overline{AB}$.
Fold vertices A and B to point C.
Measure rectangle $DEFG$ and compute its area.



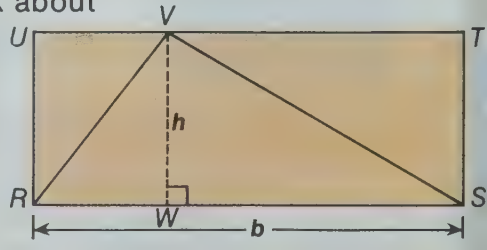
Can you determine the area of the original triangle by knowing the area of rectangle $DEFG$?

Discussing the Ideas

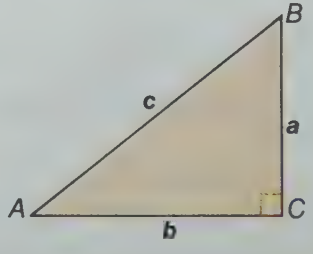
1. A Explain why the width of rectangle $DEFG$ is $\frac{1}{2} \cdot h$, which is the height of $\triangle ABC$.
   B Why is the length of the rectangle $\frac{1}{2} \cdot b$, where b is the base of $\triangle ABC$?

2. Since the area of rectangle $DEFG$ is $\left(\frac{1}{2} \cdot b\right) \cdot \left(\frac{1}{2} \cdot h\right) = \frac{1}{4}bh$, why is the formula for the area A of the triangle $A = \frac{1}{2}bh$?

3. The figure below suggests another way to think about the area of a triangle.
   A What is the area of rectangle $RSTU$?
   B Find two pairs of congruent triangles in the figure.
   C Explain the formula: Area $(\triangle RSV) = \frac{1}{2} \cdot b \cdot h$

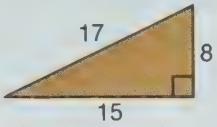


4. In right $\triangle ABC$, $\angle C$ is a right angle, a and b are legs, and c is the *hypotenuse*.
   A If b is chosen as the base, what is the height?
   B If a is the base, what is the height?
   C What rule can you state for finding the area of a right triangle?

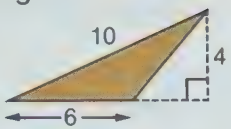


1. Find the area of each triangular region.

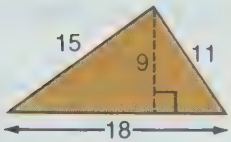
A



B

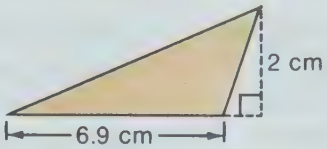


C

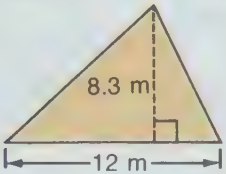


2. Give the area of each triangular region.

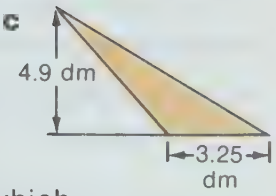
A



B

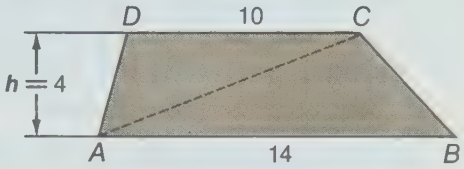


C

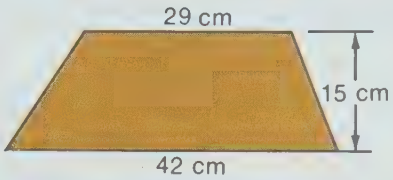


3. A **trapezoid** is a quadrilateral with a pair of parallel sides which are called the **bases** of the trapezoid.

- **A** Find the area of $\triangle ABC$.
- **B** Find the area of $\triangle ADC$.
- **C** Find the area of the trapezoid $ABCD$.

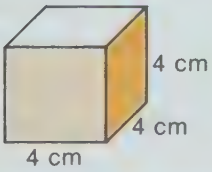


4. Use the method suggested in Exercise 3 to find the area of the trapezoid shown at the right.

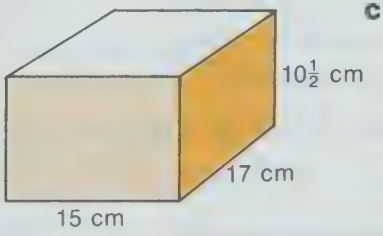


5. The **surface area** of a space figure is the sum of the areas of each face of the figure. Find the surface area of each figure.

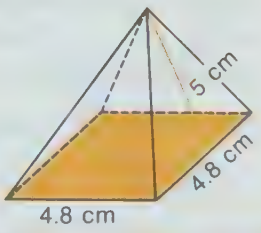
A



B



C



think

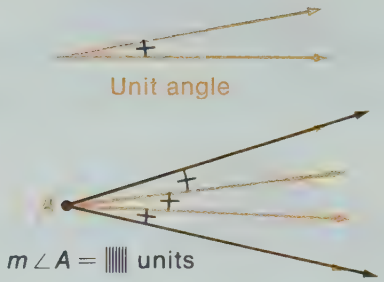
Think of any number except zero. Add 5 to it and square the sum. Subtract 16 and remember the resulting number. Call this number *A*. Now subtract 3 from the original number and square the difference. Subtract the square from *A*. Divide the resulting number by the original number. What number do you get as the answer? Try this procedure again. What is your answer this time? Do you think the result will always be the same?

More practice, page S-23, Set 43

Discussing the Ideas

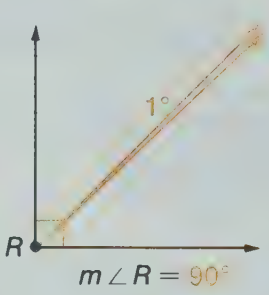
1. To measure an angle, first choose a unit angle. Then, on the given angle, "stack up" adjacent angles congruent to the unit angle. Finally, count the number of units.

   Using the given unit angle, what is the measure of $\angle A$?



2. Although a unit angle of any size may be used, the standard unit is the angle of one **degree** (1°). The size of a degree is such that the measure of a right angle is 90 degrees.

   Estimate the measure of each angle below in degrees.



3. When units smaller than one degree are needed, the unit angle of 1° is divided into 60 congruent angles, each called a **minute** (1′), and each angle of 1′ is divided into 60 congruent subunits called **seconds** (1″).

   60 minutes (60′) = 1°
   60 seconds (60″) = 1′

   What part of a degree is each of the following?

   A 30′ **B** 15′ **C** 10′ **D** 6′ **E** 1′ **F** 30″ **G** 15″

4. The example shows how sums of angle measures are found.

   Explain how 52° 105′ 81″ was rewritten as 53° 46′ 21″.

EXAMPLE:

$$\begin{array}{r} 37^\circ\ 48'\ 35'' \\ +\ 15^\circ\ 57'\ 46'' \\ \hline 52^\circ\ 105'\ 81'' \end{array} = 52^\circ\ 106'\ 21'' = 53^\circ\ 46'\ 21''$$

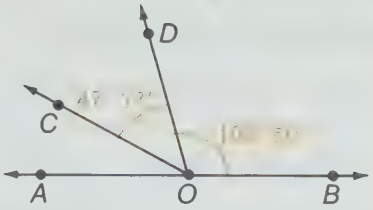
5. Explain each step in the subtraction example below.

EXAMPLE:

$$\begin{array}{r} 37^\circ\ 48'\ 35'' \\ -\ 15^\circ\ 57'\ 46'' \end{array} \rightarrow \begin{array}{r} 37^\circ\ 47'\ 95'' \\ -\ 15^\circ\ 57'\ 46'' \end{array} \rightarrow \begin{array}{r} 36^\circ\ 107'\ 95'' \\ -\ 15^\circ\ 57'\ 46'' \\ \hline 21^\circ\ 50'\ 49'' \end{array}$$

Using the Ideas

1. If $\angle AOB$ is a straight angle, what is the measure of each of the following angles?

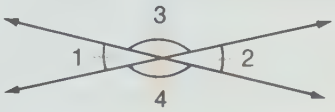


A $\angle COB$ **B** $\angle COA$ **C** $\angle AOD$

2. Two angles are **complementary** if the sum of their measures is 90°. What is the measure of an angle which is complementary to an angle whose measure is 70° 19′?

3. Two angles which are **supplementary** must have the sum of their measures equal to 180°. What is the measure of an angle which is supplementary to an angle whose measure is 125° 38′ 15″?

4. Angles 1 and 2 are **vertical angles**. Angles 3 and 4 are also vertical angles. The $m\angle 1$ is 30°.



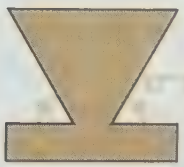
A What is $m\angle 2$? **B** What is $m\angle 1 + m\angle 3$?

C What is $m\angle 3$? **D** What is $m\angle 4$?

E Is it true that vertical angles have the same measure?

5. A ray of light striking a mirror is reflected so that the ray makes congruent angles with the mirror. Thus, $\angle APC \cong \angle BPD$. If $m\angle APC$ is 23° 40′, what is $m\angle APB$?

6. A machinist must grind the angle on a dovetail piece as shown in the figure.



A What is the greatest measure the angle may have?

B What is the smallest measure the angle may have?

7. Find the sums of the angle measures.

A
$$\begin{array}{r} 72^\circ\,49'\,31'' \\ +\ 38^\circ\ 6'\ 19'' \\ \hline \end{array}$$

B
$$\begin{array}{r} 64^\circ\,57'\,26'' \\ +\ 82^\circ\,14'\,38'' \\ \hline \end{array}$$

C
$$\begin{array}{r} 152^\circ\,50'\,10'' \\ +\ 27^\circ\ 9'\ 50'' \\ \hline \end{array}$$

D
$$\begin{array}{r} 35^\circ\,17'\,43'' \\ +\ 91^\circ\,22'\,30'' \\ \hline \end{array}$$

8. Find the differences of the angle measures.

A
$$\begin{array}{r} 84^\circ\,53'\,46'' \\ -\ 25^\circ\,39'\,37'' \\ \hline \end{array}$$

B
$$\begin{array}{r} 90^\circ \\ -\ 38^\circ\,17'\,40'' \\ \hline \end{array}$$

C
$$\begin{array}{r} 111^\circ\,11'\,11'' \\ -\ 22^\circ\,22'\,22'' \\ \hline \end{array}$$

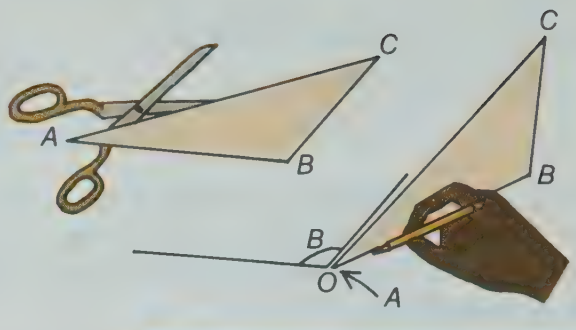
D
$$\begin{array}{r} 142^\circ\,19'\,25'' \\ -\ 106^\circ\,34'\,17'' \\ \hline \end{array}$$

More practice, page S-23, Set 44

Investigating the Ideas

Draw a triangle on a piece of cardboard and cut out the triangular region.

Mark a point O on your paper. Then use the triangular region to copy each of its three angles side-by-side so that each of the vertices is at point O.

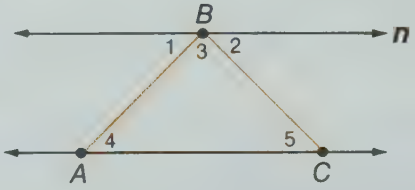


Can you try the Investigation with several triangles of different shapes?

Discussing the Ideas

1. What does the Investigation suggest about the three angles of any triangle?

2. Suppose that $n \parallel \overleftrightarrow{AC}$.



   A Is $\angle 1 \cong \angle 4$?

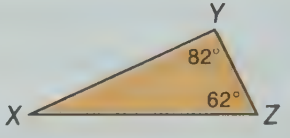
   B Is $\angle 2 \cong \angle 5$?

   C What is the sum $m\angle 1 + m\angle 2 + m\angle 3$?

   D Explain why $m\angle 3 + m\angle 4 + m\angle 5 = 180°$?

3. Complete: The sum of the measures of the angles of every triangle is ? .

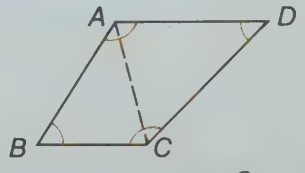
4. Explain how to find $m\angle X$ in $\triangle XYZ$.

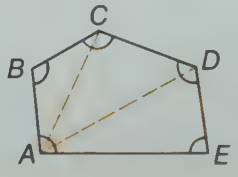


5. A What is the sum of the measures of the angles of $\triangle ABC$?

   B What is the sum of the measures of the angles of $\triangle ACD$?

   C What is the sum of the measures of the angles in any convex quadrilateral?

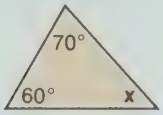




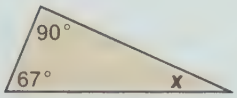
6. Explain how to find the sum of the angle measures of a pentagon.

1. Give the measure for **x** in each triangle.

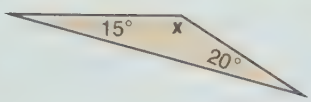
A



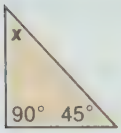
B



C



D

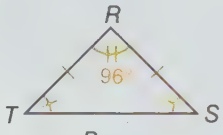


2. The measures of two angles of a triangle are given. Find the measure of the third angle.

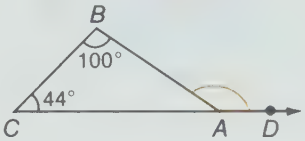
A 45°, 45°
B 50°, 50°
C 43°, 63°
D 75°, 28°
E 27° 23′, 59° 8′
F 14° 50′, 98° 15′
G 15° 28′, 69° 28′
H 60° 4′ 5″, 36° 25″

3. What is the measure of each angle of an equilateral triangle?

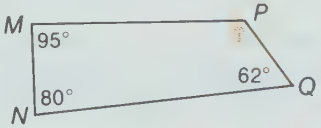
4. The **apex angle**, $\angle R$, of an isosceles triangle has a measure of 96°. What are the measures of the **base angles**?



5. An **exterior angle** at vertex A of $\triangle ABC$ is $\angle BAD$. What is $m\angle BAD$?



6. What is $m\angle P$ of quadrilateral $MPQN$?

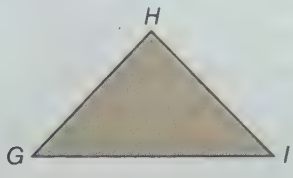


7. **A** What is the sum of the measures of the five angles of a regular pentagon?
B What is the measure of each angle?

8. **A** What is the sum of the measures of the angles of a regular hexagon?
B What is the measure of each angle?

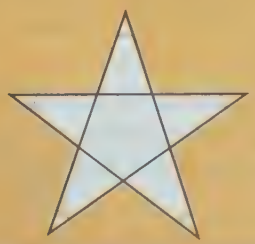
9. **A** What is the sum of the measures of the angles of a regular decagon?
B What is the measure of each angle?

10. In $\triangle GHI$, $m\angle H$ is twice $m\angle G$, and $m\angle I$ is 5° more than $m\angle G$. What are the measures of the angles?



think

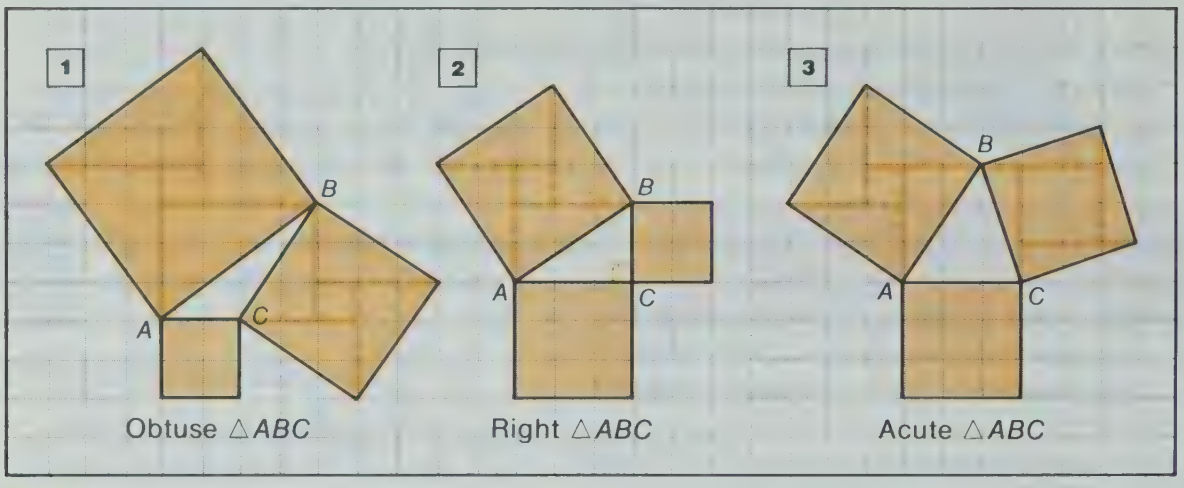
What is the sum of the measures of the angles at the tips of this regular 5-pointed star?



The Pythagorean Theorem

Investigating the Ideas

In the figures below, squares have been constructed on the three sides of obtuse, right, and acute triangles.



Can you find and record the area of each square in each of the figures?

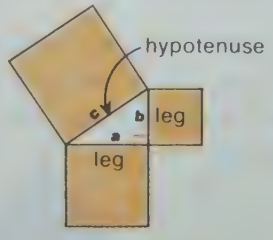
The dotted segments may help you count the unit squares.

Discussing the Ideas

1. Copy and complete the table.

	Area of square on *AB*	Area of square on *AC*	Area of square on *BC*	Sum of the areas of squares on *AC* and *BC*
Obtuse △*ABC*	25	4	13	
Right △*ABC*				
Acute △*ABC*				

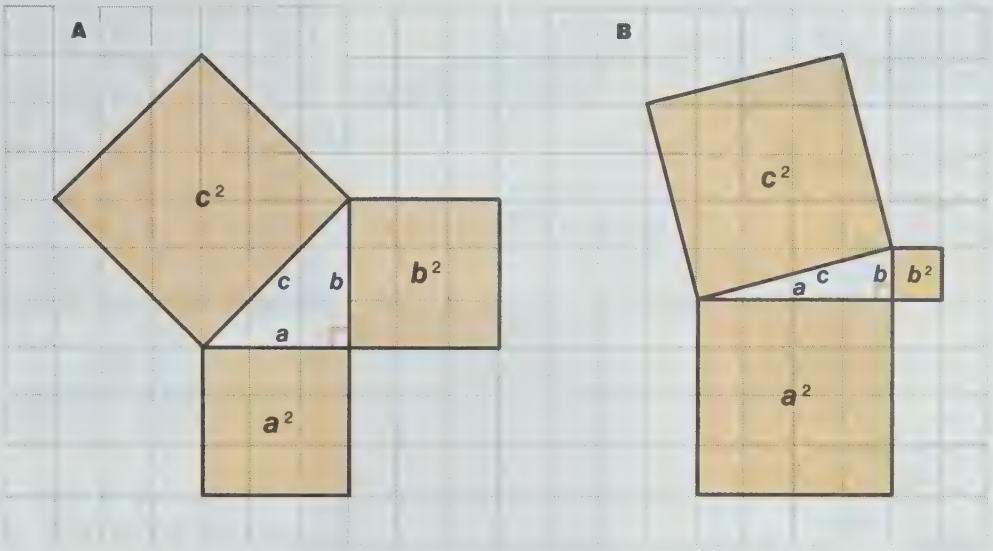
2. The Investigation suggests a generalization for right triangles called the **Pythagorean Theorem.**



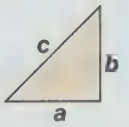
In any right triangle, the sum of the areas of the squares on the legs is equal to the area of the square on the hypotenuse.

What equation does the statement above suggest if a right triangle has legs of length ***a*** and ***b*** and hypotenuse of length ***c***?

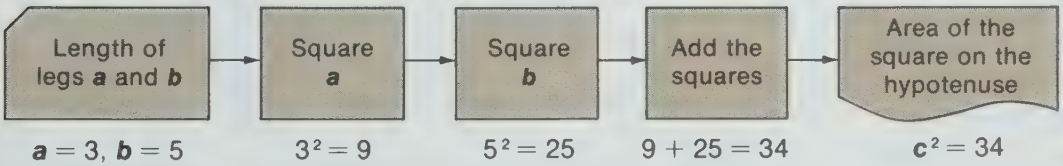
1. Find the area of each square in each figure.
Unit squares are indicated by the grid.



For Exercises 2 through 10, find the area of the square on the hypotenuse of each right triangle with legs of length ***a*** and ***b***. Study the flow chart.



$a^2 + b^2 = c^2$



2. $a = 6$ $b = 3$

3. $a = 1$ $b = 1$

4. $a = 4$ $b = 5$

5. $a = 3$ $b = 4$

6. $a = 0.6$ $b = 0.8$

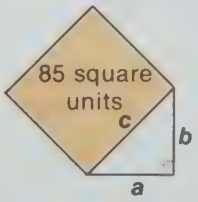
7. $a = 10$ $b = 10$

8. $a = 1\frac{1}{2}$ $b = 2$

9. $a = 5.4$ $b = 6.5$

10. $a = 12.5$ $b = 8.2$

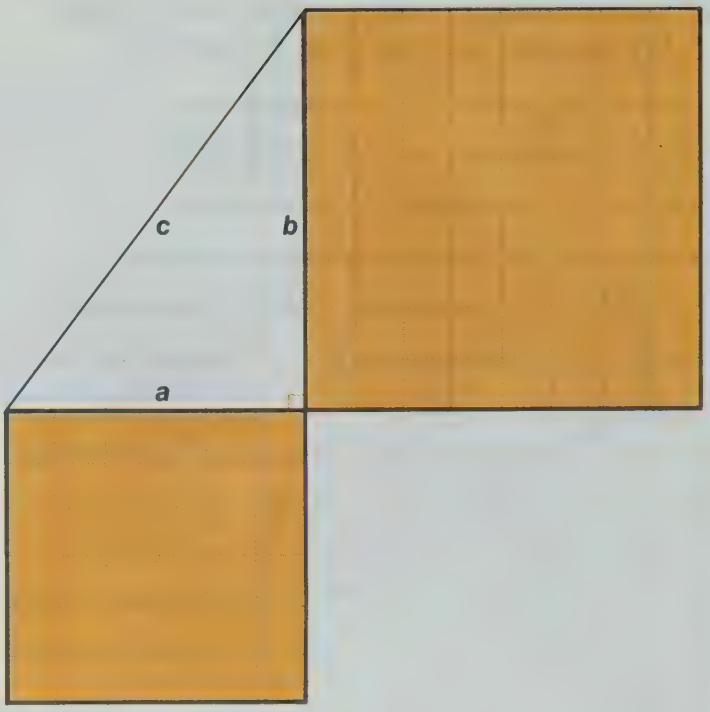
11. The area of the square on the hypotenuse of a right triangle is 85 square units. What are the possible lengths of legs ***a*** and ***b*** if both ***a*** and ***b*** are whole numbers? (HINT: There is more than one possibility.)



Investigating the Ideas

What is the area of each of the squares on legs ***a*** and ***b***?

What is the area of the square on hypotenuse ***c***?



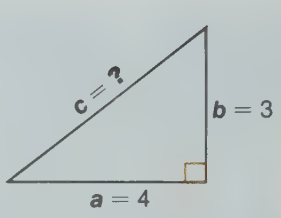
Can you find the length of the hypotenuse of this right triangle?

Discussing the Ideas

1. Explain how you found the length of the hypotenuse of the right triangle in the Investigation.

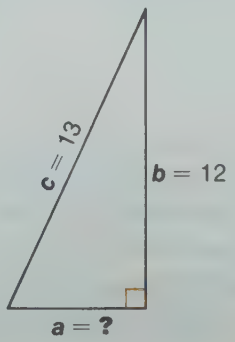
2. Study and complete the solution for finding the length of the hypotenuse of the right triangle.

$$c^2 = a^2 + b^2$$
$$c^2 = 4^2 + 3^2$$
$$c^2 = 16 + 9$$
$$c^2 = 25$$
$$c = \blacksquare$$



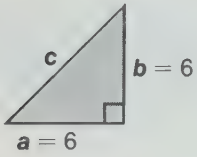
3. Study and complete the solution for finding the length of leg ***a*** of this right triangle.

$$c^2 = a^2 + b^2$$
$$13^2 = a^2 + 12^2$$
$$169 = a^2 + 144$$
$$169 - 144 = a^2$$
$$\blacksquare = a^2$$
$$\blacksquare = a$$

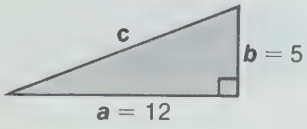


1. Find the length of hypotenuse **c** for each right triangle.

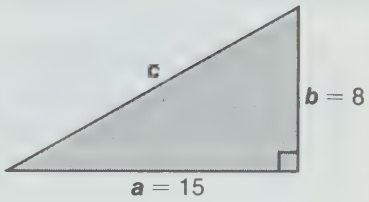
A



B

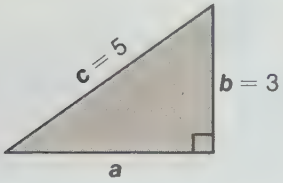


C



2. Find the length of leg ***a*** or ***b*** of each right triangle.

A



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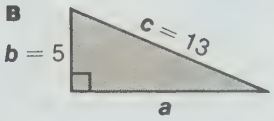
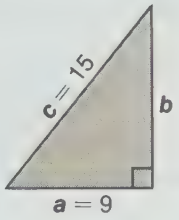
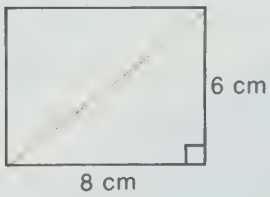


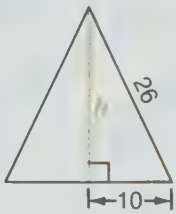
C



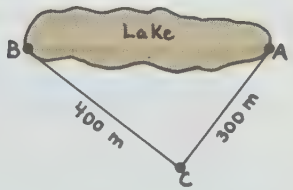
3. How long is the diagonal of a rectangle if the length is 8 cm and the width is 6 cm?



4. What is the height ***h*** of this triangle?

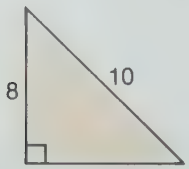


5. Point *C* is chosen so that $\angle ACB$ is a right angle. How far is it across the lake from *A* to *B* if the lengths of $\overline{AC}$ and $\overline{BC}$ are 300 m and 400 m respectively?

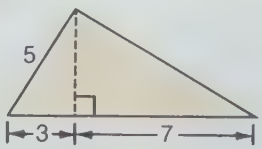


6. Find the area of each region. (HINT: You must first use the Pythagorean Theorem to find the third side of a triangle in each figure.)

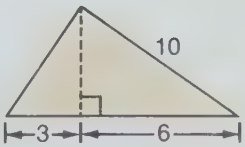
A



B



C



7. Find the length of the third side of each right triangle.

A $a = 12$ $b = 16$

B $a = 1.2$ $c = 1.3$

C $a = 7$ $b = 24$

D $a = 40$ $c = 41$

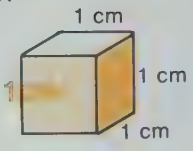
E $b = 30$ $c = 34$

F $a = 24$ $c = 26$

More practice, page S-24, Set 45

Investigating the Ideas

To find the **volume** of a space region, you can think of dividing it into unit-cube regions and counting the number of cubes it takes to fill the space.



A cubic centimetre (cm^3) is a unit of volume.

A cubic decimetre (dm^3) contains 1000 cubic centimetres.

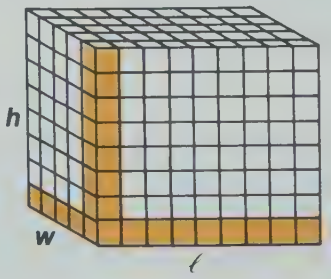
A cubic metre (m^3) contains 1000 cubic decimetres.

Can you estimate the volume of some of the space regions below and find a way to check your estimate?

1. The volume of your math book in cubic centimetres.
2. The volume of a shoe box or a grocery bag in cubic decimetres.
3. The volume of your classroom in cubic metres.

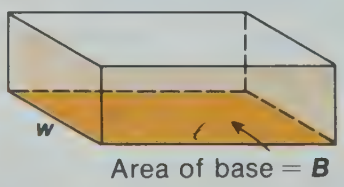
Discussing the Ideas

1. A Count the cubes along the edge ℓ.
   B Count the cubes along the edge w.
   C How many cubes in the bottom layer?
   D How many layers in height h?
   E How many cubes in the solid box?



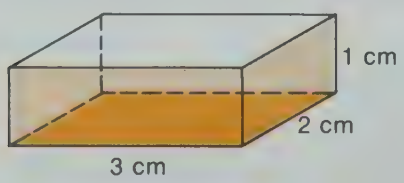
2. Explain why the volume V of the rectangular solid (box) is $V = \ell \cdot w \cdot h$

3. A What is the area B of the base of the box in terms of ℓ and w?
   B Explain why the volume of the box is given by the formula $V = B \cdot h$

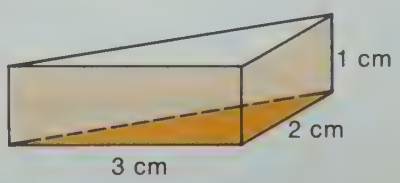


4. What is the volume of each solid?

A

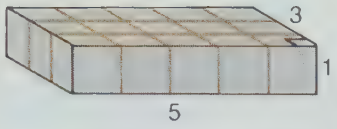


B

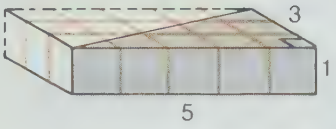


1. Find the volume of the shaded portion of each solid.

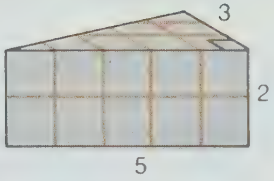
A



B

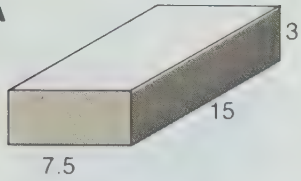


C

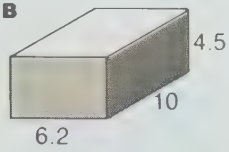


2. Find the volume. All dimensions are in centimetres.

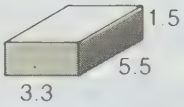
A



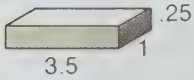
B



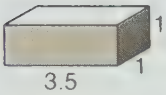
C



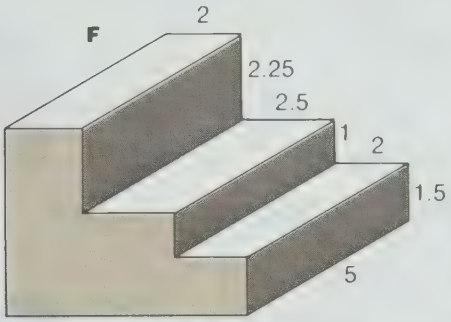
D



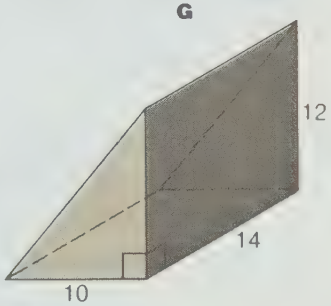
E



F

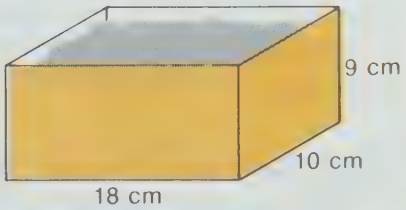


G



3. Use the dimensions on the container at the right to complete each part.

- **A** Find the volume.
- **B** How many litres of water will the container hold? (1 = 1000 cm^3)



4. This L-shaped piece has a volume of 3 cm^3. What is the volume of a similar L-shaped piece that is twice as long, twice as wide, and twice as high?

5. What is the volume of the L-shaped piece if every dimension is tripled?

think

A certain cube has the same number of units of volume as it has units of area. How long is each edge of the cube?

More practice, page S-24, Set 46

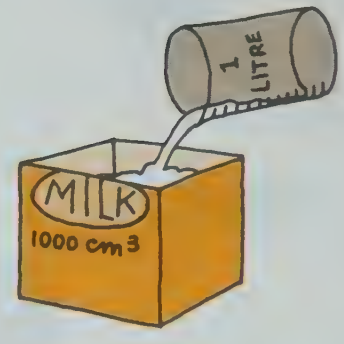
Investigating the Ideas

Obtain a large empty milk carton. Find the interior length and width of the carton in centimetres.

Can you cut off the top of the carton so that the bottom part will have a volume of 1000 cm^3?

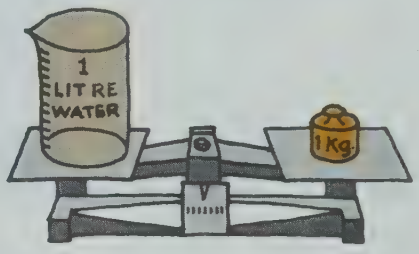
Discussing the Ideas

1. If you fill the bottom part of the carton with water, it would have a **capacity** of 1 **litre** (ℓ) of water. What part of a litre of water is 1 cm^3 of water?

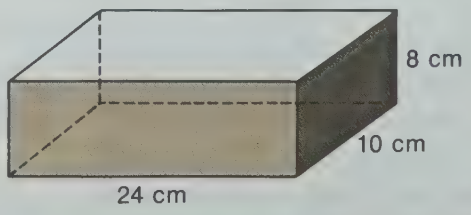


2. One cubic centimetre of water is 0.001 litre or 1 **millilitre** (ml).
   - **A** What part of a litre is 500 ml?
   - **B** What part of a litre is 10 ml?

3. The **mass** of one litre of water under standard conditions is 1 **kilogram** (kg), or 1000 grams. What is the mass of only one millilitre of water?



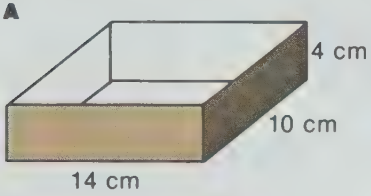
4. **A** What is the volume of the box in cubic centimetres?
   - **B** What is the capacity of the box in millilitres?
   - **C** What is the capacity of the box in litres?
   - **D** What is the mass of water that the box would contain?



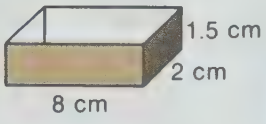
5. Find the capacity of some common containers in litres or millilitres. Then give the mass of water that each container will hold.

1. Give the capacity of each container in millilitres.

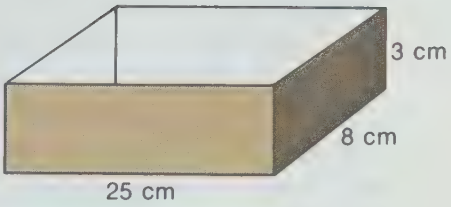
A



B

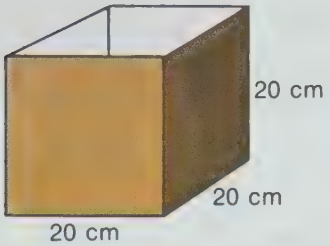


C

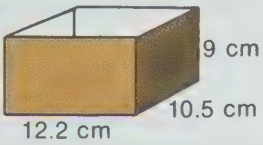


2. Give the capacity of each container in litres.

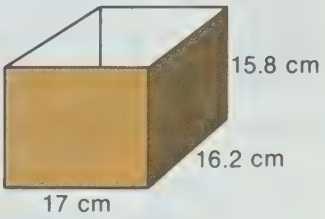
A



B



C



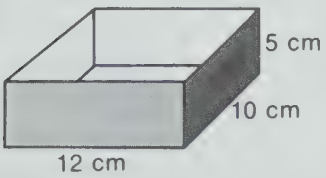
3. Give the mass of water in grams that each container in Exercise 1 will hold.

4. Give the mass of water in kilograms that each container in Exercise 2 will hold.

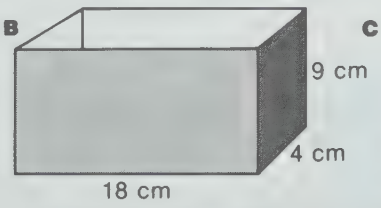
5. A **kilolitre** (kl) is 1000 litres. What is the capacity of a tank in kilolitres if the tank is a cube one metre on each edge?

6. Which container is large enough to hold 680 grams of water?

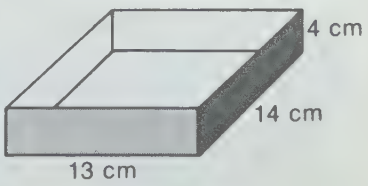
A



B



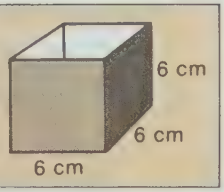
C



7. How much larger in volume must the two other containers in Exercise 6 be in order to hold 680 grams of water?

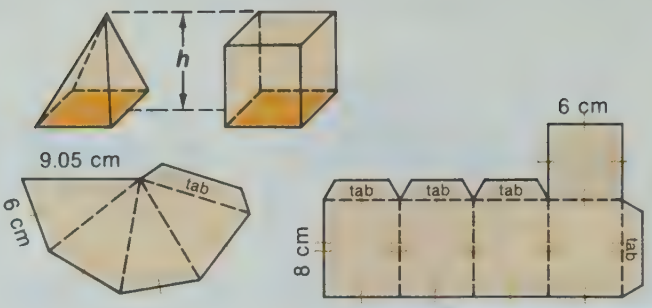
think

When the open-top cube is one twelfth full of paint, there is just enough paint to cover the five faces of the open-top cube. How thick will the paint be on the box if you could use **all** of the paint in the cube?



Investigating the Ideas

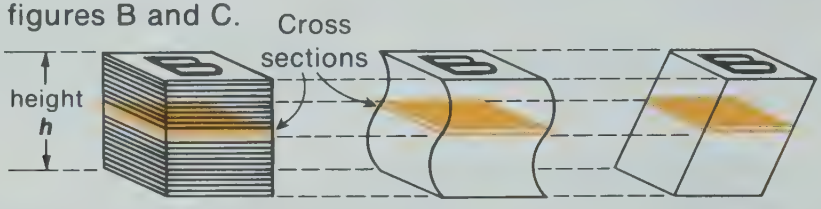
Make copies of the patterns on cardboard, cut them out, and form models for an open-top **prism** and a **pyramid** with its base removed. Both models will have the same dimensions for the base and height.



Can you find how many "pyramidsful" of sand, rice, or other fine material it takes to fill the prism?

Discussing the Ideas

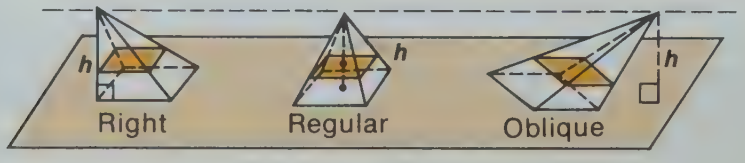
1. The prism constructed for the Investigation is a **right prism** because the lateral edges are perpendicular to the base. Imagine a right prism (figure A) made of a large number of thin sheets of steel and then slid to form figures B and C.



   - **A** Is the height of each figure the same?
   - **B** Are the cross sections of each figure the same?
   - **C** Is the volume of each figure the same?
   - **D** Explain the formula $V = B \cdot h$

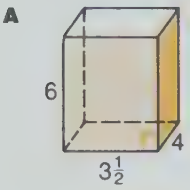
2. Explain why the formula for the volume of a pyramid with base B and height h is $V = \frac{1}{3} \cdot B \cdot h$

3. Do you think that an oblique pyramid has the same volume as a right pyramid with the same base and height? What do these figures suggest? The shaded cross sections are all at the same distance from their bases.

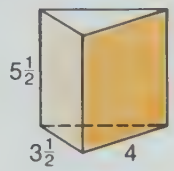


1. Find the volumes of these prisms.

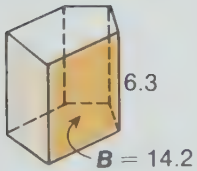
A



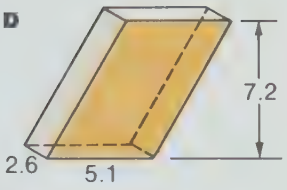
B



C

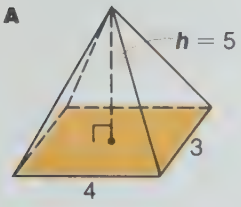


D

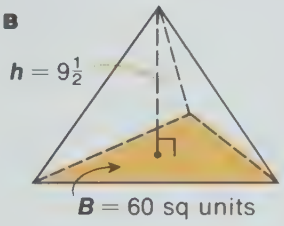


2. Use the formula $V = \frac{1}{3} \cdot B \cdot h$, where B is the area of the base and h is the height, to find the volume of each pyramid.

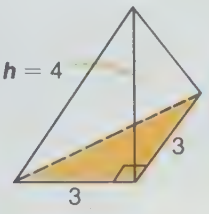
A



B



C



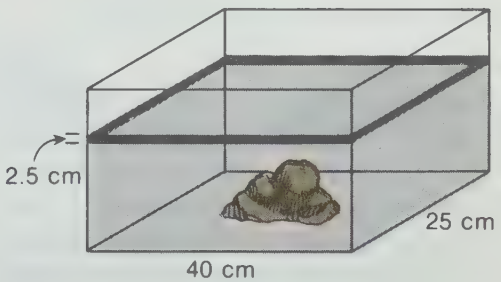
3. The Great Pyramid of Egypt had a square base of 230.4 metres on each side. Originally, the pyramid had a height of 146.5 metres. What was the volume of the pyramid?

4. A concrete sidewalk is to be constructed that is 22 metres in length, 1.5 metres wide, and an average thickness of 15 centimetres.

A About how many cubic metres of concrete are needed?

B Concrete has a mass of about 2300 kilograms per cubic metre. About how many kilograms of concrete are needed?

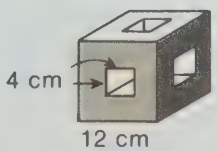
5. When an irregular object is submerged in water, it displaces a volume of water equal to its own volume. If a submerged object causes the water level in the tank shown to rise 2.5 centimetres, what is the volume of the object?



6. A solid cube 12 centimetres on each edge has a square hole cut completely through the centres of the faces of the cube.

A What is the volume of the part of the cube that was removed?

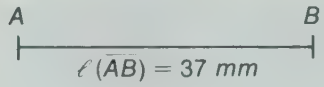
B What is the volume of the remaining part of the cube?



More practice, page S-25, Set 47

REVIEWING THE IDEAS

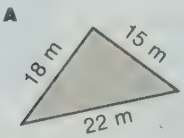
1. What is the length of $\overline{AB}$ in



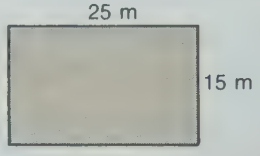
- **A** centimetres?
- **B** decimetres?
- **C** metres?

2. Find the perimeter of each region.

A

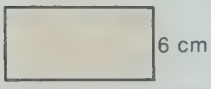


B



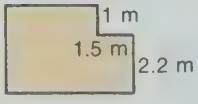
3. Find the area of each region.

A

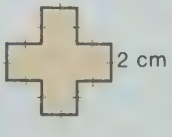


B

C



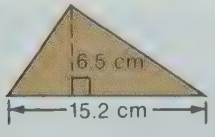
D



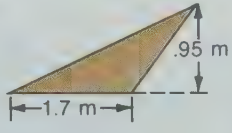
4. If a piece of adding machine tape 6 centimetres wide has an area of one square metre, how long is it?

5. What is the area of each region?

A



B



6. Find the sum and difference.

A

$$\begin{array}{r} 29°\,38'\ 7'' \\ +\,42°\,51'\,29'' \\ \hline \end{array}$$

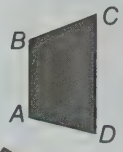
B

$$\begin{array}{r} 125°\,18'\,20'' \\ -\ 53°\,40'\,38'' \\ \hline \end{array}$$

7. What is the measure of the complement of an angle measuring $37°15'$?

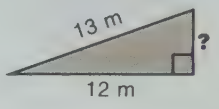
8. How large is the third angle?

9. What is the sum of the measures of angles A, B, C, and D?

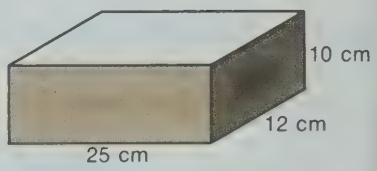


10. What is the area of the large square if □ is a unit square?

11. Find the third side of the triangle.



12. Find the volume.



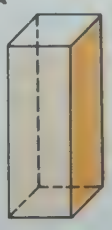
13. **A** What is the mass of 200 millilitres of water?

B What size container will hold 1000 grams of water?

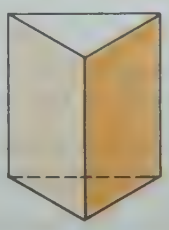
14. What is the capacity in litres of an aquarium 40 cm by 20 cm by 18 cm?

15. The base of each solid below has an area of 40 square units. The height of each solid is 12 units. Find the volume of each solid.

A



B



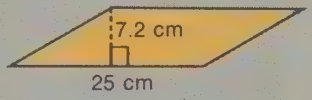
C

TEST YOURSELF

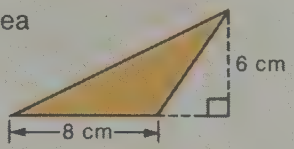
1. What is the perimeter of a rectangular plot of ground 120.8 metres long by 57 metres wide?

2. A rectangular room is 6.1 metres long and 4.3 metres wide. How many square metres of carpet are needed to cover the floor from wall to wall?

3. What is the area of the parallelogram?



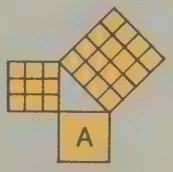
4. What is the area of the shaded region?



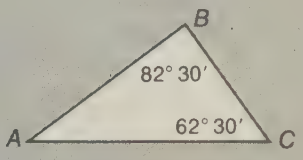
5. Find the sum and difference.

A $\begin{array}{r} 37^\circ\,14'\,34'' \\ +\,81^\circ\,29'\,30'' \\ \hline \end{array}$ B $\begin{array}{r} 90^\circ \\ -\,17^\circ\,38'\,34'' \\ \hline \end{array}$

6. What is the area of square A on the leg of the right triangle?

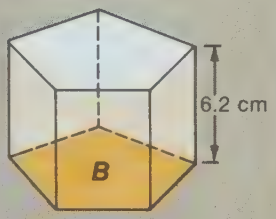


7. What is the measure of $\angle A$?



8. What is the length of each edge of a cube that will contain one litre of water?

9. The area ***B*** of the base of the prism is 66 cm². What is the volume of the prism?



10. What is the volume of a pyramid 18 cm high and with a square base 10 cm on each edge?

RESEARCH PROJECTS

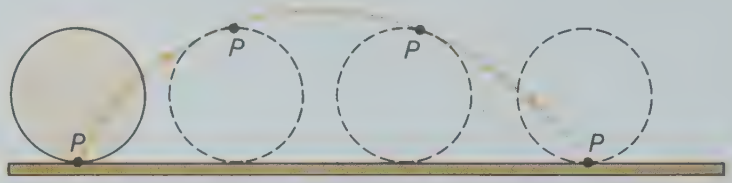
A Make a model that will demonstrate the Pythagorean Theorem. (See *Instructional Aids in Mathematics*, *34th Yearbook*; Washington, D.C.: National Council of Teachers of Mathematics, 1973, pp. 378–379.)

B Investigate Plateau's Problem, the problem of determining the least area that will span a given shape. (See *What is Mathematics*? by R. Courant and H. Robbins; New York: Oxford University Press, 1941, pp. 385–397.)

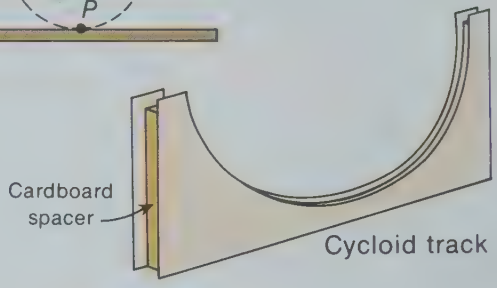
C Find out how the volume of irregular objects can be found by finding the amount of water they displace when they are submerged. Find the volume of several irregular-shaped solids whose volume determined by other methods would be different. (See *Mathematics: Man's Key to Progress* by Richard A. Denholm; Pasadena, California: Franklin Publications, 1968, pp. 84-85; available from Grolier Ltd.)

The Cycloid

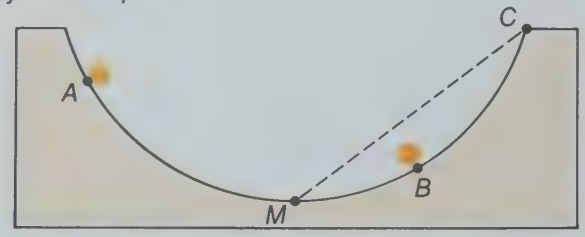
If a circular disk is rolled along a line, the path of a point P on the circle is called a **cycloid** (the curve in color). The cycloid has some interesting properties.



Cut out copies of two **cusps** of a cycloid from heavy cardboard. Glue a third piece of cardboard between them to form a cycloid track.



Try this experiment:



Place two marbles on the track on opposite sides of point M (the midpoint and lowest part of the cusp). If the marbles are released at the same time, they will reach point M at the same time even though the distances along the curve are not the same. Furthermore, a marble released at point C will roll along the curve to point M more quickly than a marble which rolls along a straight line from C to M.

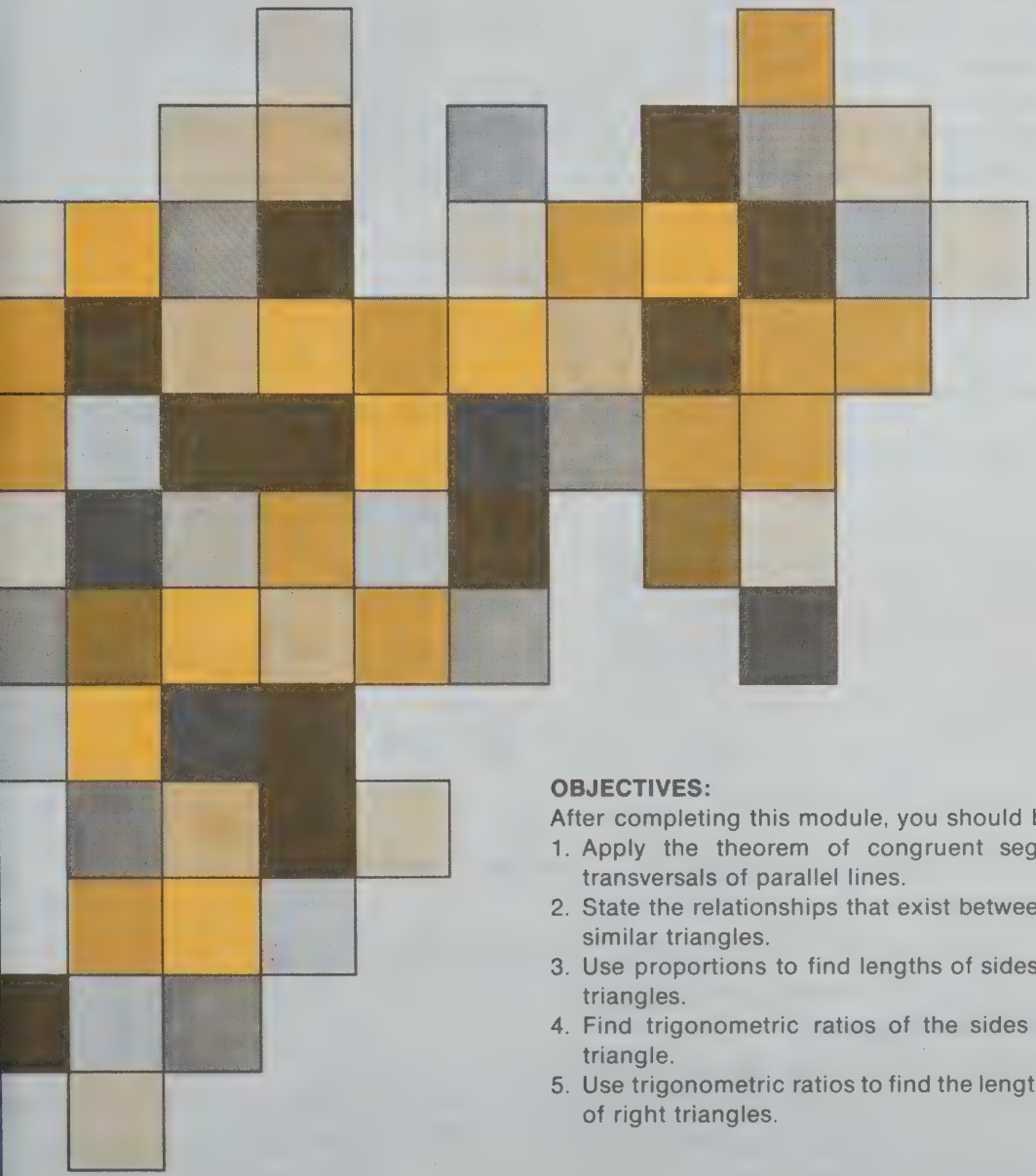
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. 355.6 m **2.** 26.23 m^2 **3.** 180 cm^2 **4.** 24 cm^2
5. A 118° 44′ 4″; B 72° 21′ 26″ **6.** 7 square units
7. 35° **8.** 10 cm **9.** 409.2 cm^3 **10.** 600 cm^3

UNIT G: *Geometry*

MODULE 3: *Similar Triangles and Trigonometry*



OBJECTIVES:

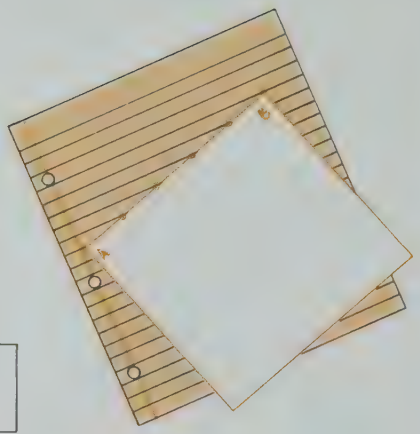
After completing this module, you should be able to:

1. Apply the theorem of congruent segments on transversals of parallel lines.
2. State the relationships that exist between a pair of similar triangles.
3. Use proportions to find lengths of sides of similar triangles.
4. Find trigonometric ratios of the sides of a right triangle.
5. Use trigonometric ratios to find the lengths of sides of right triangles.

Parallel Lines and Transversals

Investigating the Ideas

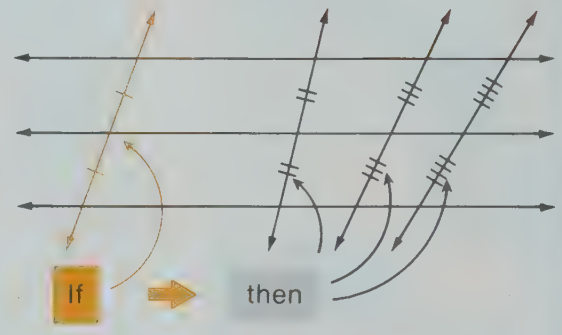
Suppose that you want to divide one edge of a small unlined card into five pieces of equal length but do not have a ruler. The figure shows how you can use a sheet of lined notebook paper to do this.



Can you use this method to divide a segment into seven pieces of equal length?

Discussing the Ideas

1. The Investigation illustrates a geometric theorem which can be stated as follows:

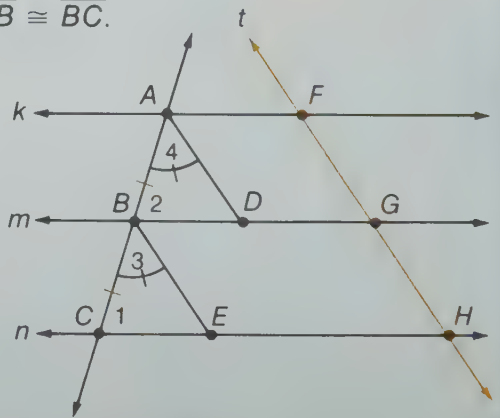


> If parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on all transversals.

Explain how this theorem justifies the results of the Investigation.

2. The following exercise may help you understand why the theorem above is true.

Suppose that k, m, and n are parallel, and $\overline{AB} \cong \overline{BC}$.



A How does $\angle 1$ compare with $\angle 2$? Explain your answer.

B Suppose that we draw $\overline{AD}$ and $\overline{BE}$ so that $\angle 3 \cong \angle 4$. Are $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ parallel?

C What can you say about $\triangle ABD$ and $\triangle BCE$? Explain your answer.

D Is it true that $\overline{AD} \cong \overline{BE}$? Explain.

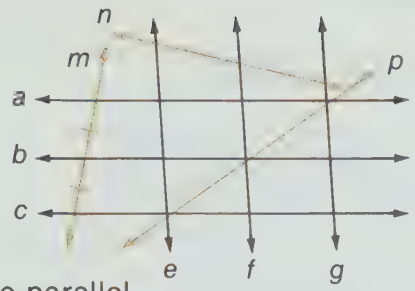
E Suppose that $\overleftrightarrow{FH}$ is a transversal parallel to $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$. What kind of figures are $ADGF$ and $BEHG$?

F Do you think $\overline{FG}$ is congruent to $\overline{GH}$? Explain your answer.

1. In the figure, $a \parallel b \parallel c$ and $e \parallel f \parallel g$. The segments on line m are congruent.

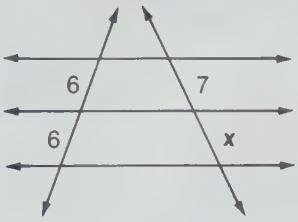
A Are the segments on line n congruent? Why?

B Are the segments on line p congruent? Why?

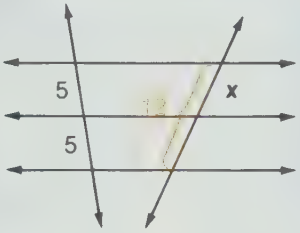


2. Find the length **x**. The horizontal lines are parallel.

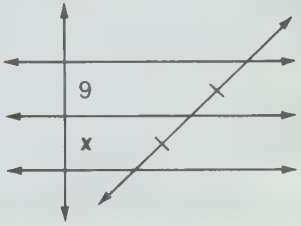
A



B

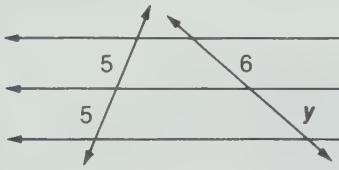


C

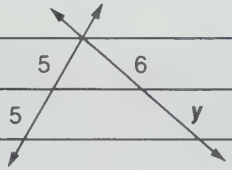


3. Find the length **y**. The horizontal lines are parallel.

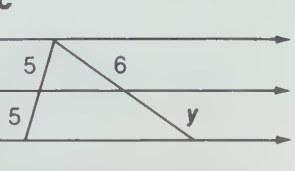
A



B

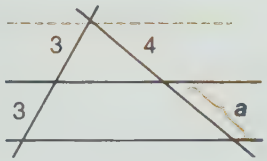


C

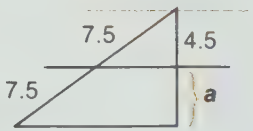


4. Find the length **a**. The horizontal lines are parallel.

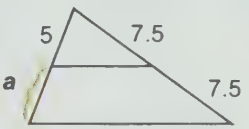
A



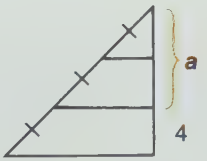
B



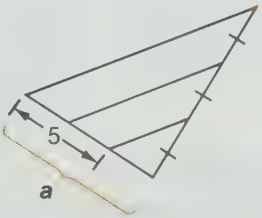
C



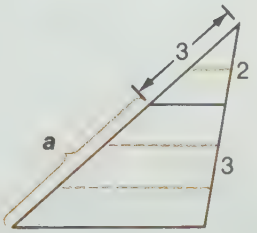
D



E

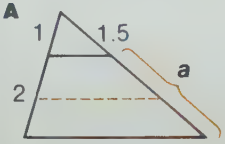


F

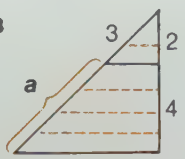


5. Sometimes it is helpful to sketch parallels as shown in parts **A**, **B**, and **C**. Assume that the horizontal lines are parallel and find the length **a**.

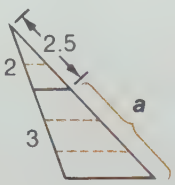
A



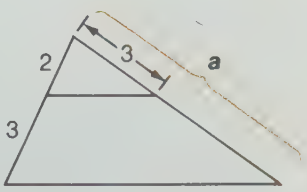
B



C



D

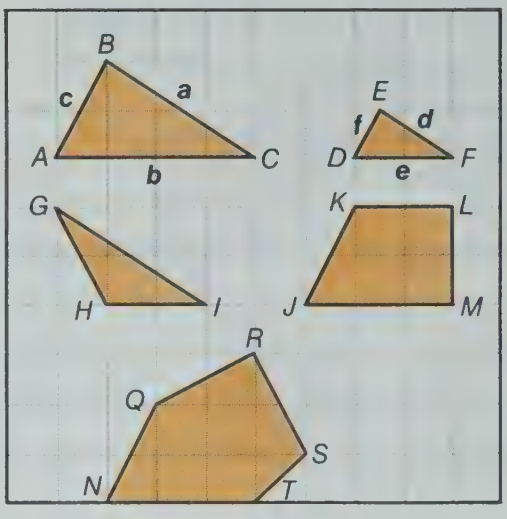


Investigating the Ideas

$\triangle ABC$ and $\triangle DEF$ have the same shape but not the same size.

Draw the other three polygons on a sheet of graph paper.

Can you draw polygons that have the same shape as these three, but not the same size?



Discussing the Ideas

1. **Similar polygons** have the same shape but not necessarily the same size. The formal definition follows.

> Two polygons are similar if there is a one-to-one correspondence between the vertices of the polygons so that corresponding angles are congruent and the lengths of corresponding sides are proportional.

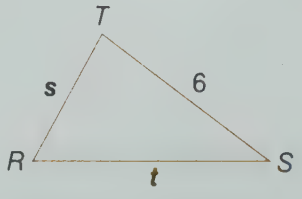
For $\triangle ABC$ and $\triangle DEF$ above, we say:

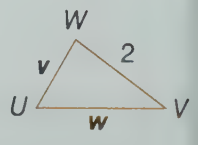
"$\triangle ABC$ is similar to $\triangle DEF$" and we write $\triangle ABC \quad \triangle DEF$.

This means: $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$ and $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$.

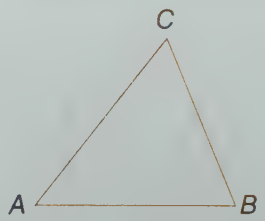
Label the vertices and sides of the other polygons you drew. Show which pairs of angles are congruent and which pairs of sides have the same ratio.

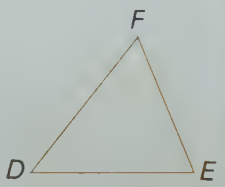
2. If $\triangle RST \sim \triangle UVW$,
   - **A** which vertices must correspond?
   - **B** which pairs of angles are congruent?
   - **C** what is the ratio of t to w? s to v?



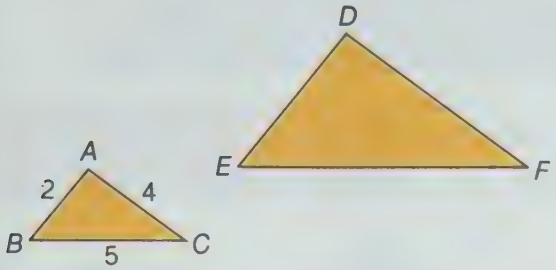


3. **A** If $\triangle ABC \sim \triangle DEF$ and $m\angle C = 62°$, what is $m\angle F$? Why?
   - **B** If the ratio of the length of $\overline{FE}$ to the length of $\overline{CB}$ is 5 to 6, what is the ratio of the length of $\overline{DF}$ to the length of $\overline{AC}$?

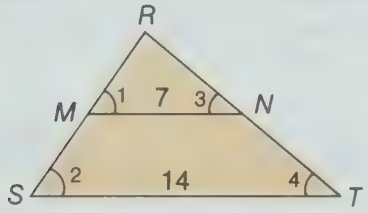




1. $\triangle ABC \sim \triangle DEF$. The length of $\overline{EF}$ is three times the length of $\overline{BC}$. What are the lengths of the sides of $\triangle DEF$?



2. In the figure, $\overline{MN} \parallel \overline{ST}$, M is the midpoint of $\overline{RS}$, and N is the midpoint of $\overline{RT}$.



A Is $\angle 1 \cong \angle 2$? Why?

B Is $\angle 3 \cong \angle 4$? Why?

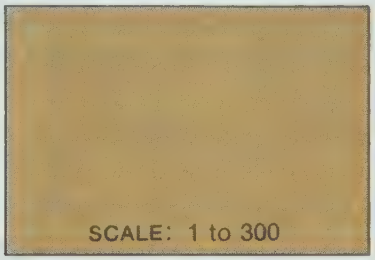
C Is $\triangle RMN \sim \triangle RST$?

3. If two triangles are equilateral, are they similar to each other? Why?

4. Suppose you know that $\triangle GHI$ is an equilateral triangle. You also know that $\triangle XYZ \sim \triangle GHI$. What is the measure of $\angle X$?

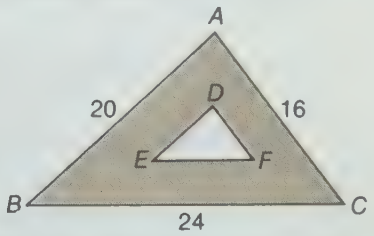
5. Suppose $\triangle JKL \sim \triangle PQR$ and $m\angle J = 27°$ and $m\angle L = 63°$. What are the measures of each of the angles of $\triangle PQR$?

6. Aretha made a scale drawing of her mathematics classroom using a scale of 1 to 300. Her drawing was 4.7 centimetres long and 3.2 centimetres wide. What were the actual dimensions of the classroom in metres?



7. A photograph of the surface of the planet Mars shows a crater 8 millimetres in diameter. If the photograph is $\frac{1}{7\,500\,000}$ actual size, about how many kilometres in diameter is the crater?

★ **8.** $\triangle ABC \sim \triangle DEF$, but the perimeter of $\triangle DEF$ is only $\frac{3}{4}$ of the perimeter of $\triangle ABC$. What is the length of each side of $\triangle DEF$? What is the ratio of the lengths of any two corresponding sides?



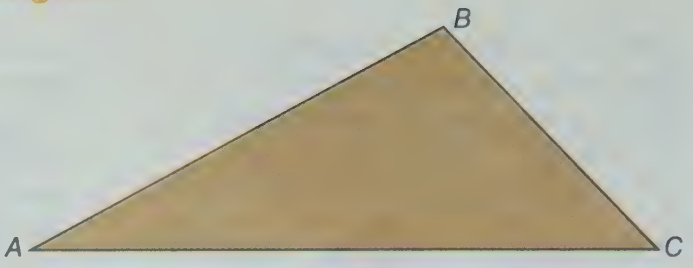
think

Arrange nine dots in

1. eight rows of three each.
2. nine rows of three each.
3. ten rows of three each.

Investigating the Ideas

Make a copy of $\triangle ABC$ on your paper.



Can you draw another triangle whose sides are twice as long as the sides of $\triangle ABC$?

Discussing the Ideas

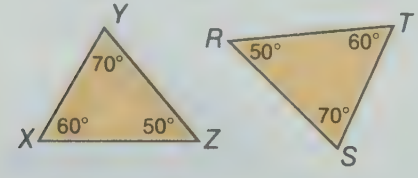
1. Do you think that the two triangles are similar to each other? Give your reasons.

2. Measure each angle of $\triangle ABC$ with a protractor. Draw a triangle whose angles have the same measure as the angles of $\triangle ABC$ but whose sides are different in length from $\triangle ABC$. Is this triangle similar to $\triangle ABC$?

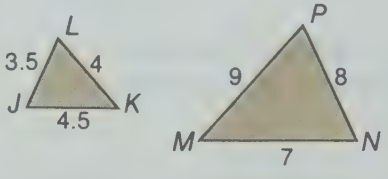
3. If two triangles are similar, then
   - [1] corresponding angles are congruent.
   - [2] corresponding sides are proportional.

   In later geometry courses, you will be able to prove that the existence of either condition *alone* is sufficient to prove that the triangles are similar.

   A Is $\triangle XYZ \sim \triangle TSR$? Why?

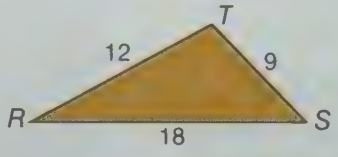


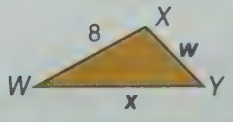
   B Is $\triangle JKL \sim \triangle MPN$? Why?



4. Suppose $\triangle RST \sim \triangle WXY$. The proportion below can be used to find length **w**.

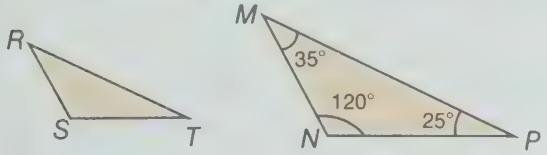
$$\frac{w}{9} = \frac{8}{12}$$
$$12w = 72$$
$$w = 6$$



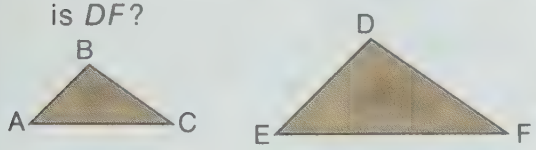


Write and solve a proportion to find length **x**.

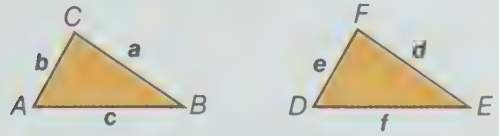
1. Suppose that $\triangle MNP \sim \triangle RST$. Give the measure of each angle of $\triangle RST$.



2. Suppose that $\triangle ABC \sim \triangle EDF$. If length $\overline{AB} = 2$, length $\overline{BC} = 3$, and $\overline{DE}$ is twice as long as $\overline{AB}$, how long is $\overline{DF}$?



In Exercises 3 through 11, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. The letters ***a***, ***b***, ***c***, ***d***, ***e***, and ***f*** represent the lengths of the sides.



EXAMPLE: Find ***a*** if $b = 3$, $d = 4$, $e = 12$

SOLUTION: $\frac{a}{d} = \frac{b}{e}$ or $\frac{a}{4} = \frac{3}{12}$

$12a = 12$, hence $a = 1$

3. Find ***f*** if $a = 6$, $d = 15$, $c = 2$.

4. Find ***e*** if $b = 5$, $c = 2$, $f = 3$.

5. Find ***a*** if $d = 4$, $b = 2$, $e = 7$.

6. Find ***c*** if $f = 4\frac{1}{2}$, $d = 5$, $a = 6$.

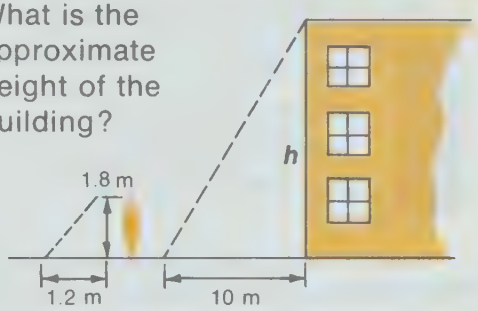
7. Find ***a*** if $b = e$, $d = 1.32$.

8. Find ***b*** if $d = 3$, $a = 2$, $e = 1$.

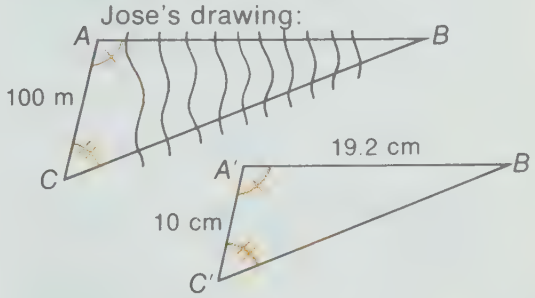
9. Find ***d*** if $c = 2$, $a = 2.1$, $f = 3.4$.

10. Find ***a*** if $d = 2\frac{1}{2}$, $c = 3\frac{1}{3}$, $f = 4\frac{1}{4}$.

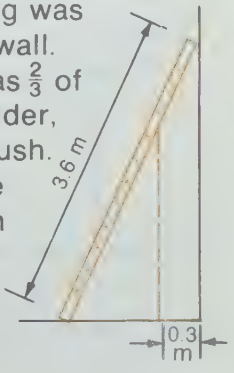
11. A woman 1.8 metres tall casts a shadow 1.2 metres long at the same time that a building casts a shadow 10 metres long. What is the approximate height of the building?



12. To measure the distance between points *A* and B on opposite sides of a river, Jose went downstream 100 m to a point *C*. Then he measured $\angle A$ and $\angle C$ and drew $\triangle A'B'C'$ similar to $\triangle ABC$, making the distance of $\overline{A'C'}$ 10 cm. Then he measured $\overline{A'B'}$ and found it to be 19.2 cm. What was the distance from *A* to *B*?



13. A ladder 3.6 m long was leaning against a wall. When a painter was $\frac{2}{3}$ of the way up the ladder, he dropped his brush. If the brush hit the ground 0.3 m from the wall, about how far was the base of the ladder from the wall?

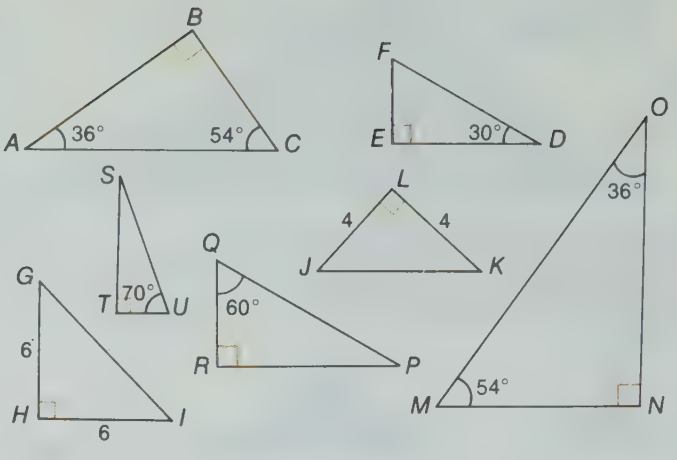


More practice, page S-25, Set 48

Investigating the Ideas

Several right triangles are shown.

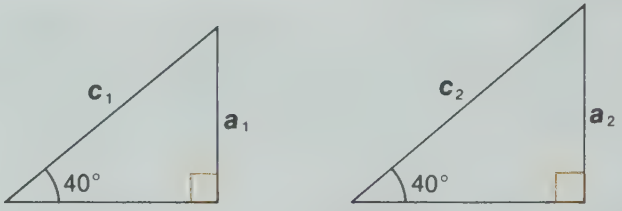
How many pairs of similar right triangles can you find?



Discussing the Ideas

1. Explain how you found the pairs of similar right triangles.

2. Explain why you can be sure that the two triangles are similar.

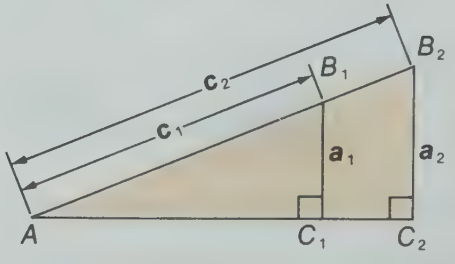


3. Since the two triangles above are similar, complete the proportions $\frac{a_1}{a_2} = \frac{c_1}{c_2}$ and $\frac{a_1}{c_1} = \frac{\blacksquare}{\blacksquare}$ and explain why they are true.

4. Explain why you can be sure that $\triangle AB_1C_1 \sim \triangle AB_2C_2$.

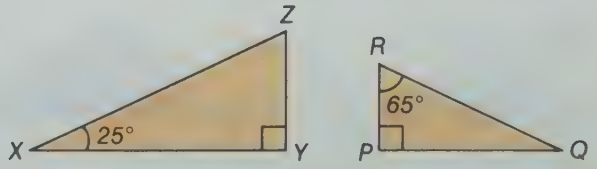
5. Complete the following proportion, constructed from the figure in Exercise 4.

$$\frac{a_1}{c_1} = \frac{\blacksquare}{\blacksquare}$$



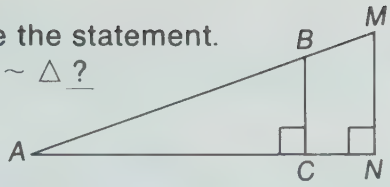
6. **A** Are the triangles similar? Why?

   B Complete the statement. $\triangle XYZ \sim \triangle \underline{?}$.



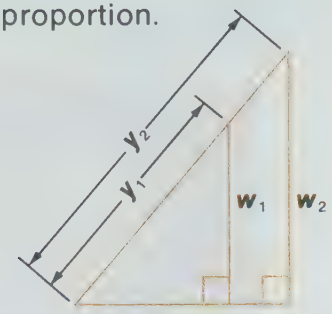
1. Complete the statement.

$\triangle ABC \sim \triangle \underline{?}$



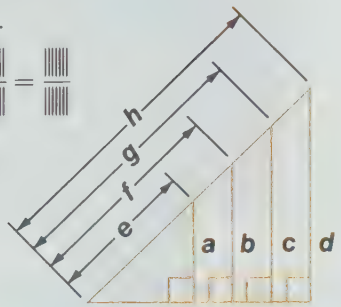
2. Complete the proportion.

$\frac{w_1}{y_1} = \frac{\blacksquare}{\blacksquare}$



3. Give the missing proportions.

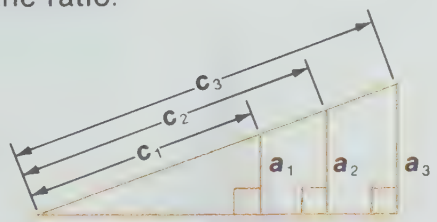
$\frac{a}{e} = \frac{b}{\blacksquare} = \frac{\blacksquare}{\blacksquare} = \frac{\blacksquare}{\blacksquare}$



4. Suppose you know that $\frac{a_1}{c_1} = \frac{1}{2}$. Give the ratio.

A $\frac{a_2}{c_2}$

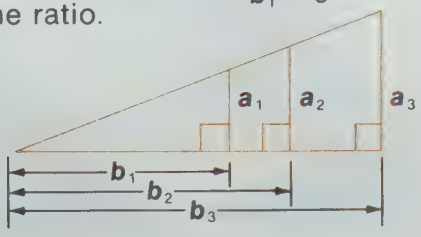
B $\frac{a_3}{c_3}$



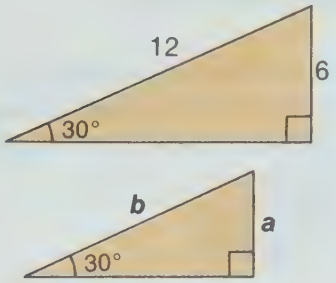
5. Suppose you know that $\frac{a_1}{b_1} = \frac{4}{5}$. Give the ratio.

A $\frac{a_2}{b_2}$

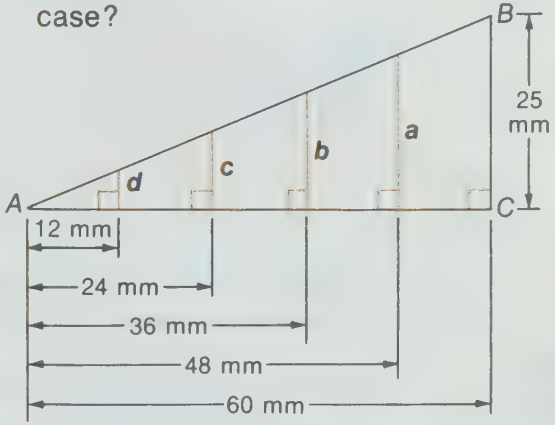
B $\frac{a_3}{b_3}$



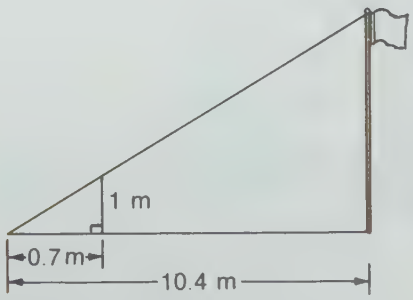
6. What is the ratio $\frac{a}{b}$?



7. Measure the lengths ***a***, ***b***, ***c***, and ***d*** in the figure to the nearest millimetre. Then use proportions to compute ***a***, ***b***, ***c***, and ***d***. Did you get the same numbers in each case?



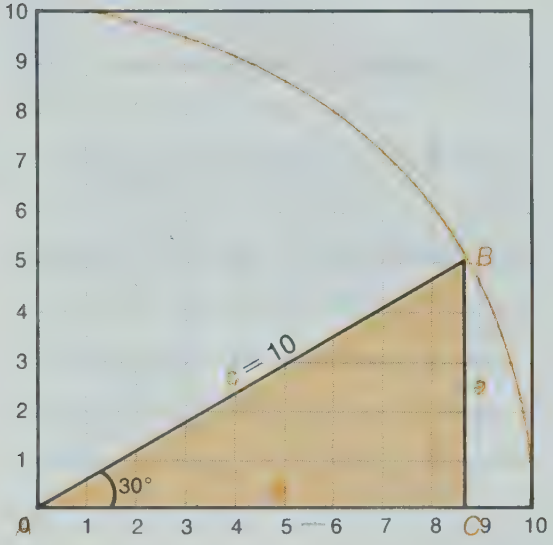
8. How tall is the flagpole in the drawing?



More practice, page S-26, Set 49

Investigating the Ideas

Right $\triangle ABC$ has hypotenuse $c = 10$ units and $m\angle A = 30°$.
The lengths of sides a and b can be estimated as $a \approx 5.0$ and $b \approx 8.7$.
Then the ratios shown in the table can be found.



m∠A			
10°			
20°			
22°			
30°	0.5	0.87	0.57
45°			
52°			
60°			
65°			
70°			

On graph paper, draw seven more right triangles with $c = 10$ units but with angles at A having the other measures shown in the table.

Can you find the ratios for each of the angles in the table?

Discussing the Ideas

1. For $m\angle A = 65°$, the ratio $\frac{a}{b} \approx 2.1$.

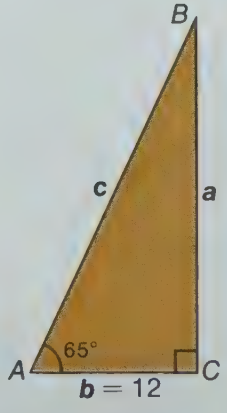
   For $\triangle ABC$ at the right,

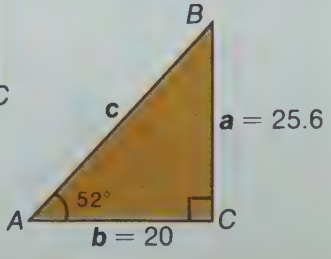
   we have $\frac{a}{12} = 2.1$.

   Hence, $a = 12 \cdot 2.1$

   $a = 25.2$

   Show how the ratio $\frac{b}{c}$ for 65° can be used to find length c in $\triangle ABC$.





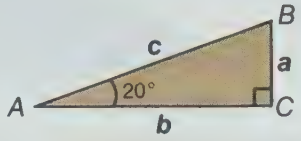
2. A Find the ratio $\frac{a}{b}$ to the nearest hundredth.
   B How can this ratio and the table above help you give the approximate measure of $\angle A$?

For these problems, you can use the table completed in the Investigation on the opposite page.

1. **A** What is $\frac{a}{b}$?

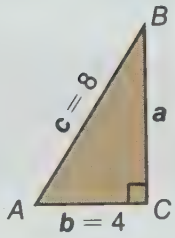
B What is $\frac{a}{c}$?

C What is $\frac{b}{c}$?



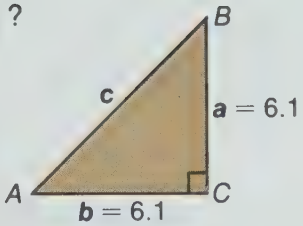
2. **A** What is $\frac{b}{c}$?

B What is $m\angle A$?



3. **A** What is $\frac{a}{b}$?

B What is $m\angle B$?

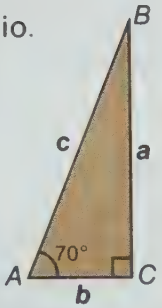


4. Give each ratio.

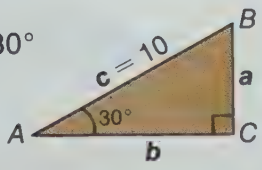
A $\frac{a}{c}$

B $\frac{b}{c}$

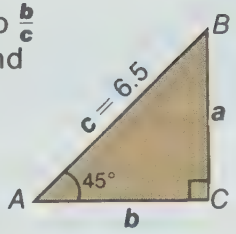
C $\frac{a}{b}$



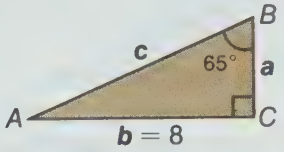
5. Use the ratio $\frac{a}{c}$ for 30° from the table to find length ***a*** in the triangle.



6. Use the ratio $\frac{b}{c}$ for 45° to find length ***b***.



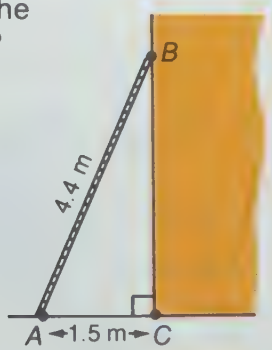
7. Find the length ***a*** for the triangle.



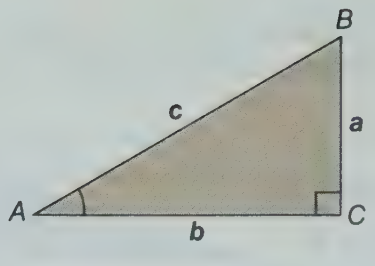
★ **8.** A ladder 4.4 metres long is leaned against a wall so that the base of the ladder is 1.5 metres from the wall.

A What is the measure of the angle the ladder makes with the ground ($m\angle A$)?

B How high up on the wall will the ladder reach?



Investigating the Ideas



In $\triangle ABC$, if $\angle C$ is a right angle, the ratio $\frac{a}{c}$ is called the **sine** (pronounced like "sign") of $\angle A$.
We abbreviate and write: $\sin A = \frac{a}{c}$

The other ratios $\frac{b}{c}$ and $\frac{a}{b}$ are called **cosine** and **tangent** respectively.

We abbreviate these and write: $\cos A = \frac{b}{c}$, $\tan A = \frac{a}{b}$

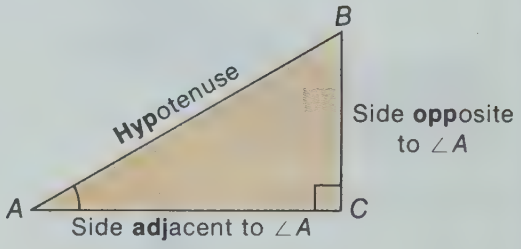
Measure each side of $\triangle ABC$ to the nearest millimetre.

Can you compute $\sin A$, $\cos A$, and $\tan A$ to the nearest hundredth for $\triangle ABC$?

Discussing the Ideas

1. The sine, cosine, and tangent ratios are called **trigonometric ratios** and can be computed with great accuracy by advanced mathematical methods. Using the table in the previous lesson, what is the approximate measure of $\angle A$?

2. It will be helpful to give names to the sides of the right triangle relative to $\angle A$. The following abbreviations may help you remember the ratios:

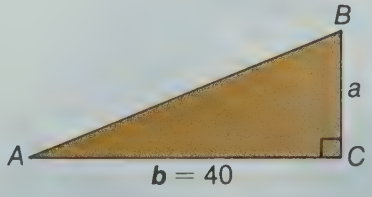


$\sin A = \frac{\text{opp}}{\text{hyp}}$ $\cos A = \frac{\text{adj}}{\text{hyp}}$ $\tan A = \frac{\text{opp}}{\text{adj}}$

   A Which ratio (sin, cos, or tan) can you find if you know the lengths of the opposite side and the hypotenuse?

   B Which ratio can you find if you know the lengths of the adjacent side and the opposite side?

3. If $\tan A \approx 0.43$ and $b = 40$,

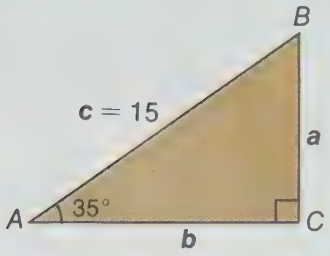


then $\tan A = \frac{\text{opp}}{\text{adj}} = \frac{a}{b}$

and $0.43 = \frac{a}{40}$.

Complete the solution to find a.

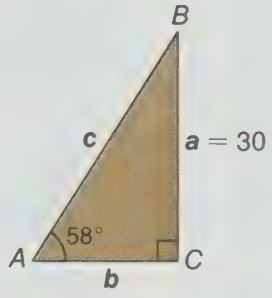
1. In right $\triangle ABC$, $c = 15$, $\sin 35° \approx 0.57$, and $\cos 35° \approx 0.82$. Complete the solution to find a and b.



A $\sin 35° = \frac{a}{15}$, therefore $\frac{a}{15} \approx 0.57$, and $a \approx$ ▮.

B $\cos 35° = \frac{b}{15}$, therefore $\frac{b}{15} \approx 0.82$, and $b \approx$ ▮.

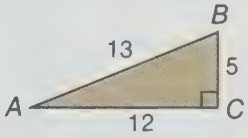
2. In the right triangle, $a = 30$, $\tan 58° \approx 1.6$, and $\sin 58° \approx 0.85$. Complete the solution to find b and c.



A $\tan 58° = \frac{30}{b}$, therefore $\frac{30}{b} \approx 1.6$, and $b \approx$ ▮.

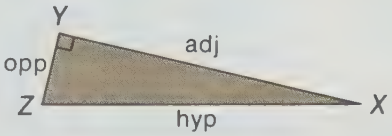
B $\sin 58° = \frac{30}{c}$, therefore $0.85 \approx \frac{30}{c}$, and $c \approx$ ▮.

3. Use the figure at the right to compute the ratios for $\angle A$ to the nearest hundredth.



A $\tan A$ **B** $\sin A$ **C** $\cos A$

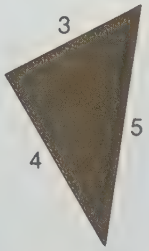
4. In right $\triangle XYZ$, $\angle Y$ is the right angle and the sides are designated as shown.



A Compute $\tan X$ if $\overline{XY}$ has a length of 40 cm and $\overline{YZ}$ has a length of 9 cm.

B Compute $\sin X$ if the length of the hypotenuse is 41 cm.

★ **5.** A right triangle has sides of lengths 3 units, 4 units, and 5 units.



A Which side is the hypotenuse?

B Let the side that is 3 units long be the side opposite $\angle A$. What is $\sin A$?

C What is $\cos A$?

D What is $\tan A$?

More practice, page S-26, Set 50

A TABLE OF TRIGONOMETRIC RATIOS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0°	0.000	1.000	0.000	45°	0.707	0.707	1.000
1°	0.017	1.000	0.017	46°	0.719	0.695	1.036
2°	0.035	0.999	0.035	47°	0.731	0.682	1.072
3°	0.052	0.999	0.052	48°	0.743	0.669	1.111
4°	0.070	0.998	0.070	49°	0.755	0.656	1.150
5°	0.087	0.996	0.087	50°	0.766	0.643	1.192
6°	0.105	0.995	0.105	51°	0.777	0.629	1.235
7°	0.122	0.993	0.123	52°	0.788	0.616	1.280
8°	0.139	0.990	0.141	53°	0.799	0.602	1.327
9°	0.156	0.988	0.158	54°	0.809	0.588	1.376
10°	0.174	0.985	0.176	55°	0.819	0.574	1.428
11°	0.191	0.982	0.194	56°	0.829	0.559	1.483
12°	0.208	0.978	0.213	57°	0.839	0.545	1.540
13°	0.225	0.974	0.231	58°	0.848	0.530	1.600
14°	0.242	0.970	0.249	59°	0.857	0.515	1.664
15°	0.259	0.966	0.268	60°	0.866	0.500	1.732
16°	0.276	0.961	0.287	61°	0.875	0.485	1.804
17°	0.292	0.956	0.306	62°	0.883	0.469	1.881
18°	0.309	0.951	0.325	63°	0.891	0.454	1.963
19°	0.326	0.946	0.344	64°	0.899	0.438	2.050
20°	0.342	0.940	0.364	65°	0.906	0.423	2.145
21°	0.358	0.934	0.384	66°	0.914	0.407	2.246
22°	0.375	0.927	0.404	67°	0.921	0.391	2.356
23°	0.391	0.921	0.424	68°	0.927	0.375	2.475
24°	0.407	0.914	0.445	69°	0.934	0.358	2.605
25°	0.423	0.906	0.466	70°	0.940	0.342	2.747
26°	0.438	0.899	0.488	71°	0.946	0.326	2.904
27°	0.454	0.891	0.510	72°	0.951	0.309	3.078
28°	0.469	0.883	0.532	73°	0.956	0.292	3.271
29°	0.485	0.875	0.554	74°	0.961	0.276	3.487
30°	0.500	0.866	0.577	75°	0.966	0.259	3.732
31°	0.515	0.857	0.601	76°	0.970	0.242	4.011
32°	0.530	0.848	0.625	77°	0.974	0.225	4.332
33°	0.545	0.839	0.649	78°	0.978	0.208	4.705
34°	0.559	0.829	0.675	79°	0.982	0.191	5.145
35°	0.574	0.819	0.700	80°	0.985	0.174	5.671
36°	0.588	0.809	0.727	81°	0.988	0.156	6.314
37°	0.602	0.799	0.754	82°	0.990	0.139	7.115
38°	0.616	0.788	0.781	83°	0.993	0.122	8.144
39°	0.629	0.777	0.810	84°	0.995	0.105	9.514
40°	0.643	0.766	0.839	85°	0.996	0.087	11.430
41°	0.656	0.755	0.869	86°	0.998	0.070	14.301
42°	0.669	0.743	0.900	87°	0.999	0.052	19.081
43°	0.682	0.731	0.933	88°	0.999	0.035	28.636
44°	0.695	0.719	0.966	89°	1.000	0.017	57.290
45°	0.707	0.707	1.000	90°	1.000	0.000	—

Using the Trigonometric Table

The table on the opposite page gives the sine, cosine, and tangent for angles whose measures are a whole number of degrees from 0° through 90°. The numbers in the table are rounded to nearest thousandth. Thus, when you read that "$\sin 27° \approx 0.454$," this means $0.4535 < \sin 27° < 0.4545$. The trigonometric ratios are useful in solving problems concerning right triangles. As you continue your study of mathematics, you will learn to use these ratios in many other situations.

1. Use the table of ratios to find the following:

A $\sin 24°$ **E** $\tan 81°$

B $\tan 46°$ **F** $\sin 27°$

C $\cos 55°$ **G** $\cos 88°$

D $\sin 37°$ **H** $\sin 35°$

2. Use the table to find the degree measures for x in each equation.

A $\sin 15° = \cos x°$ **C** $\cos 20° = \sin x°$

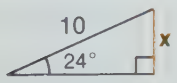
B $\sin 63° = \cos x°$ **D** $\sin 86° = \cos x°$

3. Complete this sentence.

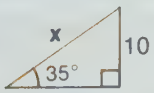
If $\sin y° = \cos x°$, then $x + y =$ ▨.

4. Find the length x.

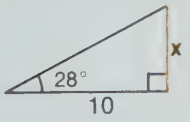
A



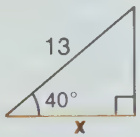
B



C

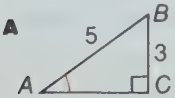


D

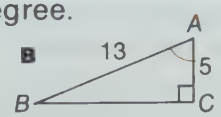


5. In each triangle, find $m\angle A$ to the nearest degree.

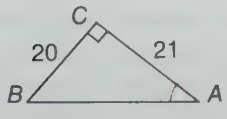
A



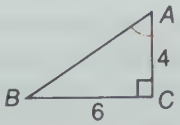
B



C



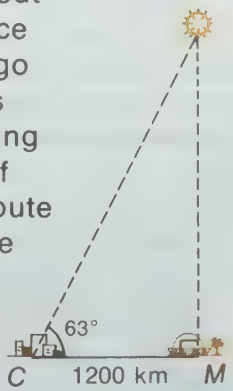
D



6. In traveling one kilometre along a road, suppose a person goes up a gradual incline of 50 metres.

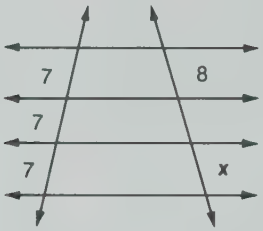
What is the approximate degree measure of the upward slope of the road?

7. A satellite was sighted in Chicago at the time that it was directly over Montreal. The angle determined by the line of sight from Chicago and the ground was about 63°. The distance between Chicago and Montreal is 1200 km. Ignoring the curvature of the earth, compute the approximate altitude of the satellite.

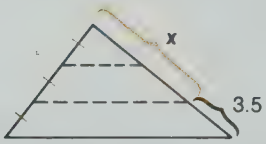


REVIEWING THE IDEAS

1. In each figure, the horizontal lines are parallel. Find length **x**.



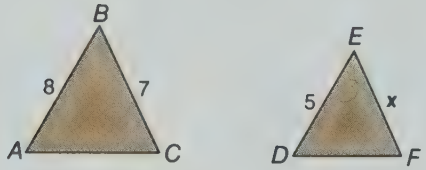
2. Find **x**. The horizontal lines are parallel.



3. $\triangle JKL \sim \triangle PQR$

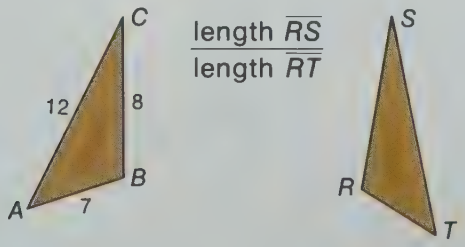
- **A** What pairs of angles are congruent?
- **B** What pairs of sides must correspond?

4. $\triangle ABC \sim \triangle DEF$. What is the measure **x**?

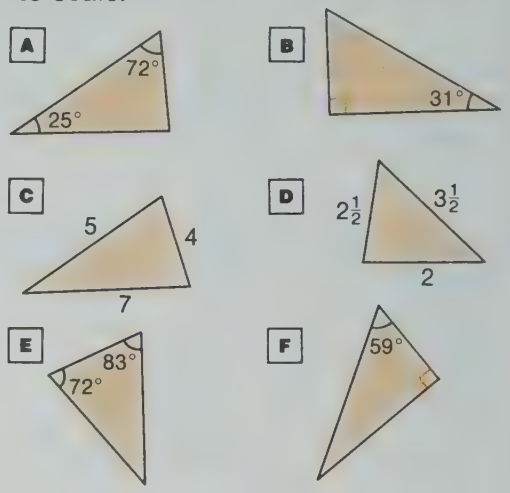


5. If $\triangle BAD \sim \triangle RSV$, then $\angle A$ is congruent to what angle of $\triangle RSV$?

6. $\triangle ABC \sim \triangle TRS$. What is the ratio?

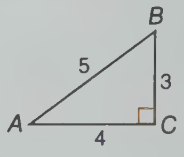


7. Which pairs of triangles are similar? The figures are not necessarily drawn to scale.



8. Using the 3-4-5 right triangle, give the name (sin, cos, or tan) for each ratio.

- **A** ? $A = \frac{3}{5}$
- **B** ? $A = \frac{4}{5}$
- **C** ? $A = \frac{3}{4}$



9. For the triangle in Exercise 8, use the table of trigonometric ratios to estimate the measure of $\angle A$ to the nearest degree.

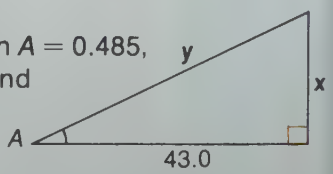
10. Use the table of trigonometric ratios on page G-68 to find each of these.

- **A** sin 27°
- **B** cos 38°
- **C** tan 19°
- **D** cos 71°
- **E** tan 48°
- **F** sin 83°

11. Use the table of trigonometric ratios to give the degree measure of $\angle A$.

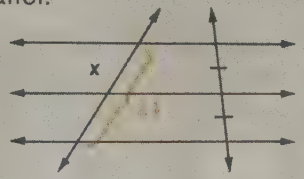
- **A** $\sin A = 0.940$
- **B** $\cos A = 0.988$
- **C** $\sin A = 0.500$
- **D** $\cos A = 0.500$

12. Suppose that $\sin A = 0.485$, $\cos A = 0.875$, and $\tan A = 0.554$. Find **x** and **y**.



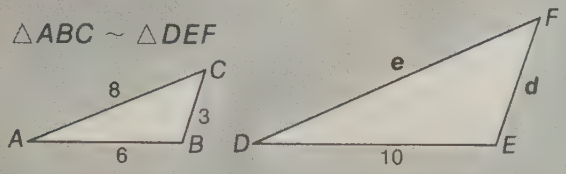
TEST YOURSELF

1. Find length **x**. The horizontal lines are parallel.



2. If $\triangle PQR \sim \triangle JKL$, what angle is congruent to $\angle R$?

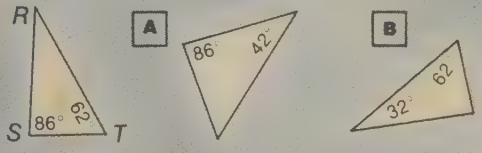
3. $\triangle ABC \sim \triangle DEF$



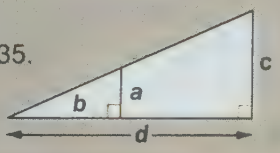
   A What is the ratio $\frac{d}{e}$?

   B Find length ***d***.

4. Which triangle is similar to $\triangle RST$?

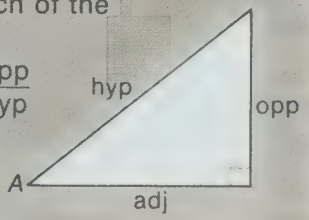


5. In the figure, $\frac{a}{b} = 0.35$. What is $\frac{c}{d}$?



6. Give the name (sin *A*, cos *A*, or tan *A*) for each of the following:

   A $\frac{\text{opp}}{\text{adj}}$ B $\frac{\text{opp}}{\text{hyp}}$ C $\frac{\text{adj}}{\text{hyp}}$

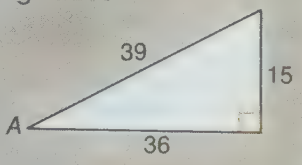


7. Give the following ratios:

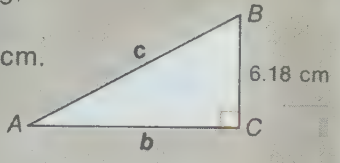
   A sin *A*

   B cos *A*

   C tan *A*

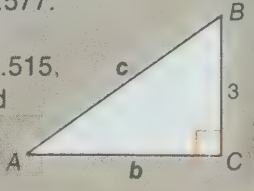


8. In the drawing, sin *A* = 0.500 and ***a*** = 6.18 cm. What is length ***c***?



9. Using the triangle in Exercise 8, find ***b*** if tan *A* = 0.577.

10. Find ***b*** if sin *A* = 0.515, cos *A* = 0.857, and tan *A* = 0.601.

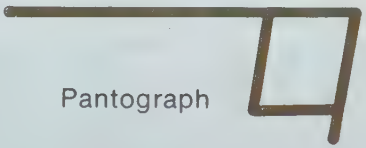


RESEARCH PROJECTS

A Find how you can use similar triangles to determine the heights of trees, buildings, and so on, by measuring their shadows. (See *Mathematics: A Man's Key to Progress* by Richard A. Denholm; Pasadena, California: Franklin Publications, 1968, pp. 53-55; Grolier Ltd.)

B Find out about the origins of trig-, onometry. (See *The Wonderful World of Mathematics* by Lancelot Hogben; Garden City, New York: Doubleday, 1955.)

C A **pantograph** is a mechanical linkage that can be used to produce drawings that are similar to a given drawing.



Pantograph

Find out how pantographs are made. Try making your own model of a pantograph and demonstrate its use. (See *The World Book Encyclopedia*; Chicago: Field Enterprises Educational Corporation, 1974, vol. 15, p. 112.)

PENTOMINOS

Pentominos are regions formed by five squares connected edge to edge. There are exactly 12 different pentominos.

Cut out the 12 pentominos from graph paper. Then try the problems below. The pieces may be turned over or rotated.

1. Make a square using pieces E, F, G, H, and J.
2. Form a 3 by 5 rectangle using pieces F, G, and H.
3. Form another 3 by 5 rectangle using pieces B, C, and D.
4. Each of the 12 pieces has the approximate shape of one or more letters of the alphabet. What are they?
5. Form a Roman numeral for 1066 using the appropriate pieces.
6. How many words can you spell with the pieces?
7. All 12 pieces can be used to form the following rectangles:
   A 3 by 20 **B** 4 by 15 **C** 6 by 10 **D** 5 by 12
   How many of these can you form?
8. Cover all of an 8 by 8 square except the four corner squares with the pentominos.
9. Choose one of the pentominos. Then use nine other pentominos to form a region whose dimensions are three times those of the pentomino selected.
10. Determine the size of the largest "field" you can enclose with the 12 pieces. Any two pieces must touch along an entire side of a square.

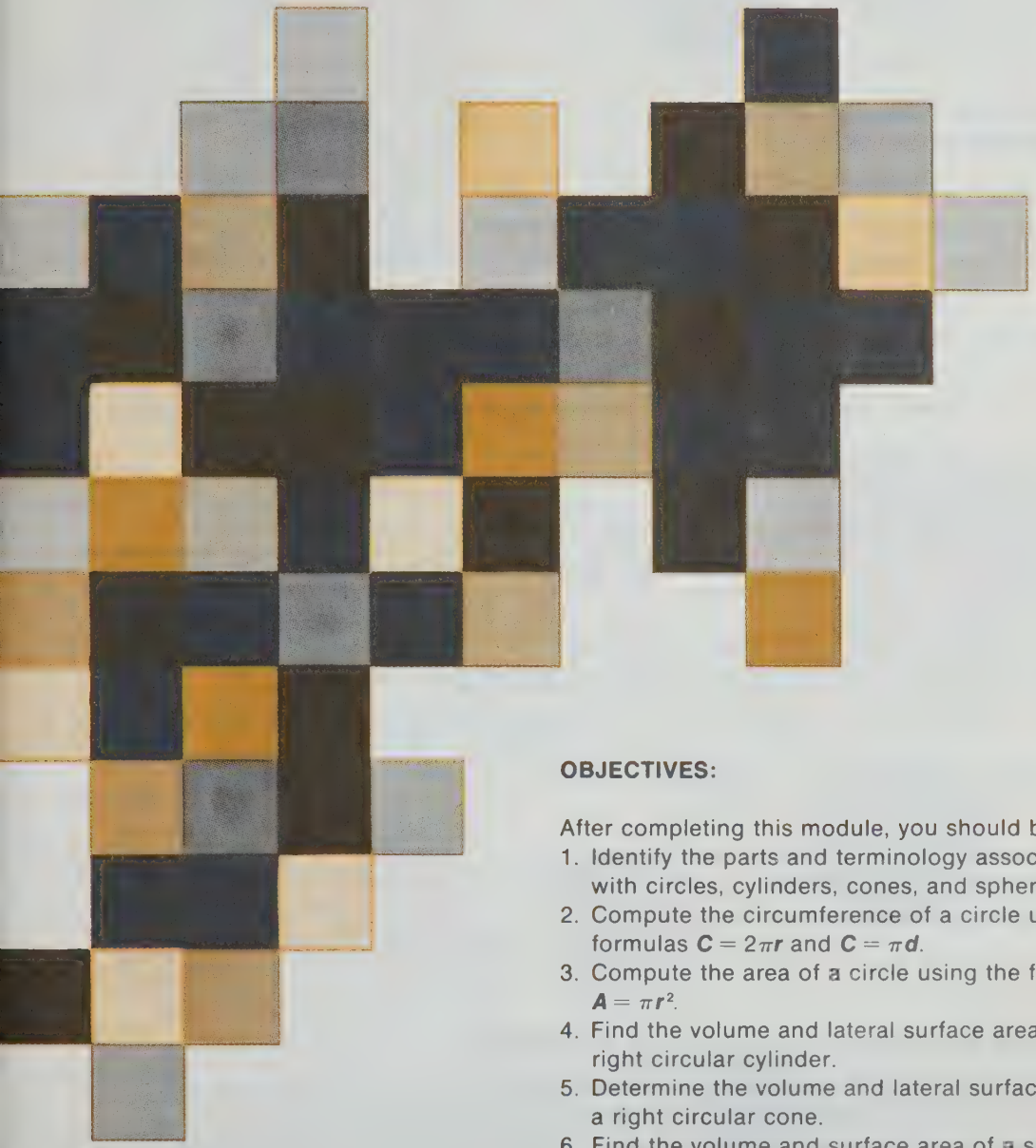
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. $x = 12$ 2. $\angle L$ 3. **A** $\frac{3}{8}$; **B** $d = 5$ 4. **B** 5. $\frac{c}{d} = 0.35$
6. **A** $\tan A$; **B** $\sin A$; **C** $\cos A$ 7. **A** $\frac{15}{39}$; **B** $\frac{36}{39}$; **C** $\frac{15}{36}$ 8. $c = 12.36$
9. $b = 10.71$ 10. $b = 4.99$

UNIT G: *Geometry*

MODULE 4: *Circles, Cylinders, Cones, and Spheres*



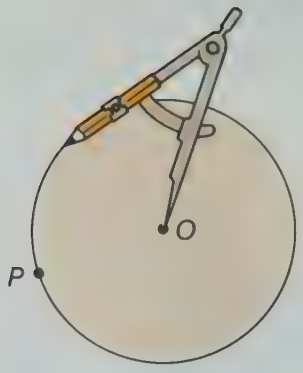
OBJECTIVES:

After completing this module, you should be able to:

1. Identify the parts and terminology associated with circles, cylinders, cones, and spheres.
2. Compute the circumference of a circle using the formulas $C = 2\pi r$ and $C = \pi d$.
3. Compute the area of a circle using the formula $A = \pi r^2$.
4. Find the volume and lateral surface area of a right circular cylinder.
5. Determine the volume and lateral surface area of a right circular cone.
6. Find the volume and surface area of a sphere.

Investigating the Ideas

Draw a circle with centre O and mark a point P on the circle.



Can you draw the circles described below?

Each of the circles must have its **centre on the circle** and must also **pass through point *P***.

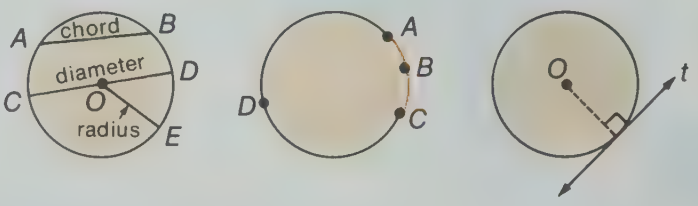
- **A** A circle that is the same size as the given circle.
- **B** A circle smaller than the given circle.
- **C** A circle larger than the given circle.
- **D** The largest of all such circles.

Discussing the Ideas

1. The curve encompassing all the circles which have centres on the given circle and pass through point P is called a **cardioid**.
   - **A** Draw several more circles of that type.
   - **B** Look up the word cardioid in a dictionary to see why this curve has this name.

2. Give your own definition of **circle**.

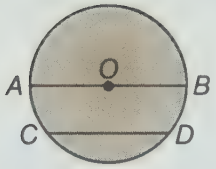
3. The figures below may help you to review some terms associated with circles.



   - **A** Give a definition of **chord**, **radius**, and **diameter**.
   - **B** The set of points on the circle shown in color is minor arc ABC (SYMBOL: $\overset{\frown}{ABC}$). Why is it a minor arc? What points on the circle could be a **major arc**?
   - **C** Line t is **tangent** to the circle. How are t and the circle related?

4. **A** What is meant by the **interior** of a circle?
   B What is the **exterior** of a circle?

1. The circle with centre O has a diameter of 2 centimetres.



A How long is a radius?

B What is $\overline{CD}$ called?

C Which is a minor arc of the circle, $\overset{\frown}{ABD}$ or $\overset{\frown}{ACD}$?

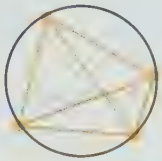
2. A circle has a radius of 5 centimetres with centre at point P. Which word (**interior**, **exterior**, or **on**) describes the location of each of the following points of the circle?

A Point X is 3.7 centimetres from P.

B Point Y is 50 millimetres from P.

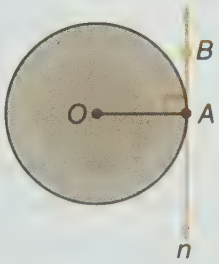
C Point Z is 6.09 centimetres from P.

3. Extend the table at the right up through 10 points on a circle.



Number of points	Number of chords
2	1
3	3
5	

4. Suppose that a line n is perpendicular to the radius $\overline{OA}$ at point A. Let B be any other point on line n.

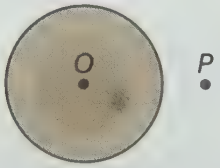


A Which is longer, $\overline{OB}$ or $\overline{OA}$?

B Does line n touch the circle in more than one point?

C Is line n tangent to the circle?

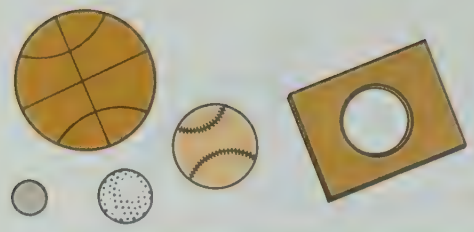
★ **5.** **A** Draw a circle with centre O and a radius of 2 centimetres. Mark a point P in the exterior of the circle 3 centimetres from O.



B Draw at least 10 circles whose centres are on the circle and which pass through point P. The curve encompassing all such circles (the *envelope* of the circles) is called a **limaçon** (pronounced "lee' ma·son").

Investigating the Ideas

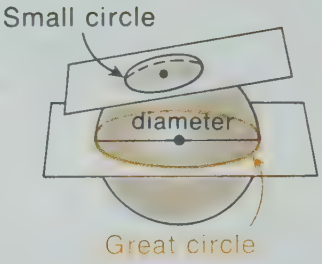
Choose a small ball such as a marble, golf ball, tennis ball, baseball, or handball.



Can you cut a circular hole in a piece of cardboard so that the ball you chose will just slip through the hole?

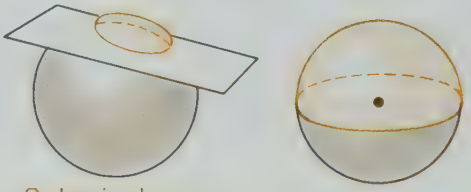
Discussing the Ideas

1. A ball is a model of a **sphere**. If the ball just slips through the circle that you cut in the cardboard, then the circle is the size of the **great circle** on the sphere.
   - **A** What is the diameter of the sphere that you chose?
   - **B** If the ball is too big for the hole, is the circle a **small circle** on the sphere?



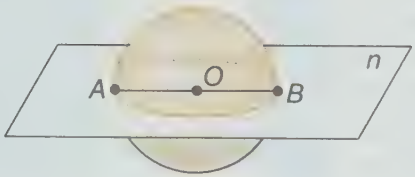
2. **A** How many points of intersection do two great circles on a sphere have?
   B How are the points of intersection related?

3. A plane cuts from a sphere a **spherical cap**. When is a spherical cap a **hemisphere** (half a sphere)?



4. If a plane and a sphere have only one point in common, then the plane is tangent to the sphere.
   - **A** What physical examples of a plane tangent to a sphere can you find?
   - **B** If a plane intersects a sphere in more than one point, what is their intersection?

1. In the figure, plane n contains point O, the centre of the sphere.



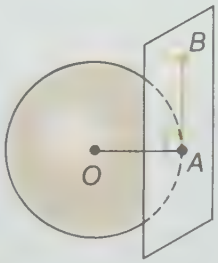
- **A** What is the intersection of plane n and the sphere?
- **B** What is $\overline{OB}$ called?
- **C** If P is a point such that $\overline{OB} \cong \overline{OP}$, where is point P?
- **D** What is the set of points on the sphere on one side of plane n called?

2. Can three points on a sphere all lie on one line? Explain.

3. Point A is on the sphere with centre O. What is the set of all points B such that $\overline{OB} < \overline{OA}$?

4.
- **A** What part of a sphere is related to a whole sphere as a semicircle is related to a whole circle?
- **B** An arc is related to a circle in the same way as what part of a sphere is related to a sphere?

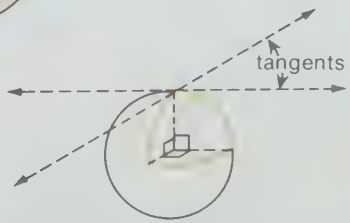
5. Suppose that a plane is perpendicular to the radius $\overline{OA}$ of a sphere at point A.



- **A** Which is longer, $\overline{OB}$ or $\overline{OA}$?
- **B** Does the plane touch the sphere at more than one point?
- **C** Is the plane tangent to the sphere?

6. Three arcs of great circles on a sphere form what is called a **spherical triangle**. The second figure shows a special spherical triangle. It encloses one eighth of the sphere's surface. Suppose you define the *measure of an angle of a spherical triangle* as the measure of the angle between the tangents of the circles at the great circle intersections.

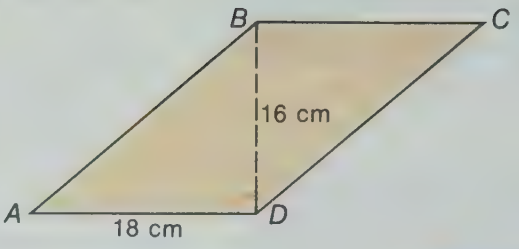
Spherical triangle



- **A** What would be the measure of each angle of this special triangle?
- **B** What is true for the sum of the angles of a plane triangle that is not true for a spherical triangle?

Investigating the Ideas

Suppose a label from a cylindrical food can is the size and shape of this parallelogram. Cut out two parallelograms like this one.



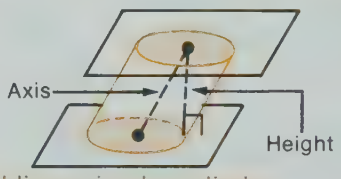
Can you show that the label above will fit around two differently-shaped cylindrical cans?

Discussing the Ideas

1. The parallelograms in the Investigation were used to form models of **right circular cylinders**. Which of the two cylinders you made do you think has the greater capacity?

Right circular cylinder

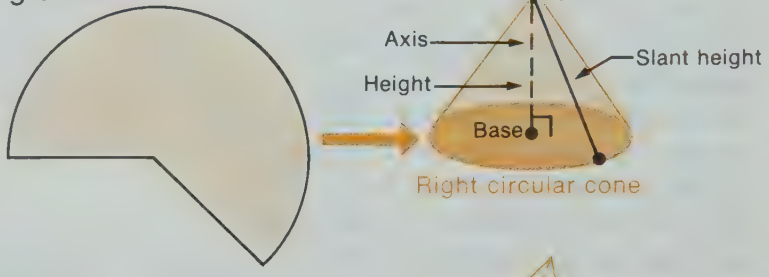
2. What is the difference between an **oblique cylinder** and a right cylinder?



Oblique circular cylinder

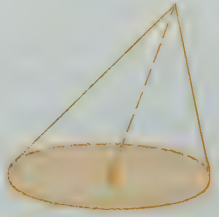
3. A portion of a circular region can be used to make a model of a **cone**.

   Make a model of a cone which has a slant height of 10 centimetres.



Right circular cone

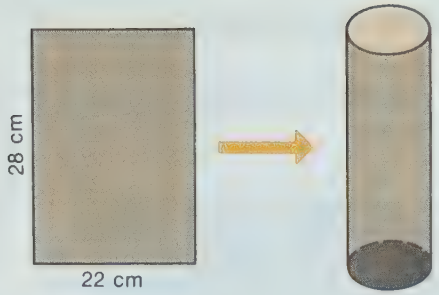
4. What do you think is the difference between an **oblique circular cone** and a right circular cone?


Oblique circular cone

1. The rectangular sheet of paper was used to form a model of a right circular cylinder.

A What is the height of the cylinder?

B Give the height of another cylinder that could be formed from this rectangular sheet of paper.



2. Patterns for models of right circular cones are shown below.

A

B

C

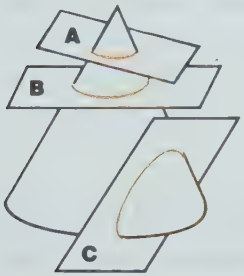
D

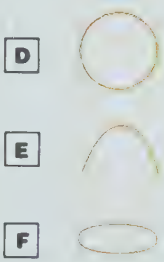
A What is the slant height of each cone?

B Which cone will have the greatest height?

C Which cone's base will have the greatest diameter?

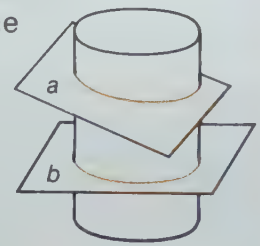
3. Three planes intersect this cone. Match each plane **A**, **B**, and **C** with the correct intersection **D**, **E**, or **F**.





4. **A** Sketch the intersection of plane *a* and the circular cylinder.

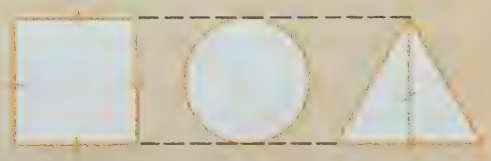
B Sketch the intersection of plane *b* and the circular cylinder.



think

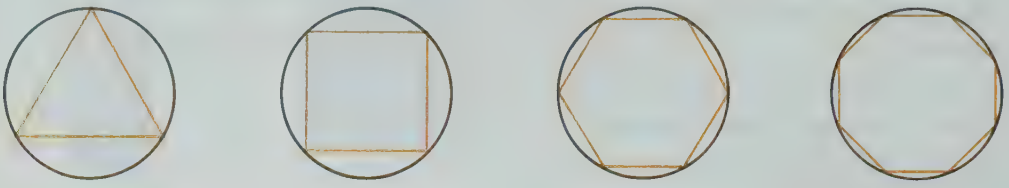
By turning it to the proper position, a solid of a certain shape can be passed through each of these three holes. At some stage of its passage through each hole, the solid will completely fill the hole.

Draw a picture or make a model of the solid.



Investigating the Ideas

Draw four circles each with a radius of 10 centimetres.



Can you inscribe regular polygons of 3, 4, 6, and 8 sides in the circles?

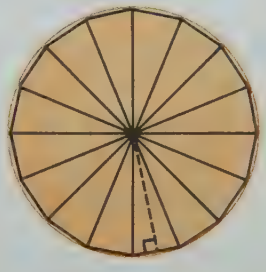
Discussing the Ideas

1. Find the measure of the sides of each inscribed polygon to the nearest tenth of a centimetre. Then record the length of one side and the perimeter of each polygon.

Polygon	Length of one side	Perimeter

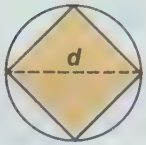
2. A Which inscribed polygon has the longest sides?
   B Which inscribed polygon has the greatest perimeter?
   C Do the perimeters of the inscribed polygons increase as the number of sides increase?

3. Consider regular inscribed polygons with a large number of sides.
   A What happens to the length of the sides as more sides are added?
   B What happens to the perimeter as more sides are added?



It is possible to prove that as the number of sides of the regular polygons increase, their **perimeters approach a definite number**. This number is the **circumference** of the circle.

1. **A** Draw circles with diameters of 4, 6, and 8 centimetres



B Inscribe squares in each circle.

C Measure the sides of each square to the nearest tenth of a centimetre and find the perimeter of each square.

D Compute the ratio $\frac{P}{d}$, when P is the perimeter of the square and d is the diameter of the circle.

E What do you notice about the ratios $\frac{P}{d}$ for the circles?

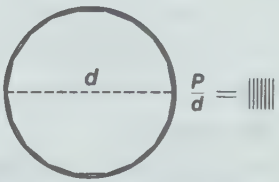
d	Length of side	P	$\frac{P}{d}$
4 cm			
6 cm			
8 cm			

2. Repeat Exercise 1, but inscribe regular hexagons in the circles.

3. Repeat Exercise 1, inscribing regular octagons in the circles.

4. Compare the ratios $\frac{P}{d}$ found in Exercises 1, 2, and 3. Does the ratio $\frac{P}{d}$ increase as the number of sides of the inscribed polygons increase?

5. It can be proved that as the number of sides of the regular inscribed polygons increase, the ratios $\frac{P}{d}$ **approach a definite number**. This number is the ratio of the circumference of a circle to its diameter. What is your estimate of this number?

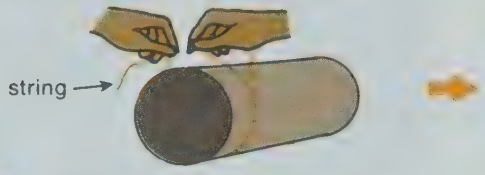


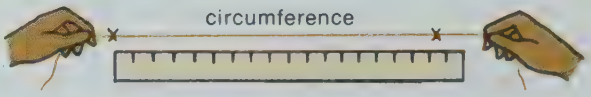
Think

The eight small equilateral triangles can be formed using 16 toothpicks. Can you remove four of the toothpicks so that only four small equilateral triangles will remain? No partial triangles allowed!

Investigating the Ideas

Choose several cylindrical objects with different diameters. The illustration below suggests a way of using a string to find the circumference of a circle.





Can you complete a table like the one at the right for the circles you chose?

Circumference (C)	Diameter (d)	$\frac{C}{d}$
62.4 cm	19.9 cm	3.135
27.5 cm	8.8 cm	3.125

Discussing the Ideas

1. A Is the last column in the table approximately the same number for all the circular objects?
   B Find the average of the numbers in the last column.

2. The ratio $\frac{c}{d}$, which is the same for all circles, is given a special name, π (pronounced "pie"). The table at the right gives various approximations for π.
   A What is π to the nearest hundredth?
   B What is π to the nearest thousandth?

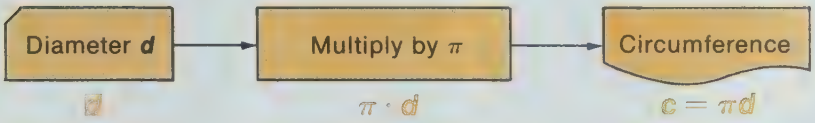
HISTORICAL APPROXIMATIONS FOR π

Babylonia, 2000 B.C., $\pi = 3$
Egypt, 1700 B.C., $\pi = \left(\frac{16}{9}\right)^2$
China, 1000 B.C., $\pi = 3$
100 A.D., $\pi = \sqrt{10}$
300 A.D., $\pi = 3\frac{1}{8}$
500 A.D., π computed to six decimal places
Greece, Archimedes,
250 B.C., $3\frac{10}{71} < \pi < 3\frac{1}{7}$
Europe, Middle Ages, $\pi = 3, 3\frac{1}{7}, \sqrt{10}$
Today, with digital computers, π has been computed to 100,000 decimal places.
$\pi = 3.141592653589\ldots$

3. Since $\frac{C}{d} = \pi$, you can find the circumference of any circle by the formula $C = \pi d$. What is the approximate circumference of a circle whose diameter is 10 centimetres? (Use $\pi \approx 3.14$.)

4. If r is the radius of a circle, then the diameter $d = 2r$. Therefore, $C = \pi \cdot d = \pi \cdot 2r = 2\pi r$. What is the approximate circumference of a circle whose radius is 4 centimetres?

1. Find the circumference of each circle with the given diameter. The flow chart may help you.



A $d = 8$ cm **B** $d = 15$ cm **C** $d = 6.2$ cm **D** $d = 120$ mm **E** $d = 0.71$ m **F** $d = 1.3$ m **G** $d = 1$ dm **H** $d = 2.4$ cm

2. A circle has a circumference of 83.21 centimetres. What is its diameter? (Use $\pi \approx 3.14$.)

3. Compute the circumference of each circle with the given radius. The flow chart may help you.

A $r = 5$ cm **B** $r = 6.8$ mm **C** $r = 37$ mm **D** $r = 0.8$ m

4. The Earth is spherical with a radius of about 6400 kilometres. What is the approximate circumference at the equator to the nearest thousand kilometres?

★**5.** Suppose a model of a cone is made from the pattern shown.

A What is the circumference of the base of the cone?

B What is the diameter of the base of the cone?

★**6.** A bicycle wheel has a diameter of 65 centimetres.

A About how many revolutions will it make per kilometre?

B At 30 km/h, about how many revolutions will the wheel make per minute?

★**7.** The best approximations for π are computed by using **infinite series**. The following is one way of getting π from an infinite series:

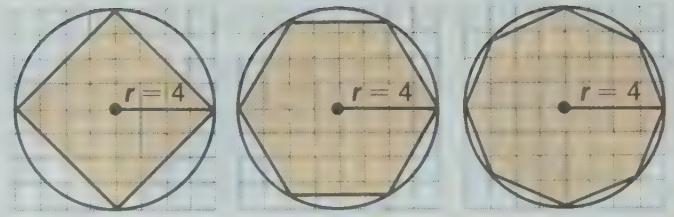
First compute $\alpha = \frac{1}{2} - \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 + \frac{1}{5} \cdot \left(\frac{1}{2}\right)^5 - \frac{1}{7} \cdot \left(\frac{1}{2}\right)^7 + \cdots$

Then compute $\beta = \frac{1}{3} - \frac{1}{3} \cdot \left(\frac{1}{3}\right)^3 + \frac{1}{5} \cdot \left(\frac{1}{3}\right)^5 - \frac{1}{7} \cdot \left(\frac{1}{3}\right)^7 + \cdots$

Then compute $4 \cdot (\alpha + \beta)$ to get an approximation for π.

Investigating the Ideas

Regular polygons of 4, 6, and 8 sides are inscribed in circles whose radii are 4 units.



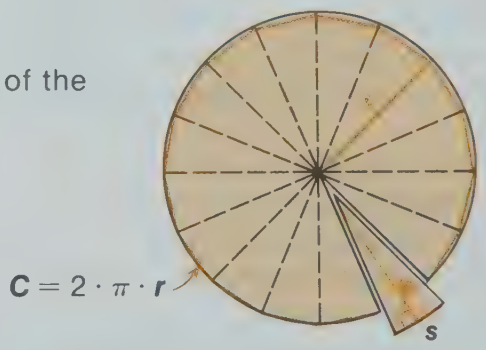
Can you estimate the area of each regular polygon by counting squares?

Discussing the Ideas

1. As the number of sides of the polygon increased, did the area also increase?

2. It can be proved that as the number of sides increase the areas of the inscribed regular polygons in a given circle approach a definite number, depending, of course, on the radius of the circle. This number is defined to be the **area of the circle**.

   What are your estimates for the areas of the circles in the Investigation?



3. The figure at the right is a regular polygon inscribed in a circle of radius r. Think of cutting the polygonal region to form the parallelogram below.
   - A Explain why the base of the parallelogram is nearly $\pi \cdot r$.
   - B Why is the height h nearly r?
   - C Explain why the area of the parallelogram is almost $\pi \cdot r \cdot r$ or $\pi \cdot r^2$.

4. The area A of any circle can be found by the formula $A = \pi r^2$ where r is the radius of the circle.
   - A What are the areas of the circles in the Investigation?
   - B How close is this to your estimate in Exercise 2?

1. Find the area of each circle with the given radius.
The flow chart may help you.

A $r = 7$ cm **B** $r = 10$ cm **C** $r = 1.2$ m **D** $r = 0.07$ mm

2. The area of a circle is 78.5 square centimetres.
What is its radius? (Use $\pi \approx 3.14$.)

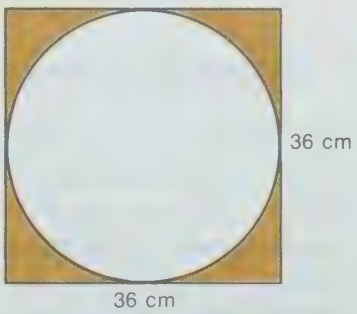
3. Copy and complete the table of data about circles. (Use $\pi \approx 3.14$.)

	RADIUS	DIAMETER	CIRCUMFERENCE	AREA
A	8.0 cm			
B		5 m		
C				12.56 cm^2
D		20 km		
E			14.13 cm	
F	0.12 cm			

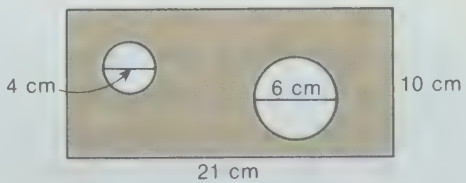
4. A circular mirror is cut from a square piece of glass 36 centimetres on each side.

A If the diameter of the mirror is as large as possible, what is its area?

B What is the area of the glass outside the circle?



★ **5.** Two circular holes are cut in a rectangular piece of sheet metal. The holes are 4 centimetres and 6 centimetres in diameter. The sheet of metal is 21 centimetres by 10 centimetres. What is the remaining area of the rectangular piece after the holes are cut?



Circumference and Area Problems

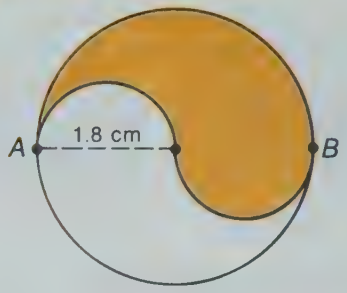
1. **A** What is the circumference of the large circular region?

B What is the distance along the S-shaped curve from *A* to *B*?

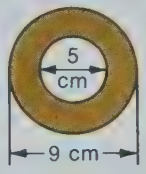
C What is the distance from *A* to *B* around the large circle?

D What is the area of the large circle?

E What is the area of the shaded portion of the circle?

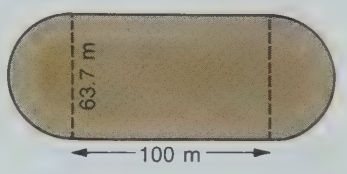


2. Suppose that the ring is stamped out of sheet metal weighing 13 grams per square centimetre. What is its mass?



3. A giant sequoia tree has a circumference of 30.9 metres. What is the diameter of the tree to the nearest tenth of a metre?

4. An oval track has two straight sides 100 metres in length. The ends of the track are semicircles with diameters of 63.7 metres. What is the distance, to the nearest metre, around the track?

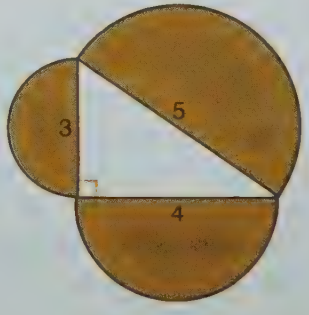


5. What is the area enclosed by the oval track in Exercise 4?

6. The Santa Maria Del Tule Tree, a Montezuma cypress in Oaxaca, Mexico, is reported to have the greatest circumference of any tree in the world. At a height of 150 centimetres above the ground, the circumference is 45.7 metres. What is the diameter of the tree to the nearest tenth of a metre?

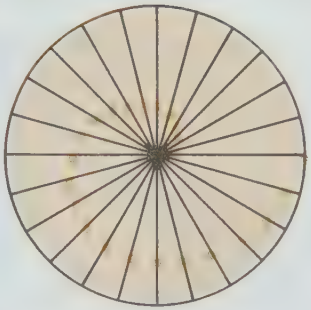
7. **A** Find the area of the semicircle on each side of the right triangle.

B Show that the sum of the areas of the two smaller semicircles is equal to the area of the semicircle on the hypotenuse.

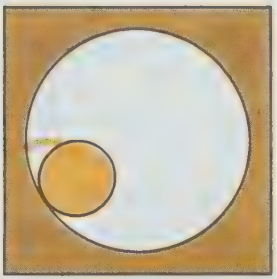


SPECIAL CURVES

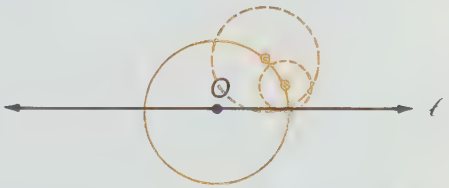
1. Draw a circle with a 10-centimetre diameter. Draw a radius every 15°. Mark a dot at the centre of the circle, then measure 0.4 centimetres out from the centre and mark a dot on a radius. Then put a dot 0.8 centimetres from the centre on the next radius, a dot 1.2 centimetres from the center on the next radius, and so on, until you have put a dot on each radius. Connect the dots with a smooth curve to form a **spiral**.



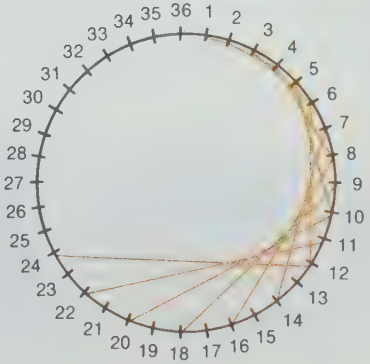
2. Cut a circular hole 12 centimetres in diameter in a piece of cardboard. Cut a circular disk 4 centimetres in diameter from the cardboard. Mark a point *P* on the edge of the disk. Roll the disk along the circular hole (without slipping) and mark the path of point *P* with dots. Connect the dots with a smooth curve. This curve is called a **deltoid**.



3. Draw a circle with centre *O* and a line ℓ containing *O*. Now construct many circles all with their centres on the first circle and tangent to line ℓ. The envelope of all these circles will be a kidney-shaped curve called a **nephroid**.

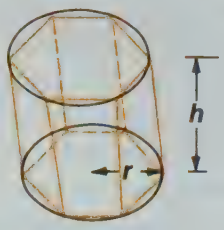


4. Use a protractor to divide a circle into 36 congruent arcs by marking points every 10° around the circle. Number the points successively from 1 through 36. Join point 1 with 2 by a segment, point 2 with 4, point 3 with 6, in general, point ***n*** with 2***n***. For points numbered more than 18, you will have to repeat. Thus, 19 will be joined with 2, 20 with 4, and so on. The curve outlined by these segments is a **cardioid**.



Discussing the Ideas

1. Think about a prism inscribed in a cylinder (as in the figure). We know that the volume of the prism is the area of the base times the altitude to the base.



Volume $= \pi \cdot r^2 \cdot h$

   A How does the volume of the prism compare with the volume of the cylinder?

   B As the number of sides of the prism is increased, how does the resulting volume compare to that of the cylinder?

   C Explain why the volume of the cylinder is

$$V = \text{area of base} \cdot \text{height} = \pi \cdot r^2 \cdot h$$

2. In the figure, a right prism is inscribed in a right circular cylinder.

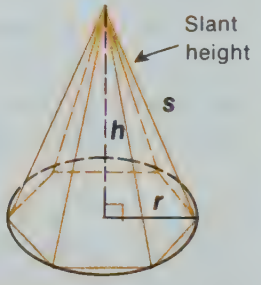
Lateral surface area $= 2 \cdot \pi \cdot r \cdot h$

   A How does the lateral surface area (surface area not including the two bases) of the prism compare to that of the cylinder?

   B As the number of sides of the prism is increased, how does the resulting lateral surface area of the prism compare to that of the cylinder?

   C The lateral surface area of a right prism is the perimeter of the base times the height. Explain why the lateral surface area of the cylinder is $2 \cdot \pi \cdot r \cdot h$.

3. In the figure, a right pyramid is inscribed in a right circular cone.



Volume $= \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$

Lateral surface area $= \pi \cdot r \cdot s$

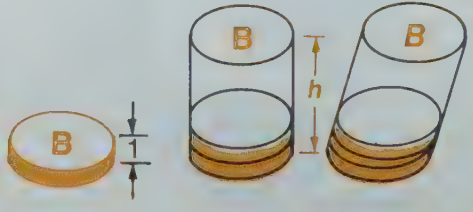
   A Explain why the volume of the cone is $\frac{1}{3} \cdot \pi \cdot r^2 \cdot h$.

   B Explain why the lateral surface area of the cone is half the circumference times the slant height, or $\pi \cdot r \cdot s$.

   C Explain why $s^2 = r^2 + h^2$.

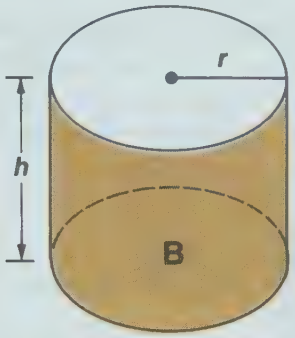
   D If $r = 3$ and $h = 4$, what is s?

4. If a cylinder is 1 unit thick, what is its volume if the area of the base is B? Explain the formula for the volume of a cylinder in terms of counting slabs of a unit thickness.

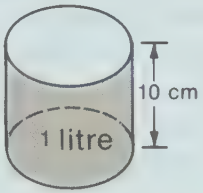


1. Copy and complete the table for right circular cylinders. (Use $\pi \approx 3.14$.)

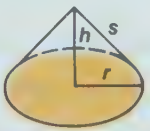
	Radius r	Height h	Area of base b	Volume V	Lateral surface
A	6 cm	10 cm			
B	12 cm	6 cm			
C	20 cm	60 cm			
D	0.5 cm	0.25 cm			
E		8 cm		220 cm³	



2. A cylindrical can is designed to hold one litre (1000 cm³). The height of the can is 10 centimetres. What is the diameter of the base to the nearest tenth of a centimetre?

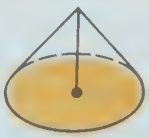


3. Copy and complete the table. The dimensions are for right circular cones. (Use $\pi \approx 3.14$.)

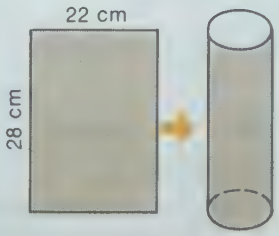


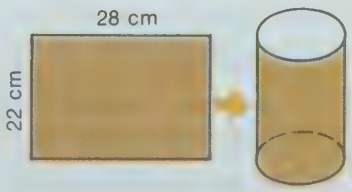
	Radius r	Height h	Slant height s	Area of base B	Circumference of base	Volume V	Lateral surface
A	6 cm	8 cm					
B	1.2 dm	1.6 dm					
C	5 m		13 m				

4. About how many cubic metres of sand are in a conical sand pile 2.5 metres in diameter and 1.2 metres high?



5. Two rectangular sheets of typing paper, each 22 by 28 centimetres, are used to form models of cylinders.





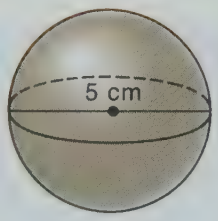
A What is the lateral surface area of each cylinder?

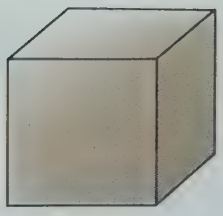
B Which cylinder has the greater volume?

Volume and Surface Area of Spheres

Investigating the Ideas

Suppose you made a model of a sphere with a diameter of 5 centimetres from modelling clay. Then you formed a cube from the clay sphere.





What is your estimate for

A the length of an edge of the cube?

B the volume of the cube?

C the surface area of the cube?

Find some ways to check your estimates.

Discussing the Ideas

1. Will the length of one edge of the cube be longer or shorter than the diameter of the sphere?

2. The volume of the cube is the same as the volume of the sphere. What is your estimate for the volume of the sphere?

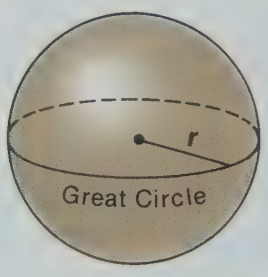
3. Do you think the surface area of the cube is the same as the surface area of the sphere? If not, which has the greater surface area?

4. We shall accept, without proof, the formula for the surface area of a sphere with radius r to be:

   $\text{Surface area} = 4\pi r^2$

   A How does the surface area of a sphere compare with the area of a great circle of a sphere?

   B What is the surface area of the sphere in the Investigation?



$\text{Surface area} = 4 \cdot \pi \cdot r^2$

$\text{Volume} = \frac{4}{3} \cdot \pi \cdot r^3$

5. The formula for the volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$

   A Use this formula to compute the volume of the sphere in the Investigation.

   B How close was your estimate for the volume of the cube to the calculated volume of the sphere?

1. Copy and complete the table for spheres. (Use $\pi \approx 3.14$.)

	Radius	Volume	Surface Area
A	3 cm		
B	6 mm		
C	11.4 cm		
D	8 cm		
E	1 m		

2. What is the mass of a solid iron ball with a 10-centimetre diameter if one cubic centimetre of iron has a mass of 7.8 grams?

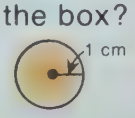
3. The Earth is nearly a sphere with a radius of 6400 kilometres.

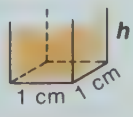
- **A** What is its surface area?
- **B** About 70.6% of the Earth's surface is covered by water. How many square kilometres is this?
- **C** What is the Earth's volume?

4. The radius of Earth's moon is about 1728 kilometres.

- **A** What is the surface area of the moon?
- **B** What is its volume?

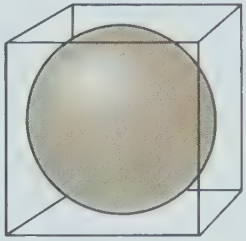
5. A ball of wax has a radius of 1 centimetre. When it is softened, it is pushed into a box with a square base 1 centimetre on each side. About how high will the wax be in the box?



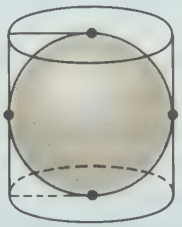


6. A basketball has a radius of approximately 12 centimetres.

- **A** What is the volume of the ball?
- **B** How much larger is the volume of a cube that will just hold the ball?
- **C** What is the surface area of the ball?
- **D** How much larger is the surface area of the cube than that of the ball?



7. Archimedes, the greatest of the Greek mathematicians, took pride in his proof that the ratio of the volume of the sphere to the volume of the touching cylinder is $\frac{2}{3}$. Show that this is so.

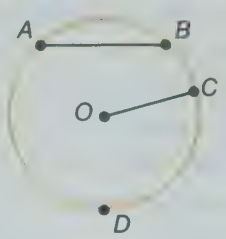


8. One sphere has twice the radius of another. How do their volumes compare? How do their surface areas compare?

9. What happens to surface area and volume if the radius of a sphere is tripled?

REVIEWING THE IDEAS

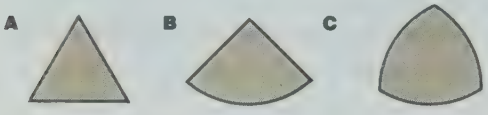
1. Name each of the following on the circle.



- **A** A chord.
- **B** A radius.
- **C** The centre.
- **D** A minor arc containing point *C*.

2. If a plane intersects a sphere and contains the centre of the sphere, what is this intersection called?

3. Which pattern will form a model for a right circular cone?



4. What is the ratio of the circumference of a circle to its diameter?

5. State the formula for the circumference of a circle.

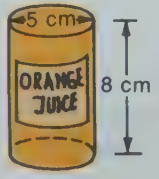
6. Find the circumference of a circle with a radius of 12.3 centimetres. (Use $\pi \approx 3.14$.)

7. If a circle has a circumference of 17.36 metres, what is the radius of the circle to the nearest tenth of a metre?

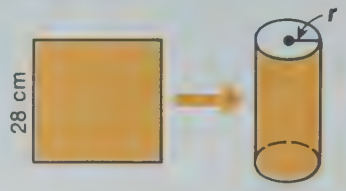
8. What is the formula for the area of a circle with radius r?

9. Find the area of a circle with a radius of 6.12 centimetres.

10. What is the area of a rectangular piece of paper that will just fit around the juice can?



11. A model of a right circular cylinder is made from a square sheet of paper 28 centimetres on each side.

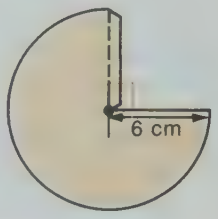


What is the lateral area of the cylinder?

12. What is the radius of the base of the cylinder in Exercise 11?

13. A model of a circular cone is made from $\frac{3}{4}$ of a circular region with a radius of 6 centimetres.

- **A** What is the slant height?
- **B** What is the circumference of the base?
- **C** What is the radius?



14. A cylindrical tank has an inside diameter of 2.1 metres and a height of 1.5 metres. What is the volume of the tank?

15. What is the volume of a cone whose base and height are the same as the cylinder in Exercise 14?

16. What is the lateral surface area of a right circular cone with a height of 12 centimetres and the diameter of the base 14 centimetres?

17. A golf ball has a radius of about 2.1 centimetres.

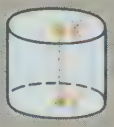
- **A** Find the volume.
- **B** Find the surface area.

TEST YOURSELF

1. Give the name of $\overline{XY}$ for each figure.

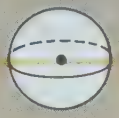
A

B



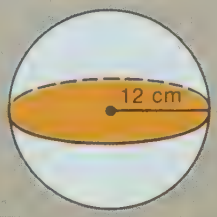
C

D



2. If a circle has a diameter of 20 centimetres, what is its circumference? (Use $\pi \approx 3.14$.)

3. A sphere has a radius of 12 centimetres. What is the circumference of a great circle of the sphere?

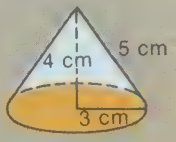


4. What is the area of one side of this coin to the nearest hundredth of a centimetre?

5. A cylinder has a radius of 4 centimetres and a height of 6.5 centimetres. What is the lateral surface area?

6. What is the volume of the cylinder in Exercise 5?

7. What is the lateral surface area of this circular cone?



8. What is the volume of the cone in Exercise 6?

9. What is the surface area of a sphere with a radius of 2 metres?

10. What is the volume of a ball whose radius is 7 centimetres? Round your answer to the nearest hundredth of a cubic centimetre.

RESEARCH PROJECTS

A Investigate several different kinds of mathematical curves such as the **astroid**, the **deltoid**, **parabola**, **ellipse**, and **hyperbola**. Show constructions of the curves. (See *Book of Curves* by E. H. Lockwood; Cambridge, Massachusetts: C.U.P., 1963; Macmillan.)

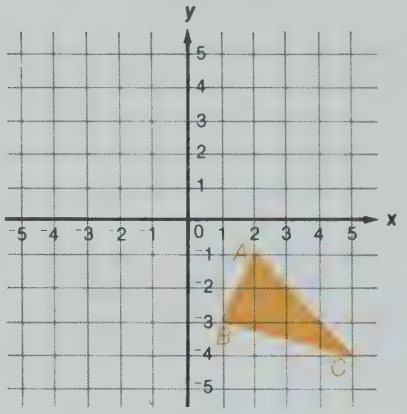
B Make a collection of ordinary objects that are examples of cylinders, cones, and spheres. Find the dimensions of each and then use the appropriate formulas to find the area and volume of each object. Make a display of them.

C Make a geometric pattern using colored threads and heavy paper. (See *Curves in Space* by Donovan A. Johnson; St. Louis: Webster, 1961; McGraw-Hill Ryerson.)

D Mosaic designs are often formed from polygons and other geometric figures. Using a straightedge and a compass, construct your own mosaic design. (See *Mathematical Recreations* by Maurice Kraitchik; New York: Dover Publications, 1953, pp. 193–213.) General Publishing.)

CUMULATIVE REVIEW

For Exercises 1, 2, and 3, draw $\triangle ABC$ in the position shown on the grid.



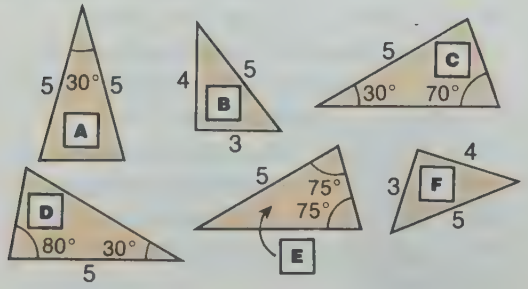
1. Show the slide image of $\triangle ABC$ after a translation of "left 3, up 5."

2. Show the reflection image of $\triangle ABC$ if the **x**-axis is the reflecting line.

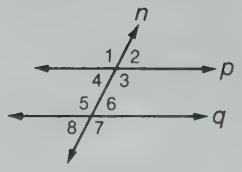
3. Using the origin (0.0) as a centre of rotation, a rotation image of point A is $(1, 2)$. What are the rotation images of B and C for the same rotation?

4. Suppose $\triangle PQR \cong \triangle XYZ$.
   A $\angle Q \cong$ ▮▮▮ **B** $\overline{PQ} \cong$ ▮▮▮

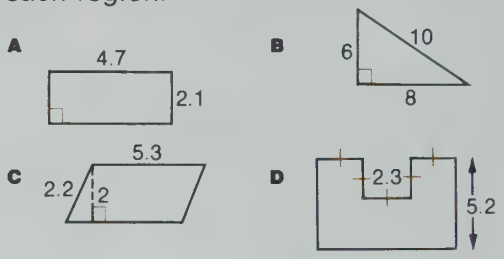
5. State which pairs of triangles are congruent and give the congruence theorem (SSS, SAS, or ASA) that applies for each pair.



6. Lines p and q are parallel. Name the angles congruent to $\angle 1$.

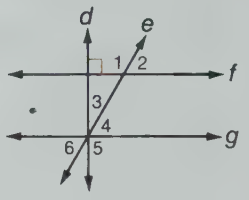


7. Find the area and perimeter of each region.

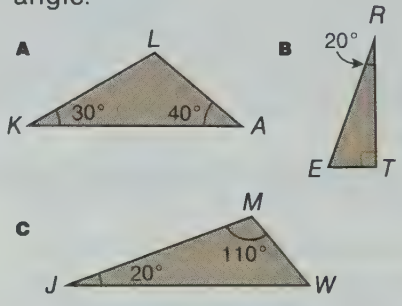


8. Suppose $f \parallel g$ and $m\angle 2 = 66°$.

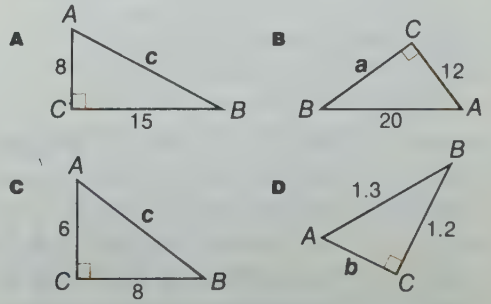
   What is the measure of each of the other angles?



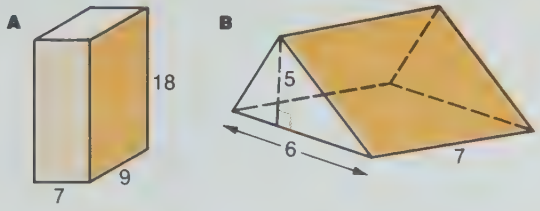
9. Find the measure of the third angle.



10. Find the length of the third side of each triangle.

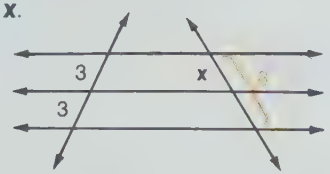


11. Find the volume of each solid.

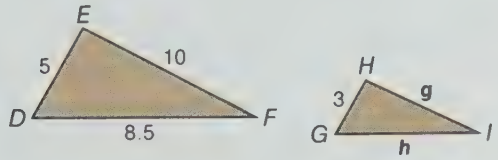


12. If a pyramid has a base area of 276 square metres and a height of 15 metres, what is the volume?

13. The horizontal lines are parallel. Find length **x**.



14. In the figure, $\triangle DEF \sim \triangle GHI$. Find the length of ***g*** and ***h***.

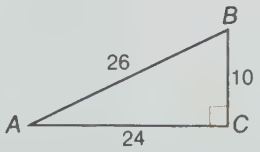


15. $\triangle ABC$ is a right triangle. Give each ratio.

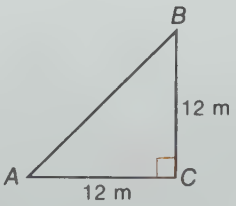
A $\sin A$

B $\cos A$

C $\tan A$



16. If $\cos A = 0.707$, what is the length of the hypotenuse to the nearest hundredth of a metre?



17. In every circle, what is the ratio of the circumference to the diameter?

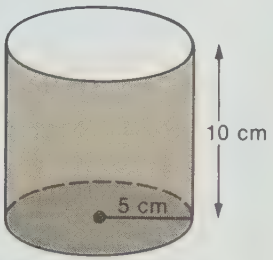
18. What is the circumference of a circle whose diameter is 10 centimetres?

19. What is the area of a circle whose radius is 6 metres?

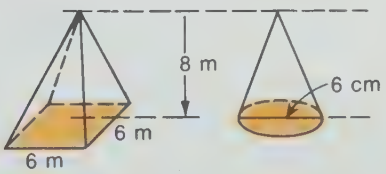
20. **A** What is the volume of this cylinder?

B What is its lateral surface area?

C What is its total surface area?



21. A pyramid has a square base 6 metres on each side and a height of 8 metres. A circular cone has a diameter of 6 metres and a height of 8 metres.



A Find the volume of the pyramid.

B Find the volume of the cone.

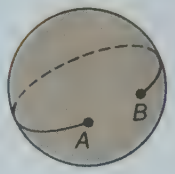
22. A sphere has a radius of 2 centimetres.

A Find the volume.

B Find the surface area.

GEODESICS

A **geodesic** is the shortest path between two points on the *surface* of a geometric figure. On a sphere, a geodesic is an arc of one of the great circles of the sphere. Airline pilots flying long distances generally fly along the geodesics of the Earth.



If you stretch a string tightly between two points on a globe, the string will fall along the geodesic for the two points. Use a piece of string and a globe to find the geodesic between each pair of cities in the first list. Then choose the name of the third city, from the second list, that also lies almost directly on the geodesic of the pair in the first list.

Geodesic determined by:	Third city on geodesic
1. Vancouver, B.C. and Miami, Florida	**A** Bangalore, India
2. New York City and Berlin, Germany	**B** Berlin, Germany
3. Peking, China and London, England	**C** Calcutta, India
4. Mexico City and New York City	**D** Fairbanks, Alaska
5. London, England and Sidney, Australia	**E** Glasgow, Scotland
6. Rome, Italy and Chicago, Illinois	**F** Havana, Cuba
7. Toronto, Canada and Tokyo, Japan	**G** Helsinki, Finland
8. Portland, Oregon and Melbourne, Australia	**H** Honolulu, Hawaii
9. Addis Ababa, Ethiopia and Singapore, China	**I** Le Mans, France
10. Madrid, Spain and Seoul, Korea	**J** New Orleans, Louisiana
11. Copenhagen, Denmark and Denver, Colorado	**K** Reykjavik, Iceland
12. Detroit, Michigan and Lima, Peru	**L** Topeka, Kansas

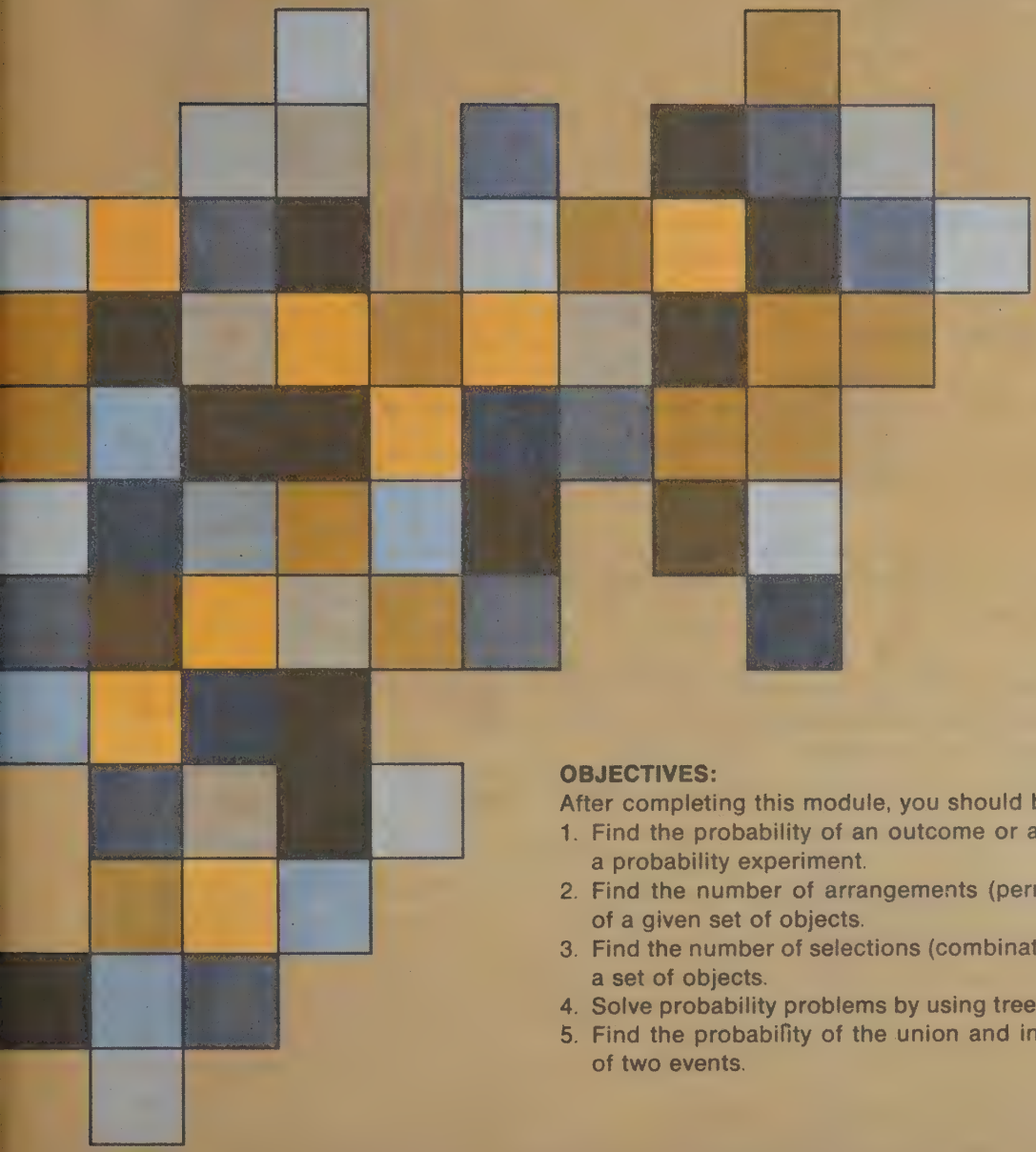
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A chord; B axis; C slant height; D diameter **2.** 62.8 cm
3. 75.36 cm **4.** 3.46 cm^2 **5.** 163.28 cm^2 **6.** 326.56 cm^3
7. 47 1 cm^2 **8.** 37.68 cm^3 **9.** 50.24 m^2 **10.** 1436.03 cm^3

UNIT H: *Some Special Topics in Mathematics*

MODULE 1: *Probability*



OBJECTIVES:

After completing this module, you should be able to:

1. Find the probability of an outcome or an event in a probability experiment.
2. Find the number of arrangements (permutations) of a given set of objects.
3. Find the number of selections (combinations) from a set of objects.
4. Solve probability problems by using tree diagrams.
5. Find the probability of the union and intersection of two events.

Investigating the Ideas

Suppose the jar at the right holds 650 orange beads and 350 black beads, and they are well mixed. Suppose you grab a handful of beads and find that you drew out 43 beads.

Can you predict how many orange beads and how many black beads you would get?

Discussing the Ideas

1. **A** What are your predictions of the number of beads of each color for some different-sized handfuls?
   B How did you make your predictions?

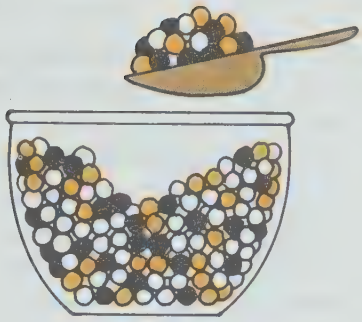
2. Do you think it is very likely that someone could grab a handful of beads that are all the same color? Give your reasons.

3. There are 650 *chances* in 1000 of drawing an orange bead. The **probability** of drawing an orange bead is $\frac{650}{1000}$. We write: $P(\text{orange bead}) = \frac{650}{1000} = 0.65$.
   A What are the chances of drawing a black bead?
   B What is $P(\text{black bead})$?

4. Suppose you had a container which would hold exactly 100 beads when full.
   A About how many orange beads would you expect to see if you poured out the beads?
   B About how many black beads?

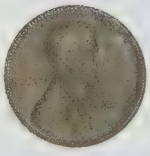
5. The spinner dial is divided into four parts of equal size.
   A What are the chances of the arrow stopping in region 1?
   B What is $P(1)$?
   C About how many times would you expect the arrow to stop in region 3 in 200 spins?

1. A container has 500 black beads, 300 orange beads, and 200 white beads. Suppose you can scoop out 100 beads at a time.



- **A** How many beads of each color would you expect in a scoop of 100 beads?
- **B** The chances of drawing a black bead are ? in 1000. P(black bead) = ?.
- **C** The chances of drawing an orange bead are ? in 1000. P(orange bead) = ?.
- **D** The chances of drawing a white bead are ? in 1000. P(white bead) = ?.

2. Suppose a coin is tossed 100 times.



- **A** About how many times would you expect it to come up heads?
- **B** P(Heads) = ? and P(Tails) = ?.

3. **A** When a die is tossed, there is 1 chance in ? that the face with 5 dots will come up.

- **B** What is the probability of getting a 5 with one toss of a die?

4. The spinner dial is divided into three parts of equal size.

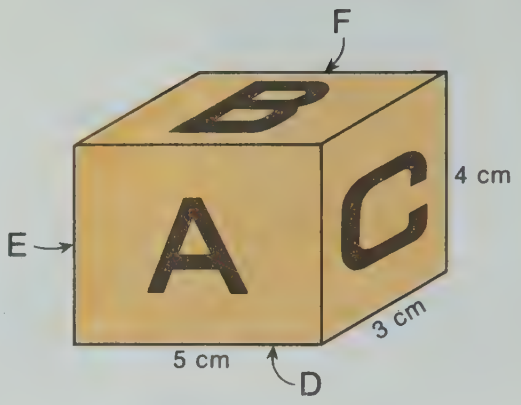
- **A** When the arrow is spun, the chances of its stopping in region 2 are 1 in ?.
- **B** The probability of the arrow stopping in region 2 is ?.
- **C** How many times would you expect the arrow to stop in region 3 in 60 spins?
- **D** How many times would you expect the arrow to stop in region 1 in 300 spins?

think

Arrange the nine prime numbers 5, 17, 29, 47, 59, 71, 89, 101, and 113 to form a three-by-three magic square.

Investigating the Ideas

Suppose you had a rectangular block as shown and labelled the six faces A, B, C, D, E, and F. Suppose you tossed the block and recorded the letter of the face that came up. How many times would you predict that each letter would come up in 100 tosses?



Can you check your prediction by trying the experiment?

Discussing the Ideas

1. Should the letters that are on the opposite faces, such as A and F, come up about the same number of times? Why?

2. The set of all possible outcomes for a probability experiment is called a **sample space** for the experiment. The sample space for the experiment in the Investigation could be designated as {A, B, C, D, E, F}.
If two outcomes have the same chances of occurring, the outcomes are **equally likely outcomes.**
   - **A** Which outcomes do you think are equally likely for the experiment?
   - **B** Do you think outcomes A and C are equally likely? Why or why not?

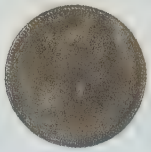
3. The table shows the results of 100 trials of the experiment above.
   - **A** Complete the table.
   - **B** If the experiment were repeated, do you think the experimental probabilities would be different?

Outcome	Occurrences	Experimental Probability of the Outcome
A	38	$\frac{38}{100}$
B	12	$\frac{12}{100}$
C	3	
D	10	
E	5	
F	32	

4. **A** Is each of the outcomes for tossing a die equally likely?
   - **B** Do equally likely outcomes have equal probabilities?

1. A sample space for a probability experiment is given. Give the probability of each outcome in the sample space.

A Toss a coin.



SAMPLE SPACE: $\{H, T\}$

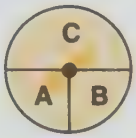
PROBABILITY: $\frac{1}{2}$ $\frac{1}{2}$

B Toss a die.

SAMPLE SPACE: $\{$ ⚀ ⚁ ⚂ ⚃ ⚄ ⚅ $\}$

PROBABILITY: $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

C Spin the arrow.



SAMPLE SPACE: {A, B, C}

PROBABILITY: $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

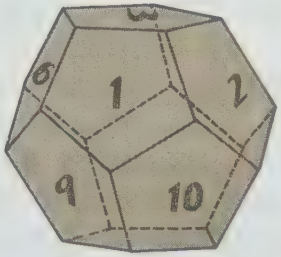
D Draw a card.

SAMPLE SPACE: {J, Q, K, A}

PROBABILITY: $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

2. What is the sum of the probabilities of the outcomes in a sample space?

3. A regular dodecahedron has its faces labelled 1 through 12. It is rolled and the number on the top face is recorded.



A What is a sample space for the experiment?

B Are the outcomes equally likely?

C What is the probability of each outcome?

4. The table shows the results of an experiment with three outcomes.

Outcome	Tally of Occurrences
A	𝍸 𝍸 𝍸 | | |
B	𝍸 |
C	|

A How many trials does the table show?

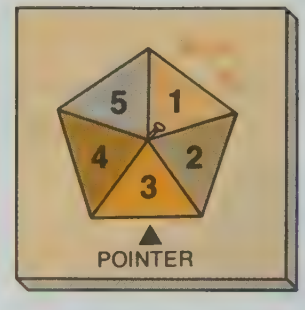
B Give the experimental probability for each outcome.

C On the basis of the experiment, how many times would you expect each outcome to occur in 100 trials of the experiment?

Probability of an Event

Investigating the Ideas

Suppose a probability experiment consists of spinning the pentagon and then recording the number of the region indicated by the pointer.



Can you predict about how many times in 100 spins the pointer would show an odd-numbered region?	Try the experiment to check your prediction.

Discussing the Ideas

1. The **event** of getting an odd-numbered outcome has occurred if any number in the set $\{1, 3, 5\}$ is an outcome of the experiment. The probability of an event is the sum of the probabilities of each outcome in the event. What is the probability of an odd-numbered outcome?

$$\text{Event: } \{1, \quad 3, \quad 5\}$$
$$\text{P(Event): } \tfrac{1}{5} + \tfrac{1}{5} + \tfrac{1}{5} = \blacksquare$$

2. A What subset of the sample space for the spinner experiment is the event for an even-numbered outcome?
   B What is the probability of the event?

3. The examples above illustrate three important ideas:
   1 Any subset of a sample space is an **event.**
   2 The **probability of the event** is the sum of the probabilities of the outcomes in that event.
   3 The sum of the probabilities of the outcomes in a sample space is 1.

   Suppose $F = \{1, 2\}$. Is F a subset of the sample space of the spinner experiment above? What is $P(F)$?

4. If an event is *certain* to happen, its probability is 1. Is the event of getting a number greater than zero in the spinner experiment certain?

5. If an event can *never* happen, the probability of the event is 0. The event of getting a number greater than 5 is the empty set ϕ. What is the probability of the event?

1. Complete the table for the spinner experiment shown.

Experiment: Spin the arrow.
Sample space: {1, 2, 3, 4}
Probability: $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

	Description of event	Subset of the sample space (Event)	Probability of the event
A	The event of getting an odd number as an outcome.	$A = \{1, 3\}$	$P(A) =$
B	The event of getting 1 as the outcome.	$B = \{1\}$	$P(B) =$
C	The event of getting a number between 1 and 4.	$C = \{?\}$	$P(C) =$
D	The event of getting a number greater than 0.	$D = \{?\}$	$P(D) =$
E	The event of getting a number greater than 4.	$E = \{?\}$	$P(E) =$

2. An experiment of tossing a die has a sample space {1, 2, 3, 4, 5, 6}. Give the probability that each event below will occur.

A {1, 3, 5} B {1, 4} C {6} D {1, 2, 3, 4, 5, 6} E {1, 2, 4, 6}

3. The example below shows how algebraic symbolism can be used to describe events.

Example (using the experiment in Exercise 2): $P(x > 3) = P(\{4, 5, 6\}) = \frac{1}{2}$

Give the probability.

A $P(x = 1 \text{ or } x = 4)$
B $P(x \text{ is even or } x > 3)$
C $P(x \neq 1)$
D $P(x \text{ is even and } x > 3)$
E $P(x = 1 \text{ or } x = 4 \text{ or } x = 6)$
F $P(x \neq 1 \text{ and } x \neq 2)$

4. The numerals 1 through 9 are written on slips of paper and placed in a container. An experiment consists of drawing one of the slips from the container. For each event described, give the subset of the sample space and the probability of the event. Part A is worked as an example.

A An odd number.
SOLUTION: $P(\{1, 3, 5, 7, 9\}) = \frac{5}{9}$
B An even number.
C A multiple of 4.
D A factor of 4.
E A square number.
F A prime number.
G A factor of 6.
H $P(x > 7)$
I $P(x \geq 7)$
J $P(x < 7)$
K $P(x \leq 7)$
★ L $P(x \text{ is even or } x < 7)$
★ M $P(x \text{ is even and } x < 7)$
★ N $P(x < 5 \text{ and } x \neq 3)$

More practice, page S-28, Set 54

Investigating the Ideas

Suppose you toss a penny and a nickel 100 times. How many times do you think you would get two heads?

Try the experiment and keep a tally of the events in a table like the one shown.

EVENT	TALLY
Two heads	
One head, one tail	
Two tails	

Can you predict the number of times each event would occur if you repeated the experiment 1000 times?

Discussing the Ideas

1. Steve thought that each of the three outcomes was equally likely, so he predicted that both coins would come up heads about 33 times in 100 tosses. Do your results agree with Steve's prediction?

2. Paula said that Steve was not correct and used the following chart to explain her reasoning:

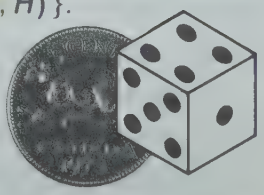
Nickel	Sample Space	
H	$\{(\ , H)$	The probability of both coins being heads is $\frac{1}{4}$.
T	$(\ , T)$	
H	$(\ , H)$	We write:
T	$(\ , T)\}$	$P(\ , H) = \frac{1}{4}$
Each of the **four** pairs is equally likely.	The set of four equally likely outcomes	

   A Each outcome in the sample space is denoted by an ordered pair. What is the probability of each outcome?

   B The event {one head, one tail} is the subset $\{(\ , T), (\ , H)\}$. Why are there two ordered pairs in this event?

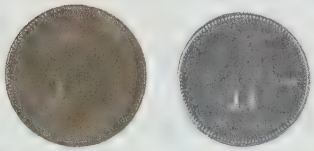
   C What is the probability of this event?

3. How many outcomes are there in the sample space for tossing a coin and a die at the time?



Outcome: $(T, 4)$

1. An experiment consists of tossing a penny and a dime. Two of the elements in a sample space for this experiment are (H, T) and (H, H) where the first letter in each pair indicates the outcome for the penny and the second letter the outcome for the dime.



Outcome: (, T)

A Give the complete sample space for the experiment.

B What is the probability of each outcome in the sample space?

2. Using the coin-tossing experiment in Exercise 1, give the subset of the sample space (event) described in each part.

A Penny heads and dime tails.

B Penny comes up heads.

C Dime comes up tails.

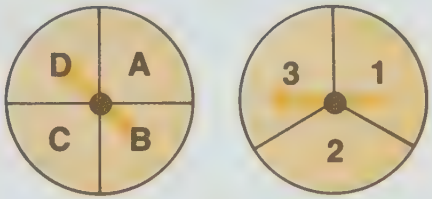
D At least one head comes up.

E One head and one tail come up.

F The two coins match.

3. Give the probability of each event in Exercise 2.

4. Two spinners like those shown are used in a probability experiment. Each outcome results in a letter and a numeral.

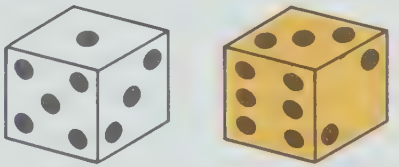


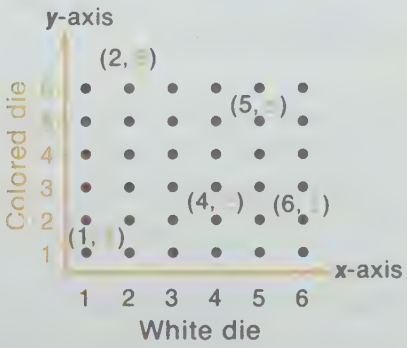
A What outcome is shown on the spinners?

B List all the outcomes as ordered pairs. How many are there in all?

C What is $P(\mathbf{C}, 2)$?

5. Two dice, one white and one colored, are tossed. The co-ordinate system shown at the right suggests how the sample space for the experiment can be found. Suppose **x** denotes the number on the white die and **y** denotes the number on the colored die. Compute the probabilities.





A $P(x + y = 2)$

B $P(x + y = 5)$

C $P(x + y = 10)$

D $P(x + y \neq 7)$

E $P(x + y = 8)$

F $P(x + y = 7)$

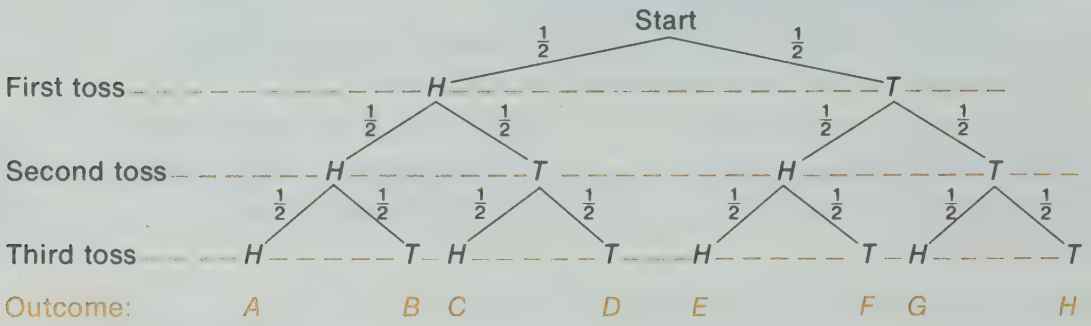
G $P(x + y = 12)$

H $P(x > y)$

Using Tree Diagrams in Probability

Discussing the Ideas

1. **Tree diagrams** are helpful in counting outcomes and computing probabilities of certain events. The tree diagram below illustrates all the outcomes when a coin is tossed three times in succession.



   A How many different outcomes are possible if a coin is tossed three times in succession?

   B Outcome *C* can be denoted by (*HTH*). What does the symbol stand for?

   C Describe outcome *A*.

2. Each branch of the tree is labelled "$\frac{1}{2}$" because the probability of tossing either a head or a tail is $\frac{1}{2}$. The tree diagram shows that there are 8 outcomes for three tosses of a coin. Since each outcome is equally likely, $P(A) = P(HHH) = \frac{1}{8}$ and $P(B) = P(HHT) = \frac{1}{8}$. Note that the probability of each outcome can be found by multiplying the probabilities along each branch of the tree. Thus $P(C) = P(HTH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

   A Which point in the tree diagram is associated with the outcome *THT*?

   B What is $P(THT)$?

3. **A** Explain how the tree diagram could be extended to show the outcomes for four tosses of a coin.

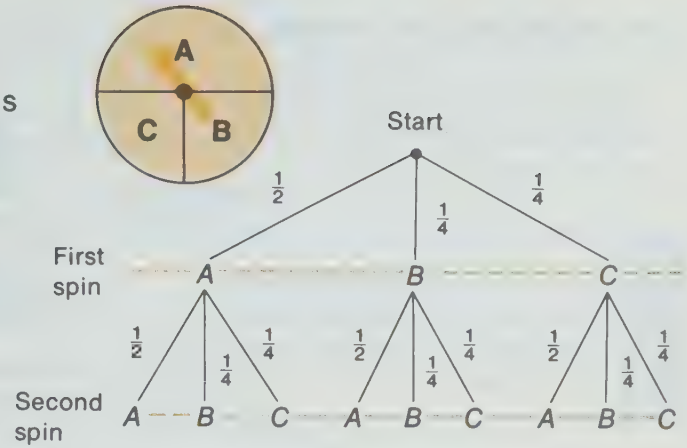
   B How many outcomes would there be for four tosses of a coin?

   C What is the probability of each outcome?

4. **A** How many outcomes would there be for five tosses of a coin?

   B What would be the probability of each outcome?

1. A spinner and a tree diagram for the experiment of flipping the spinner is shown at the right. The sample space for the experiment is $\{A, B, C\}$. $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{4}$, and $P(C) = \frac{1}{4}$. A symbol like AC represents the event of getting A on the first spin and C on the second spin. Give the probability of each event.



A AA **D** BA **G** CA
B AB **E** BB **H** CB
C AC **F** BC **I** CC

2. Two hockey teams are to play a series of 3 games. The teams are evenly matched, so for each game the probability that each team will win is $\frac{1}{2}$. Make a tree diagram for the three games, using W to represent the outcome of the home team winning and using L to represent the outcome of the home team losing. Label each branch with the probability $\frac{1}{2}$.

3. Using the tree diagram of Exercise 2, give the probability of each event.

A WW: the home team wins the first two games.
B WWL: the home team loses only the third game.
C WLW
D LWW
E The home team loses one game of the three.
F The visiting team wins the first and third games.

4. Three marbles, two colored and one black, are placed in a box. An experiment consists of drawing one marble from the box, recording its color, replacing the marble, drawing again and recording the color, and so on. Make a tree diagram for three draws of this experiment.

5. Use the tree diagram in Exercise 4 to find the probability of each event.

A BC **C** CC **E** CB **G** BBB
B BB **D** CCB **F** CCC **H** BCC

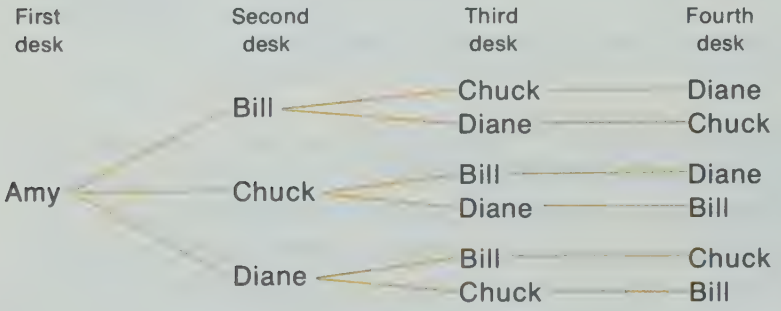
Investigating the Ideas

Amy, Bill, Chuck, and Diane are to be seated at four desks in one row.

How many different ways can you find for the four students to arrange themselves in a row?

Discussing the Ideas

1. One way of counting the number of arrangements for the students in the desks is to construct a tree diagram. Part of a tree diagram is shown below in which Amy occupies the first desk.



A How many seating arrangements does the tree show?

B Make a tree that shows Bill in the first desk.

C Make a tree that shows Chuck in the first desk.

D Make a tree that shows Diane in the first desk.

2. How can you use the tree diagrams in Exercise 1 to find the total number of arrangements of the four students at the four desks?

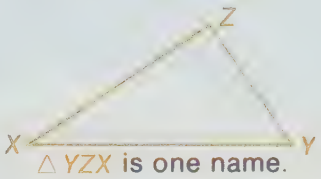
1. Three students, Jim, Sarah, and Michelle, are running for the offices of president, vice president, and secretary of a club. The person with the most votes becomes president, the person with next highest number of votes becomes vice president, and the third person becomes secretary.

- **A** Make a tree diagram that shows all the possible arrangements of the three students for the three offices.
- **B** How many arrangements are possible?

2. How many three-digit numerals can you form using each of the digits 2, 7, and 4 once in each numeral?

3. Four girls, Maria, Helen, Irene, and Jessie, are on a relay team. Each person runs one lap of the relay. How many different arrangements of order are possible for the four runners?

4. A triangle has its vertices labelled *X*, *Y*, and *Z*. How many different ways could the triangle be named using the three letters?



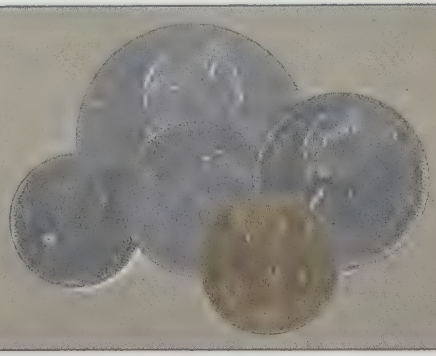
△*YZX* is one name.

★ **5.** The five cards are to be stacked on top of each other. How many different arrangements of the five cards are possible?

★ **6.** The names of six students are written on slips of paper, put into a box, and mixed thoroughly. The names are then drawn from the box and recorded in order. How many different possibilities are there for the order of the names?

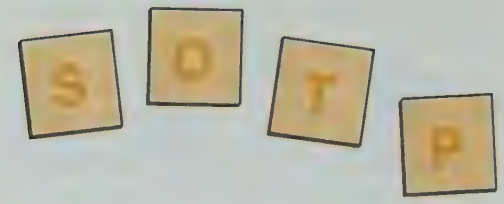
think

From the 5 coins shown, you can select any 3 of the coins. How many different amounts of money can be selected?



Investigating the Ideas

How many different 4-letter arrangements can you make using the letters on these cards?



How many meaningful words do your arrangements form?

Discussing the Ideas

1. An arrangement of the elements of a set in a certain order is called a **permutation** of the set. Thus, SOTP is one permutation of the set of letters above.

   How many permutations did you find for the four letters in the Investigation?

2. Study the table below.

Set	Permutations of the Set	Number of Permutations
A	*A*	1
A, B	*AB, BA*	$1 \cdot 2 = 2$
A, B, C	*ABC, ACB, BAC, BCA, CAB, CBA*	$1 \cdot 2 \cdot 3 = 6$

   A Does the product $1 \cdot 2 \cdot 3 \cdot 4$ give the number of permutations of a 4-element set?

   B How many permutations does a 5-element set have?

3. In general, the product $1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n$ gives the number of permutations of a set of n elements.

   A How many permutations for a set of 6 elements?

   B How many permutations for a set of 7 elements?

4. The product of all the whole numbers from 1 through n inclusive is called n **factorial** and is denoted by $n!$.

   $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n$

   Explain how to compute each of the following:

   A $3!$ **B** $5!$ **C** $10!$ **D** $3! \cdot 4!$ **E** $5! \div 4!$

1. Extend the table of factorials at the right to 12.

2. How many different arrangements of the letters are possible for the word DEN?

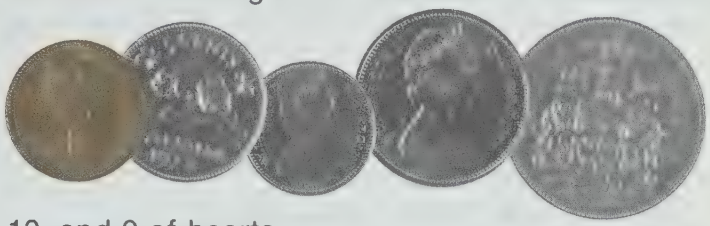
TABLE OF FACTORIALS

Number of objects	Number of arrangements	Symbol
1	1	1!
2	$1 \cdot 2 = 2$	2!
3	$1 \cdot 2 \cdot 3 = 6$	3!
4	$1 \cdot 2 \cdot 3 \cdot 4 = 24$	4!

3. Use the extended table of factorials to find the number of possible batting orders for a baseball team of nine players.

4. Seven students sit in a row of seven seats. In how many ways can these students be arranged in the seven seats?

5. How many ways can these coins be arranged in a row?



6. The ace, king, queen, jack, 10, and 9 of hearts are placed in a line on a table. In how many ways can they be placed in a line?

★ **7.** Solve the equations, avoiding computation if you can.

EXAMPLE: $6 \cdot 5! = n$ $\quad n = 6!$

A $10 \cdot 9!$

B $7 \cdot 6 \cdot 5!$

C $12 \cdot n = 12!$

D $n \cdot 4! = 5!$

E $n \cdot 3! = 5!$

F $\frac{6!}{5!} = n$

G $\frac{10!}{8!} = n$

H $\frac{2! \cdot 6!}{4!} = n$

I $\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = n$

J $\frac{8!}{n} = 8$

K $\frac{6!}{n} = 30$

L $\frac{4!}{n!} = \frac{1}{30}$

M $7! \cdot n = 9!$

N $\frac{n!}{(n-1)!} = 5$

think

The 10 letters in the box are arranged to correspond to a well-known set of objects. Write the next 4 letters in the arrangement.

OTTFFSSENT

More practice, page S-29, Set 55

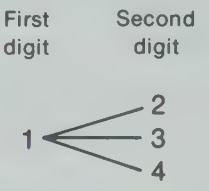
Investigating the Ideas

Suppose you had these four cards. You use two of the cards at a time to form a 2-digit numeral.

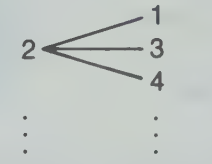
How many different 2-digit numerals can you form using the four cards?

Discussing the Ideas

1. You could use a tree diagram to help you solve the Investigation problem. Complete the tree diagram showing the cases where 3 and 4 are the first digits.



2. **A** How many different choices are there for the first digit?
   B After the first digit has been chosen, how many choices are there for the second digit?
   C How can your answers to parts **A** and **B** be used to find the total number of arrangements?



3. Exercise 2 illustrates an important principle.

THE COUNTING PRINCIPLE

If an operation can be done in any one of n ways and a second operation can be done in any one of m ways, then the two operations can be performed in $n \cdot m$ ways.

Explain how this principle could be used to find the number of 2-digit numerals that could be formed from the set $\{1, 2, 3, 4, 5\}$ if no digit is repeated.

4. The previous exercises illustrate problems of a **permutation of n objects taken r at a time**, when $r \leq n$. For example, using the counting principle repeatedly, we can find the number of arrangements or permutations of the four letters W, X, Y, and Z when taken three at a time to be $4 \cdot 3 \cdot 2 = 24$.

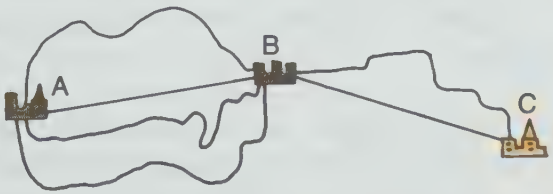
How many permutations are there of five objects taken two at a time?

1. From the set {A, B, C, D, E}, a 2-letter arrangement is to be selected.
   - **A** How many choices are there for the first letter?
   - **B** How many choices are there for the second letter after the first letter has been selected?
   - **C** How many 2-letter arrangements (permutations) are possible? (Use the counting principle.)

2. In a classroom, there are five seats in the row next to the windows, but only three students sit in these seats. In how many ways can the students arrange themselves in this row? (HINT: The first student can sit in any 1 of 5 seats; the second student can sit in any 1 of 4 seats; the third student can sit in any 1 of 3 remaining seats. Now use the counting principle.)

3. In how many different ways can a president, vice president, secretary, and treasurer be elected from a class of 20 students?

4. There are 4 roads from City A to City B and 2 roads leading from City B to City C. How many different routes are there from City A to City C?



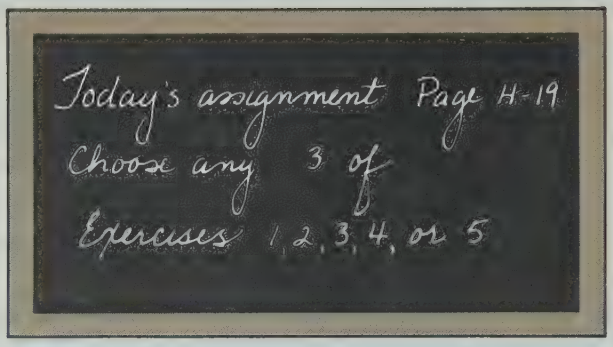
5. Suppose 2-digit numerals are composed from the digits {0, 1, 2, 3 4, 5 6, 7 8, 9}.
   - **A** How many different digits can be used as the tens' digit?
   - **B** How many digits can be used as the units' digit?
   - **C** How many 2-digit numerals in all? Check by direct counting.

6. How many 3-digit numerals are there?

★ 7. How many permutations of 3 different letters of the alphabet are posssible?

Investigating the Ideas

Suppose your teacher made an assignment like this.



How many different choices of problems do you have?

Discussing the Ideas

1. Did you find a systematic way to list the choices of three problems? Describe your method.

2. How is the Investigation problem different from a permutation of 5 things taken 3 at a time?

3. A **combination** is a subset of a given set. The Investigation involved choosing subsets of 3 elements each from a set containing 5 elements. The number of such 3-element subsets is the number of combinations of 5 elements taken 3 at a time. What is the number of combinations?

4. Study the solution below to see how permutations can be used to solve the Investigation problem.

> Number of permutations of 5 elements taken 3 at a time: $5 \cdot 4 \cdot 3 = 60$
>
> Each 3-element subset has $3! = 6$ permutations.
>
> Therefore, the number of combinations of
>
> 5 things taken 3 at a time is: $\frac{5 \cdot 4 \cdot 3}{3!} = \frac{60}{6} = 10$

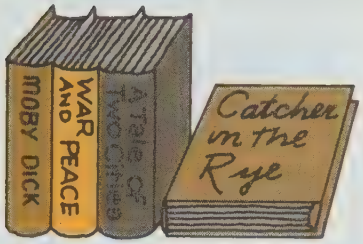
Show that the number of combinations of 5 elements taken 2 at a time is 10.

5. The symbol ${}_nC_r$ means the number of combinations of n elements taken r at a time. Thus, ${}_6C_2 = \frac{6 \cdot 5}{2!} = \frac{30}{2} = 15$.

Show how to find the following combinations:

A ${}_4C_2$ **B** ${}_6C_3$ **C** ${}_5C_1$ **D** ${}_7C_6$

1. From a list of 4 books a student can choose to read any 2 of them. How many choices does the student have?



2. If the student may read any 3 books of the 4, how many choices does the student have?

3. **A** If the student may read any one of the 4 books how many choices does the student have?

B If the student must read all 4 books, how many choices does the student have?

4. A student must choose any 2 of the following activities for Club Day at school:

Dance	Hobbies	Math Club	School Newspaper
Drama	Photography	Science Club	

How many different choices are possible?

5. On a 10-problem quiz, Carrie missed only 2 problems. How many different pairs could she possibly have missed?

6. Compute the number of combinations indicated.

A ${}_3C_2$ **B** ${}_6C_4$ **C** ${}_5C_4$ **D** ${}_{10}C_3$

7. Six people are waiting for a cab. When the cab arrives, it can take only four of the people. How many different groups of four people could ride together in the cab?

★ **8.** From a deck of 52 cards, three cards are chosen. How many different combinations are possible?

think

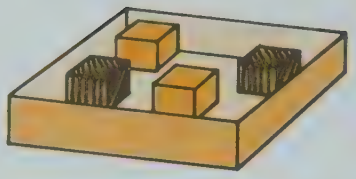
What whole number is n in this equation?

$$n! = (n + 2) \cdot (n + 3) \cdot (n + 4)$$

More practice, page S-29, Set 56

Investigating the Ideas

Put four objects that are the same size and shape into a box. Two of the objects should be the same color, the other two another color. The experiment consists of drawing two of the objects out of the box and recording their colors. Repeat the experiment 100 times.



Can you predict about how many times two objects will be the same color in the 100 trials?

Discussing the Ideas

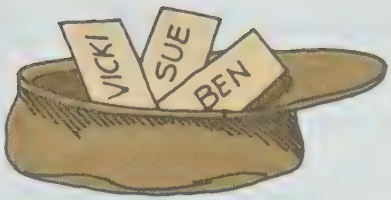
1. How close was your prediction to the actual results of the experiment?

2. You can use combinations to find the *theoretical probability* of drawing two of the objects that are the same color.
   - **A** How many sets of 2 can be selected from a set of 4 objects? That is, what is ${}_4C_2$?
   - **B** Since there are only two different sets of 2 which will match in color, the probability of drawing two objects that match in color is
   $$P(\text{same color}) = \frac{2}{{}_4C_2}.$$
   What is this probability?

3. What is the probability of drawing two of the objects that do not match in color?

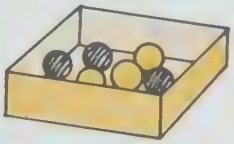
4. Suppose five objects are placed in a box. Three are white and two are black.
   - **A** How many different sets of three could be drawn?
   - **B** What is the probability of drawing the three white objects from the box?

1. Three students, two girls and a boy, each write their names on a slip of paper and put them into a hat. Two of the names are drawn by a blindfolded student.



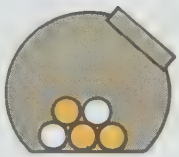
- **A** Suppose the students' names are Vicki, Sue, and Ben. List all the different pairs of names that could be drawn.
- **B** How many pairs consist of a girl's name and a boy's name?
- **C** What is the probability that a girl's name and a boy's name will be drawn?
- **D** What is the probability that the two girls' names will be drawn?

2. Six objects of the same size and shape are placed in a container. Three are one color and the other three are another color. The objects are mixed and three of them are drawn out of the container.



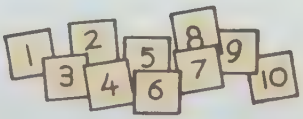
- **A** How many different sets of 3 can be drawn out?
- **B** In how many instances will the set of 3 be the same color?
- **C** What is the probability of drawing three of the objects that have the same color?

3. Three orange marbles and two white marbles are placed in a container and mixed. Then three of the marbles are withdrawn.



- **A** How many different sets of 3 could be drawn?
- **B** What is the probability that the three orange marbles will be drawn?

4. The numerals 1 through 10 are written on cards. The cards are mixed and two of the cards are drawn.



- **A** How many different pairs could be selected? $({}_{10}C_2)$
- **B** How many of the pairs will be pairs of even numbers? $({}_{5}C_2)$
- **C** What is the probability of drawing a pair of even numbers?
- **D** What is the probability of drawing an odd-number pair?
- **E** What is the probability of drawing an odd number and an even number?

Investigating the Ideas

An experiment consists of flipping a spinner like the one in the figure. A sample space for the experiment is

$$S = \{1, 2, 3, 4, 5\}.$$

The probability of each outcome in S is $\frac{1}{5}$. Suppose that E is the event $\{1, 2, 3\}$ and F is the event $\{2, 5\}$.

An outcome of the experiment is in event E **or** F if it is in $E \cup F$.

An outcome is in E **and** F if it is in $E \cap F$.

Event	Probability
$E = \{1, 2, 3\}$	$P(E) =$
$F = \{2, 5\}$	$P(F) =$
$E \cap F = \{2\}$	$P(E \cap F) =$
$E \cup F = \{1, 2, 3, 5\}$	$P(E \cup F) =$

Can you find the probability of each event in the table?

Discussing the Ideas

1. What is $P(E) + P(F)$?

2. Explain why $P(E \cup F)$ is not the same as $P(E) + P(F)$.

3. Can you tell how to find $P(E \cup F)$ if you know $P(E)$, $P(F)$, and $P(E \cap F)$?

4. The previous exercise may help you to understand how to find the probability of the union of two events.

> If S is a sample space and E and F are events in S, then $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Can you solve this problem about the spinner experiment?

What is the probability of getting either an odd number or a number greater than 3?

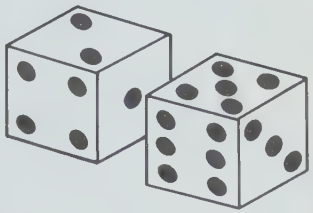
1. Suppose an experiment consists of tossing a die. The sample space is $\{1, 2, 3, 4, 5, 6\}$. E is the event of getting an even number and F is the event of getting 3 or less.

- **A** List the elements in E and F. Then find $P(E)$ and $P(F)$.
- **B** Find $E \cap F$ and give $P(E \cap F)$.
- **C** Find $E \cup F$ and give $P(E \cup F)$.
- **D** Compute $P(E \cup F)$ using the formula $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

2. A spinner is divided into 10 parts of equal size and numbered 1 through 10.

- **A** Let E be the event of getting a multiple of 3. What is $P(E)$?
- **B** Let F be the event of getting a multiple of 2. What is $P(F)$?
- **C** What is the probability of getting a multiple of 3 **and** a multiple of 2?
- **D** What is the probability of getting a multiple of 3 **or** a multiple of 2?

3. An experiment consists of tossing two dice and recording the sum of the number of dots on the two top faces. A sample space and the probability for each outcome are shown.



Sample Space: $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Probability: $\frac{1}{36}\ \frac{1}{18}\ \frac{1}{12}\ \frac{1}{9}\ \frac{5}{36}\ \frac{1}{6}\ \frac{5}{36}\ \frac{1}{9}\ \frac{1}{12}\ \frac{1}{18}\ \frac{1}{36}$

Let E be the event of getting a sum between 3 and 9.
Let F be the event of getting a sum less than 6.

- **A** Give the probability of each outcome in event E.

  $E = \{4, 5, 6, 7, 8\}$

  Probability: ▮ ▮ ▮ ▮ ▮
- **B** What is $P(E)$?
- **C** Give the probability of each outcome in event F.

  $F = \{2, 3, 4, 5\}$

  Probability: ▮ ▮ ▮ ▮
- **D** What is $P(F)$?
- **E** List the elements in $E \cup F$.
- **F** What is $P(E \cup F)$?
- **G** List the elements in $E \cap F$.
- **H** What is $P(E \cap F)$?
- **I** Find the probability of $E \cup F$ using the formula $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

More practice, page S-30, Set 57

REVIEWING THE IDEAS

1. A coin is tossed 150 times. About how many times would "tails" be expected to come up?

2. A mixture of 100 white beads and 300 red beads is in a jar.
   - **A** What is the probability of drawing a white bead?
   - **B** If you drew out a handful of beads and found that you had 39 beads, about how many of each color would you expect to get?

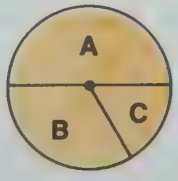
3. A sample space for flipping the spinner is $\{1, 2, 3, 4, 5, 6\}$. List the elements in the following events.

   - **A** Event A: The event of getting a number greater than 3
   - **B** Event B: The event of getting a number less than 5
   - **C** Event C: The event of getting an even number
   - **D** Event D: The event of getting a prime number

4. Compute each probability, where A, B, C, and D are the events given in Exercise 3.

   A $P(A)$ **B** $P(B)$ **C** $P(C)$ **D** $P(D)$

5. A sample space for an experiment of flipping the spinner is $\{A, B, C\}$, and $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(C) = \frac{1}{6}$. A symbol such as AC will be used to represent the event of getting A on the first flip and C on the second flip. Make a tree diagram for two flips of the spinner. Label the outcomes and the probabilities along each branch of the tree.



6. Use the tree diagram of Exercise 5 to find the probability of each event below.

   A AA **C** BC **E** CA
   B AB **D** BB **F** CC

7. Four persons run a 1500-metre race. If there are no ties, in how many orders (permutations) can they finish the race?

8. A car manufacturer advertises that one make of car is available in 3 body styles, 4 kinds of engine, and 6 colors. How many different models are available?

9. Determine the number of combinations.

   A ${}_4C_2$ **B** ${}_6C_3$ **C** ${}_5C_4$

10. From a group of 5 people, 2 are to be selected. How many different pairs can be selected?

11. Solve each equation.

   A $4 \cdot 3! = n$ **C** $(3!)^2 = n$
   B $6! \div 5! = n$ **D** $\dfrac{10 \cdot 9 \cdot 8}{3!} = n$

12.

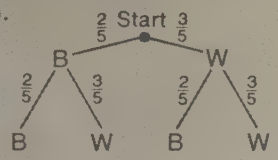
   - **A** Let E be the event of getting **B** on the first dial. List the elements of E and compute $P(E)$.
   - **B** Let F be the event of getting **2** on the second dial. List the elements in F and compute $P(F)$.
   - **C** What is $P(E \cap F)$?
   - **D** What is $P(E \cup F)$?

TEST YOURSELF

1. A What is $P(2)$?
   B What is the probability of getting a number greater than 2?

2. A What is the sample space for this spinner experiment?
   B Give the probability of each outcome.

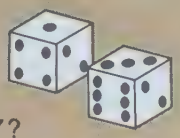
3. Two black marbles and three white marbles are placed in a box. One marble is drawn, its color recorded, replaced in the box, and a second marble is drawn.



   A What is the probability of drawing two white marbles in a row?
   B What is the probability of drawing two black marbles in a row?

4. Two coins are tossed. What is the probability that both will come up heads?

5. Two dice are tossed. What is the probability of getting a sum of 7?



6. Four people are leaving a room one at a time through one door. In how many different orders can they leave the room?

7. There are 6! permutations of the letters of the word NOTICE. How many permutations is this?

8. Two people are to be selected from a committee of eight people. How many different pairs are possible?

9. From 5 different cards, 3 are to be selected.

   How many different combinations are possible?

10. E and F are events in which $P(E) = \frac{1}{2}$, $P(F) = \frac{1}{3}$, and $P(E \cap F) = \frac{1}{6}$. Find $P(E \cup F)$.

RESEARCH PROJECTS

A Find out what **random numbers** are and how tables of random numbers are developed and used. (See *Probability: A First Course*, Revised Edition, by Frederick Mosteller, Robert E. K. Rourke, and George B. Thomas, Jr.; Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1970.)

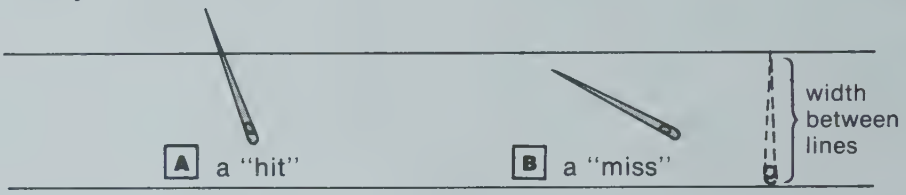
B Create a probability baseball game based on tosses of two dice. For example, Outs: 6, 7, and 8; Singles: 5 or 9; Doubles: 10; Triples: 3; Home runs: 11; Walks: 4; Errors: 2; Sacrifice fly: 12. Draw a baseball diamond on a piece of cardboard and use checkers as base runners. Make up your own rules for playing the game.

Buffon's Needle Problem

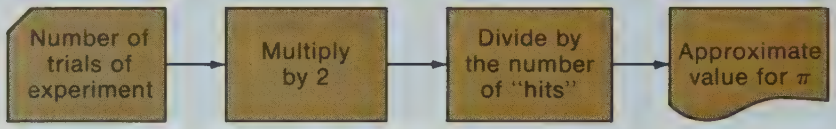
A French naturalist, Comte de Buffon (1707–1788), found an interesting way to find an approximate value for π based upon probability. Buffon's method is described below.

Draw a large number of parallel lines on a paper that are just as far apart as the length of some needle. Then perform the needle experiment as follows:

Drop the needle at random from some height above the paper and record whether the needle hits one of the lines (as in A) or misses (as in B). Repeat the experiment many times.



Follow the flow chart instructions to find an approximate value for π.



What is your approximation for π?

Combine your results with those of several classmates. Do you get a better approximation for π?

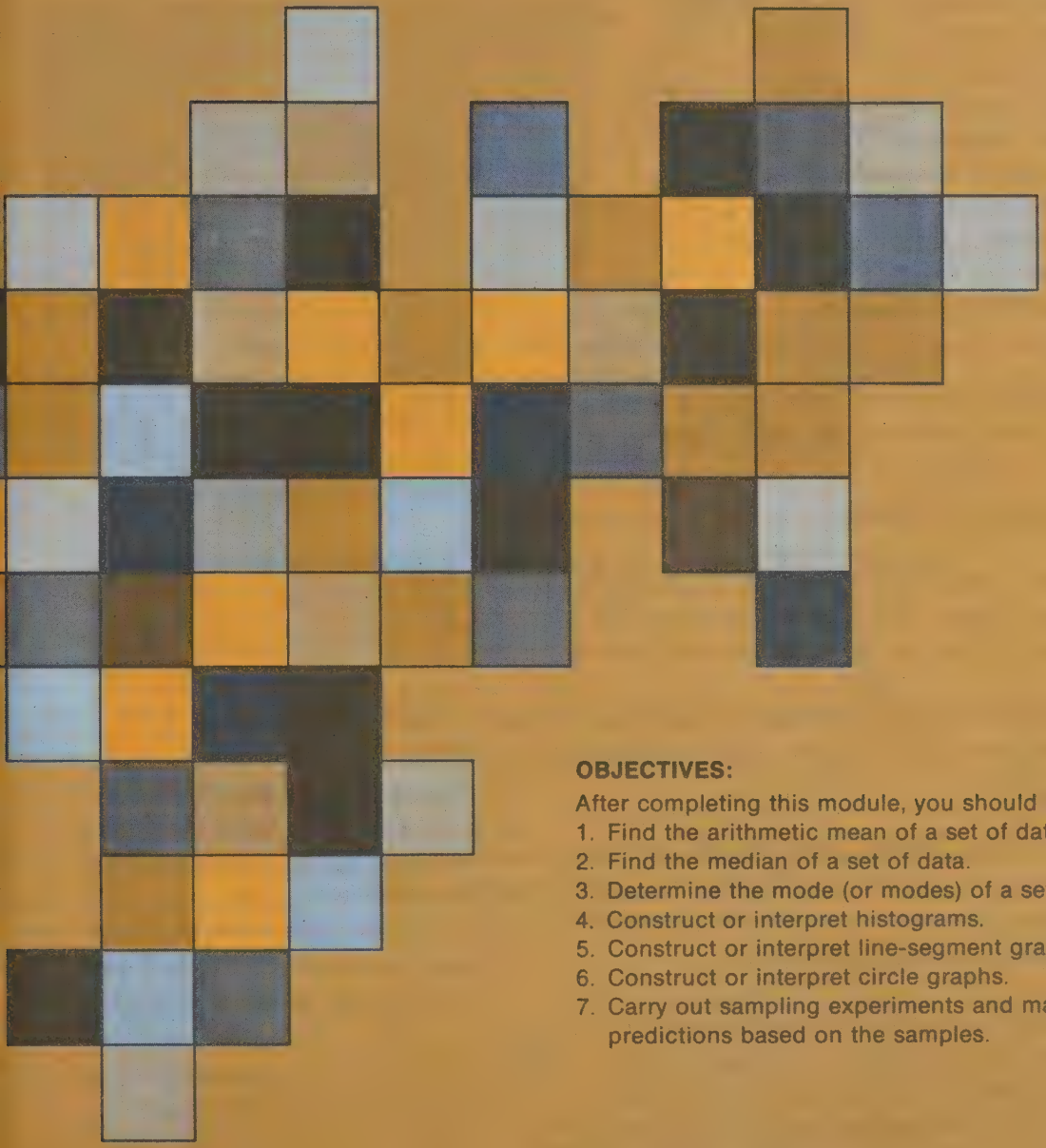
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A $\frac{1}{6}$; B $\frac{5}{6}$ **2.** A $\{1, 2, 3, 4, 5\}$; B $P(1) = \frac{1}{4}$; $P(2) = \frac{1}{4}$; $P(3) = \frac{1}{6}$; $P(4) = \frac{1}{6}$; $P(5) = \frac{1}{6}$ **3.** A $\frac{9}{25}$; B $\frac{4}{25}$ **4.** $\frac{1}{4}$ **5.** $\frac{1}{6}$ **6.** 24 **7.** 720 **8.** 28 **9.** 10 **10.** $\frac{2}{3}$

UNIT H: *Some Special Topics in Mathematics*

MODULE 2: *Statistics*



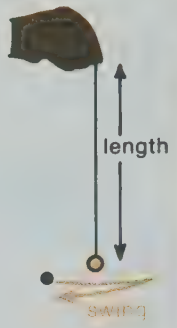
OBJECTIVES:

After completing this module, you should be able to:

1. Find the arithmetic mean of a set of data.
2. Find the median of a set of data.
3. Determine the mode (or modes) of a set of data.
4. Construct or interpret histograms.
5. Construct or interpret line-segment graphs.
6. Construct or interpret circle graphs.
7. Carry out sampling experiments and make predictions based on the samples.

Investigating the Ideas

Make a **pendulum** by tieing a small weight to a piece of string. Hold the string at a measured distance from the weight and count the number of complete swings (back and forth) that the pendulum makes in one minute. Repeat the experiment three times for each length shown in the table.



Length of pendulum	Frequency (Swings per minute)		
10 cm			
20 cm			
30 cm			
40 cm			
50 cm			

Can you estimate how long a pendulum should be in order to make one swing per second?

Discussing the Ideas

1. Did you always get the same number of swings for the same length of the pendulum? Find the average of the number of swings per minute for the three trials.

2. What relationship does your experiment show between the length of a pendulum and its frequency in swings per minute?

3. What was your estimate for the length of a pendulum that should make one swing per second? Compare your estimate with those of your classmates. Check your estimate by actual experiment.

4. If F is the frequency of a pendulum in swings per minute, then the approximate length ℓ of the pendulum in centimetres is given by the formula $\ell \approx \frac{90\,000}{F^2}$.
Thus, if $F = 30$ swings per minute, then
the length of the pendulum is $\frac{90\,000}{30^2} = \frac{90\,000}{900} = 100$ cm
Show how to use the formula to compute the length of a pendulum that makes 60 swings per minute.

5. Do you think the amount of weight used on the pendulum affects the frequency of the pendulum?

1. Which playground swing, **A** or **B**, will make more swings per minute?

2. Use the formula $\ell \approx \frac{90\,000}{F^2}$ to compute the approximate length in centimetres of each pendulum whose frequency F in swings per minute is given below.

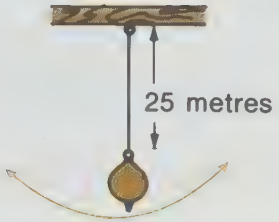
A $F = 40$ **C** $F = 50$ **E** $F = 12$

B $F = 15$ **D** $F = 100$ **F** $F = 300$

3. A long pendulum on display in a science museum has a frequency of 6 swings per minute.

A About how long does it take the pendulum to make one swing?

B What is the approximate length of the pendulum in *metres*?



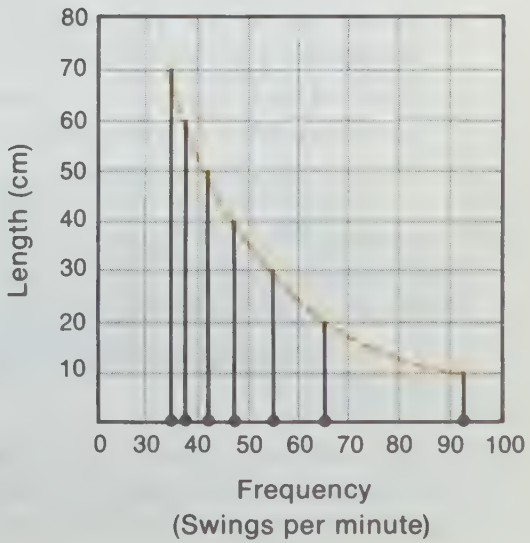
4. A rope is suspended from a beam in a gymnasium. You could make 10 swings per minute on the rope. How long is the rope?

5. A group of students made a graph by taping their pendulums perpendicular to the line of frequency and at the frequency they found for the pendulum. Use the dotted curve connecting the ends to make estimates of the following:

A The length of a pendulum which will make 60 swings per minute.

B The frequency of a pendulum 15 centimetres long.

C The length of a pendulum with a frequency of 50 swings per minute.



Investigating the Ideas

One student on this list said, "If everyone on the list were as tall as I am, our total height would be just the same as it is now."

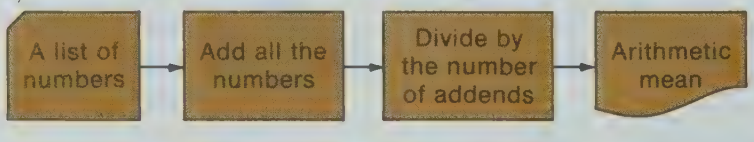
Can you find the name of the student who could have made this remark?

Name	Height (cm)	Weight (kg)
Paula S.	150	43
Jack T.	159	46
Donna M.	160	53
Dale K.	161	56
Kris E.	163	57
Eva B.	165	61
John R.	168	60
Anne W.	169	68
Dick U.	174	73
Tom T.	181	76

Discussing the Ideas

1. Explain how you found the student's name.

2. The student who made the remark above had a height which was the **arithmetic mean** of the heights of all the student heights. The flow chart below may help you to recall how to compute the arithmetic mean of a list of numbers.



   A Guess which student's weight is nearest to the arithmetic mean of the group.

   B Use the flow chart to compute the arithmetic mean of the weights of the students to the nearest tenth of a kilogram.

3. The **range** of a set of data is the difference between the largest number and the smallest number in the set. Since the heights shown in the table vary from 150 cm to 181 cm, the range in height is 31 cm. What is the range in weight?

4. Guess which student in your class is nearest to the arithmetic mean in height for all the students in the class. Find a way to check your guess.

1. The average distance from the sun to each of the nine planets is given in the table.

Planet	Distance (million km)
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	777
Saturn	1426
Uranus	2869
Neptune	4494
Pluto	5198

- **A** What is the range of the distances from the sun?
- **B** Find the mean distance of the nine planets from the sun.
- **C** Which planet is nearest to the mean distance from the sun?

2. Linda and Mary each bowled three games.

Bowling Score Sheet				
Name	Game 1	Game 2	Game 3	3-game total
Linda	136	148	128	
Mary	158	139	145	

- **A** Find their 3-game totals.
- **B** Find their average (mean) scores. Who had the higher average?
- **C** The "handicap" for each bowler is found by the formula $h = \frac{3}{4} \cdot (200 - A)$ where h represents the handicap and A is the person's average score. Compute each bowler's handicap to the nearest whole number.

3. The scores for a mathematics test given to a class are 82, 88, 72, 94, 90, 69, 74, 81, 89, 85, 72, 92, 80, 66, 75, 85, 96, 72, 83, 87, 84, 82, 77, and 78.

- **A** What is the range of the scores?
- **B** What is the mean score?

4. The weights (in kilograms) of five students are given in the table.

Name	Weight (kg)
Jean	45.9
Arthur	53.1
Jill	43.2
Chip	57.1
Jody	37.8

- **A** Find the range of the weights.
- **B** What is the arithmetic mean of their weights to the nearest tenth of a kilogram?

★ **5.** Melissa's test scores on four tests are 82, 86, 84, and 81. What is the lowest score she can make on the next test to average at least 85 for all five tests?

More practice, page S-30, Set 58

Investigating the Ideas

The 15 words in the table were selected at random by dropping a pin on a newspaper and then recording the word that was nearest to the pinpoint. When the words were listed in a table in order from the longest to the shortest, the word large was found to be the word of "middle" length. Just as many words were longer than it as were shorter.

Word	Number of letters
1. possibility	11
2. education	9
3. invested	8
4. programs	8
5. spending	8
6. instead	7
7. united	6
8. large	5
9. here	4
10. said	4
11. the	3
12. are	3
13. and	3
14. new	3
15. is	2

Can you choose 15 words of your own and find the number of letters in the "middle" word?

Discussing the Ideas

1. Does your "middle" word have about the same number of letters as the "middle" word in the table above?

2. The *middle number* in a set of numerical data arranged in order of size is called the **median**. The number "5" in the table above is the middle number of the 15 numbers. The word of median length is "large."
   - **A** What is the length of the word that has the median length in your data?
   - **B** Is your result about the same as your classmates'?

3. On four tests Jody scored 96, 88, 82, and 80. There is no middle number in this set of data. To find the median, find the arithmetic mean of the two numbers nearest the middle.

$$\text{Median} = \frac{88 + 82}{2} = \frac{170}{2} = 85$$

   - **A** If 25 test scores were arranged in order, which score would be the median score?
   - **B** If there were 30 scores, which two scores would be used to find the median score?

1. The variation in the price of a certain article in several supermarkets is shown in the table. What is the median price?

Market	Price (¢)
A	83
B	81
C	79
D	77
E	75
F	75
G	73

2. **A** Arrange the numbers in the set in order.
65, 76, 72, 100, 80, 65, 67, 65, 68, 40, 59

B Find the median of the set.

3. On a class test of 25 students, Marjorie found that her score was the median score. No one else made that score.

A How many students had scores higher than Marjorie?

B How many students had scores lower than Marjorie?

4. **A** What is the median amount of rainfall per month?

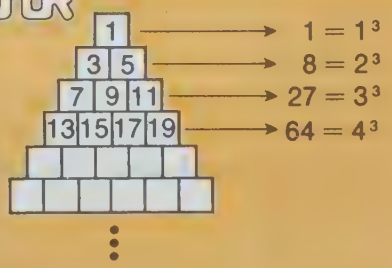
B Find the arithmetic mean of the rainfall measurements.

C Which is greater, the median or the mean rainfall per month? How much greater is it?

Month	Rainfall (cm)
November	7.3
December	5.9
January	7.5
February	5.5
March	8.8
April	9.2
May	10.6
June	11.0
July	9.1
August	6.6
September	6.9
October	7.1

5. The median income per family in a certain city is $9737. If the set of data is based upon 8461 families, about how many families earned less than $9737?

think



Try some more rows.
Does the pattern continue?

★ **6.** The median of the four numbers 48, 37, **x**, and 19 is 30.

A What number must **x** be?

B What is the mean of the four numbers?

More practice, page S-31, Set 59

Investigating the Ideas

The ratio of the circumference of any circle to its diameter is the number denoted by π. An approximation of π correct to 150 decimal places is given below.

$\pi \approx 3.14159$	26535	89793	23846	26433	83279
50288	41971	69399	37510	58209	74944
59230	78164	06286	20899	86280	34825
34211	70679	82148	08651	32823	06647
09384	46095	50582	23172	53594	08128

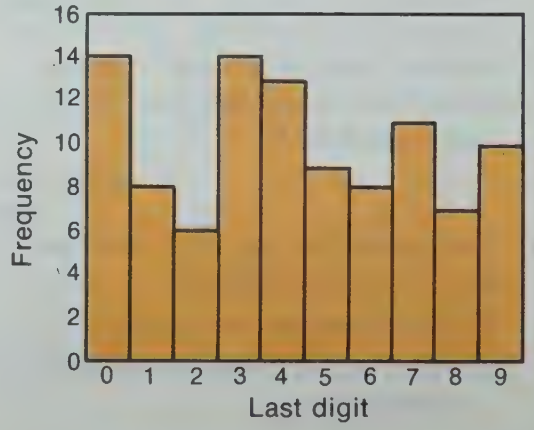
Can you find which digit (or digits) occurred most often in the expression for π?

Discussing the Ideas

1. The number of times each digit of π occurred in the Investigation is called the **frequency** of that digit.
   - A Which digit occurred most often?
   - B Which digit occurred the least number of times?
   - C What was the frequency for the digit 3?

2. The **mode** of a set of numbers is the most frequently occurring number or numbers in a set, if there are such numbers. A number must occur more than once to be a mode.
   - A Does the set of numbers in the table above have a mode?
   - B Does it have more than one mode?

3. The graph shows the frequency of the "last digit" of 100 telephone numbers chosen at random. What mode (or modes) does the graph show for "last digits"?



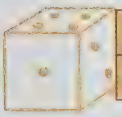
1. The table shows the ages and frequency of the students in a school band. Which age is the mode of all the ages?

Age	11	12	13	14
Frequency	2	17	23	8

2. Rita made a list of the ages of 25 of her classmates' mothers. Which age is the mode of the list?

42, 40, 36, 38, 51, 39, 38, 50, 45
34, 38, 43, 49, 42, 37, 39, 55, 35,
40, 44, 47, 38, 41, 48, 52

3. In 60 tosses of a die, a student found the frequencies for each outcome as given in the table.



Outcome	⚀	⚁	⚂	⚃	⚄	⚅
Frequency	11	12	7	8	12	10

A Which outcome (or outcomes) occurred most frequently?

B What are the modes for the experiment?

4. The results of a student survey concerning the amount of money earned from part-time work per week is shown in the table.

Amount per week	Under $2	$2–4	$4–6	$6–8	$8–10	Over $10
Frequency	10	16	12	9	7	5

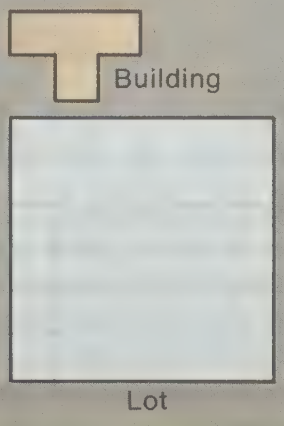
A Which amount per week did the greatest number of students earn?

B Make a graph of the data in the table.

think

1. A man plans to place as many of the T-shaped buildings on the square lot as possible. What is the greatest number of buildings the lot will hold?

2. An "Open Space Committee" insists that the buildings be placed so that there is room for the fewest possible T-shaped buildings. How many is this?

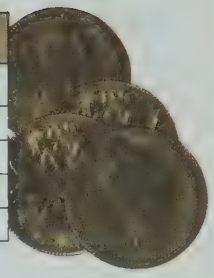


More practice, page S-31, Set 60

Investigating the Ideas

Suppose you tossed four coins at a time and kept a tally of the number of heads and tails for each toss. About how many times do you think each outcome would occur in 100 trials?

Outcome	Tally
4*H*, 0*T*	
3*H*, 1*T*	
2*H*, 2*T*	
1*H*, 3*T*	
0*H*, 4*T*	



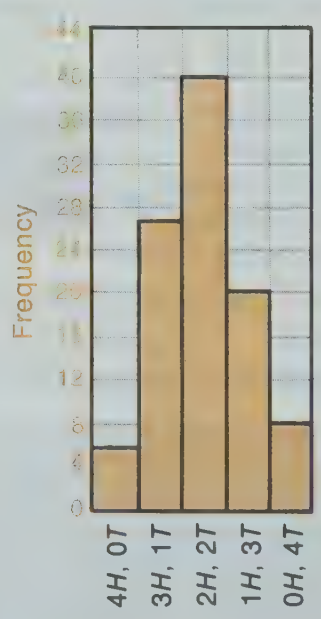
Can you carry out the experiment and check your estimate of the outcomes?

Discussing the Ideas

1. The *bar graph* at the right is a **histogram** of an experiment. Each vertical bar represents the frequency or number of times each outcome occurred.

   A If you made a histogram of your results of the experiment above, in what ways would it be like this one?

   B In what ways would your histogram be different?



2. The theoretical frequency of the outcomes for 64 trials of the experiment is shown in the table. How would a histogram for the experimental data differ from a histogram of the theoretical data?

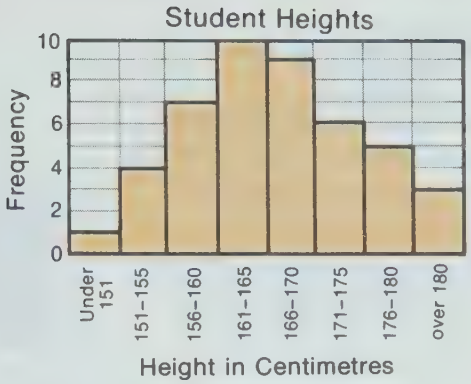
Outcome	Theoretical Frequency (64 trials)
4*H*, 0*T*	4
3*H*, 1*T*	16
2*H*, 2*T*	24
1*H*, 3*T*	16
0*H*, 4*T*	4

1. **A** How many students are represented by the histogram?

B Give the frequency for each height interval.

C What percent of the students are in the 161–165 centimetre interval?

D What percent are over 180 centimetres tall?



2. Two students tossed a pair of dice 120 times and made a tally of the frequency of the dice totals as shown in the table. Make a histogram of this data.

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	4	6	7	10	19	20	20	17	6	9	2

3. Two white marbles and four black marbles are placed in a box and thoroughly mixed. Two marbles are drawn out at random. The experiment is repeated 30 times. Make a histogram of the outcomes.

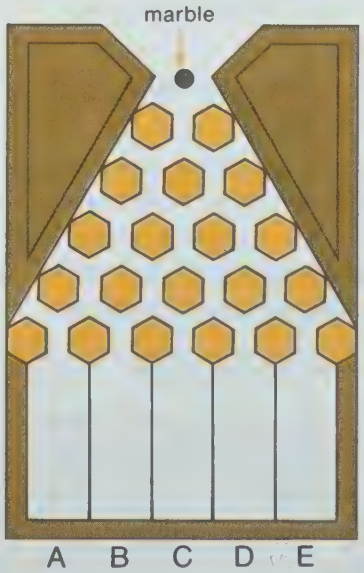
Outcome	Tally
2B	
1B, 1W	
2W	

4. A student made this device for a probability experiment. A marble is dropped in the slot at the top. The probability of falling to the left or to the right of each hexagon is $\frac{1}{2}$.

A Find the probability of a marble reaching each of the slots A, B, C, D, and E.

B Suppose 80 marbles are dropped in the slot. About how many marbles would you expect to fall in each slot?

C Make a histogram of the theoretical frequency of marbles in each slot.



Investigating the Ideas

A group of students planted some bean seeds for a science project. They kept a record of the plants' growth. The chart shows the record of one plant's growth by the week.

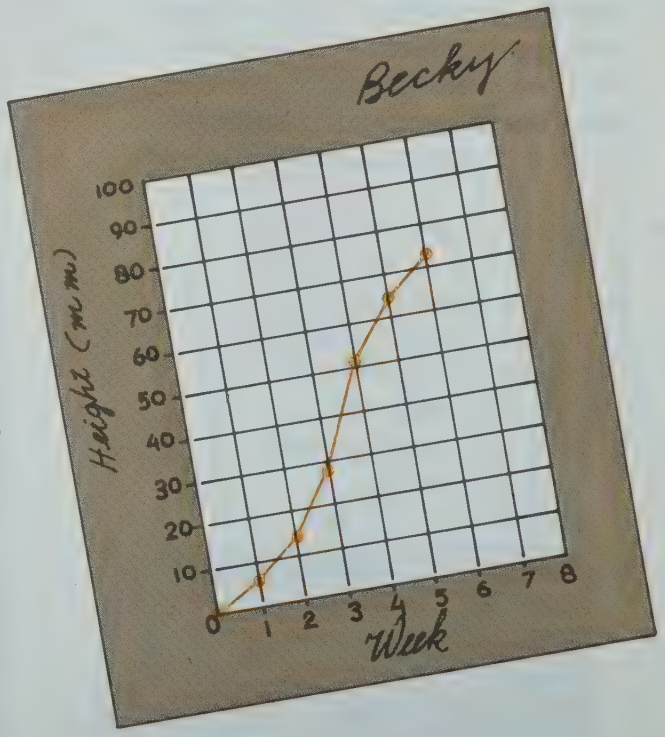
Week	1	2	3	4	5	6
Height (mm)	6	14	29	52	66	72

Can you estimate the height of the plant at the end of seven weeks?

Discussing the Ideas

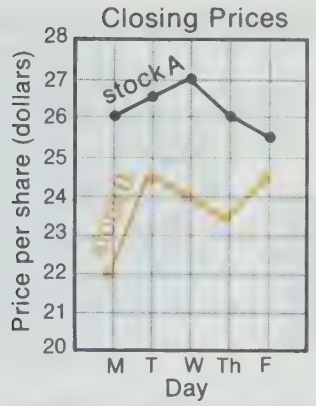
1. Explain how you made your estimate of the height of the plant after seven weeks of growth.

2. Becky made this **line-segment** graph to show the plant's growth for each week.
   - A Use the graph to tell in which week the plant grew the most.
   - B In which week did the plant grow the least?
   - C Assumimg growing conditions remain the same, how can you use the line-segment graph to estimate the plant's height at the end of seven weeks?

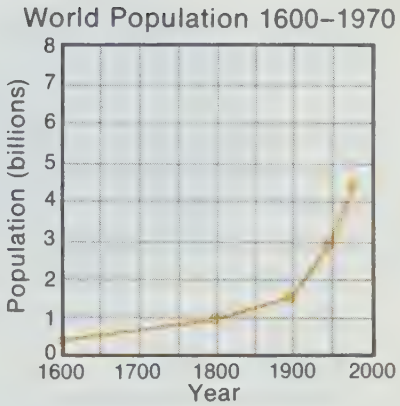


3. A What is your estimate of the plant's height at the end of eight weeks?
   - B Explain how you made your estimate.

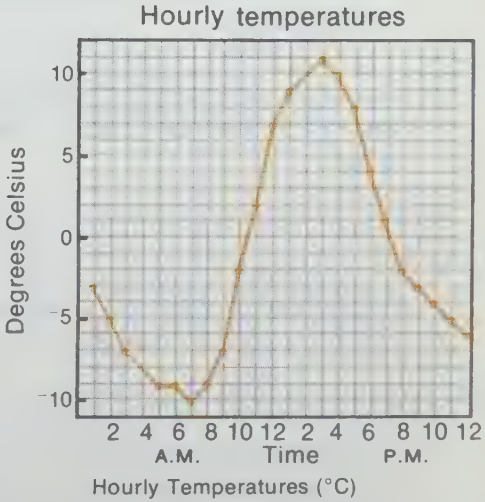
1. The line-segment graph shows the closing prices of two stocks for one week.
   - **A** What were the closing prices for stocks A and B on Monday?
   - **B** What was the highest closing price for each stock during the week?
   - **C** What was the lowest closing price for each stock during the week?
   - **D** Give the range of the closing prices for each stock during the week.



2. The line-segment graph shows the estimated world population from 1600 to the present.
   - **A** Give an estimate of the world populations for 1600, 1700, 1800, 1900, and 1950.
   - **B** The latest estimate shown on the graph is for the year 1970. What is the population for that year?
   - **C** What is your estimate of the world population for the year 2000?



3. The line-segment graph shows the hourly temperatures for a certain day.
   - **A** What was the coldest temperature during the day? At what hour was this temperature recorded?
   - **B** What was the highest temperature?
   - **C** What is the difference between the highest and the lowest temperature?
   - **D** How many degrees did the temperature fall between 3 P.M. and midnight?



4. Make a line-segment graph from the information in the table.

Hourly Temperatures (°C)

7 A.M.	17	10 A.M.	19	1 P.M.	25	4 P.M.	26
8 A.M.	17	11 A.M.	21	2 P.M.	24	5 P.M.	25
9 A.M.	16	Noon	22	3 P.M.	25	6 P.M.	25

Investigating the Ideas

Suppose you have $1.50 to spend at the Drive-Inn Diner.

Can you choose several different items and spend exactly $1.50?

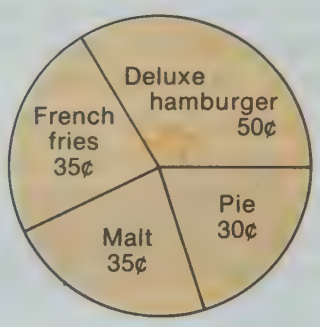
Drive-Inn Menu

Hamburger	35¢	Cola	20¢
Deluxe hamburger	50¢	Root beer	20¢
Fish-wich	50¢	Milk shake	30¢
Ham sandwich	60¢	Malt	35¢
Hot dog	40¢	Pie	30¢
French fries	35¢	Coffee	20¢
Onion rings	45¢	Milk	20¢

Discussing the Ideas

1. What fractional part of the $1.50 did you spend for each item?

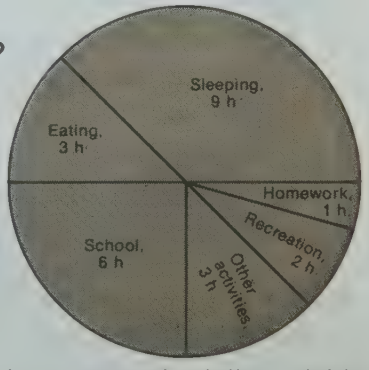
2. The **circle graph** at the right shows one way you could spend $1.50. The measure of the central angle for the sector of the circular region for the deluxe hamburger is 120° because 50¢ is $\frac{1}{3}$ of the total amount spent and $\frac{1}{3} \cdot 360° = 120°$.
   What are the measures of the central angles for each of the other items in the graph?



3. Suppose one of your items was a hot dog. Then the central angle for the item should be $\frac{40}{150} \cdot 360°$. How many degrees is this?

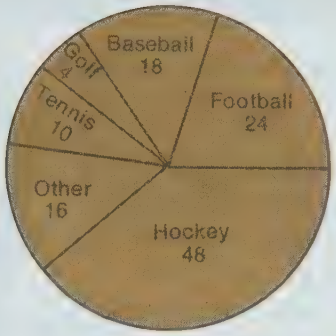
4. What would be the measure of the central angle of a circle graph for each of your choices from the menu? Explain or show how to make a circle graph of your selections from the menu.

5. The circle graph shows the hours spent in daily activities.
   - **A** What percent of each day (24 hours) is spent in each activity?
   - **B** What is the measure of the central angle of the sector for each activity?



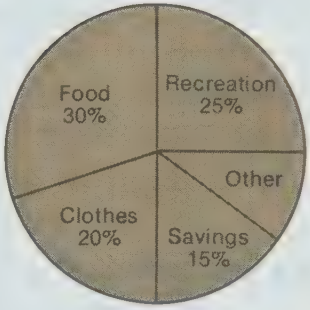
Hours spent in daily activities

1. A survey of 120 students was made to find out which sports were their favorites. The circle graph at the right shows the number of students who selected a particular sport as their favorite.



- **A** What fractional part of the circular region is represented by each sport?
- **B** Express each fractional part as a percent.

2. The circle graph shows how a student accounted for his earnings of $12 for one week.



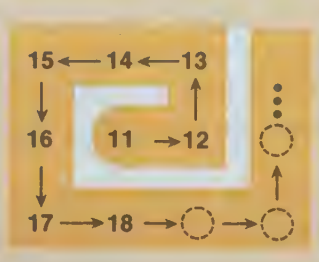
- **A** How many dollars were spent on recreation?
- **B** How many dollars were spent for clothes?
- **C** What percent is allotted to "other"? How much money is this?

3. In a school of 840 students, 52% come to school by bus, 28% walk, 12% ride bicycles, and the rest of them come by automobile. Find the number of students that come by each mode of transportation.

4. Make a circle graph of the data in Exercise 3.

think

- **A** Begin with 11 and write consecutive whole numbers in a spiral (as in the figure) until you obtain a square with a prime number in each corner. How large is the square?
- **B** Can you find a line of prime numbers in the square?
- **C** Make the square larger. What interesting patterns can you notice?



More practice, page S-32, Set 61

Investigating the Ideas

Open one end of a paper clip like this:

Toss the paper clip on the front page of a newspaper. Note the letter nearest to the point of the paper clip and tally the letter. Repeat this experiment until some letter is tallied 20 times.

Letter	Tally	Letter	Tally
A	///	N	/
B		O	
C	/	P	/
D	/	Q	/
E	////	R	//
F	/	S	//
G		T	#####
H	//	U	
I	/	V	
J		W	
K		X	
L	/	Y	
M	/	Z	

Which letter do you think will get 20 tallies first?

Discussing the Ideas

1. Which letter had 20 tallies first? Was it the letter you expected?

2. Which letter got the second most tallies?

3. If you carried out the experiment until some letter got 40 tallies, do you think you would have got a different letter than you did? Try it.

4. The letters that you tallied are a **sample** of all the letters on the newspaper page. By examining a sample of the letters, we may **infer** or conclude some things about all the letters on the page (the **population**). What things do you think you can conclude about the letters on the page, based on your sample?

5. Samples must be chosen at **random** from the population. Otherwise, a **biased sample** may result and inferences or predictions made from such samples may be unreliable. Can you describe some things that might make a sample for your experiment a biased sample?

1. In a sample of 362 letters from a newspaper, the ten most frequently occurring letters were found with the frequencies shown in the table.

Letter	Frequency	Letter	Frequency
E	50	N	27
T	34	A	26
R	32	S	25
O	29	D	15
I	28	H	12

A What percent of the letters were E's?

B Suppose the page had about 115 000 letters. About how many letters would be E's?

C About how many of the 115 000 letters would be T's?

2. Suppose you wish to take a survey of the students in your school to see which sport is their favorite. Tell whether each sample below is likely to be biased or unbiased.

A The set of boys in your class.

B Every 5th student passing by a certain corner in the hall.

C The first ten students to leave the school building at the end of the day.

D The set of girls who play violins in the school orchestra.

E Every fourth student who holds a season ticket to hockey games.

F Every 20th student taken from a list of all students arranged alphabetically.

3. Suppose it is expected that 10 000 000 people will vote in a national election. A professional poll samples 5000 voters a week before the election. The table at the right gives the number of people in the sample who planned to vote for each of the three candidates.

Candidate	[illegible]
A	2380
B	1900
C	120
Undecided	600

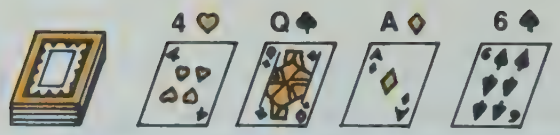
A On the basis of this information, would you say that candidate *A* will probably win the election?

B Candidate *B* concedes that candidate *A* may get as many votes as predicted by the poll taker, but still claims that he will win the election. Can you find some reasons that might make his claim true?

C Do you think it is very likely that candidate *C* will win the election?

D Assuming that the sample is a fairly accurate indicator of voter preference, give the approximate minimum and maximum number of votes each of the three candidates can expect to receive.

Investigating the Ideas



Shuffle a deck of playing cards very well. Turn up the first four cards. Make a tally of the number of aces you turn up in the sample of four cards.

Number of Aces	Tally
0 aces	
1 ace	
2 aces	
3 aces	
4 aces	

Can you estimate about how many times you will get at least one ace in the sample of four if you try the experiment 50 times?

Discussing the Ideas

1. How well did your experimental results compare with your estimate?

2. Since there are four aces in a deck of 52 cards, the probability of **not** getting an ace on the first card turned up is $\frac{48}{52}$. The probability of **not** getting an ace on the second card turned up is $\frac{47}{51}$. Why?

3. What is the probability of not getting an ace on either of the first two cards turned up? (Multiply the probabilities.)

4. What is the probability of not getting an ace on the third card that is turned up? of not getting an ace on the fourth card?

5. The product of the four probabilities in Exercises 2 and 4 is the probability of getting no aces in a 4-card sample.

   $P(\text{no aces}) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} \approx 0.718$

   Therefore, the probability of getting at least one ace in a 4-card sample is

   $1 - P(\text{no aces}) = 1 - 0.718 \approx 0.282.$

   A Did you get at least one ace in 0.282 (about one fourth) of your trials of the experiments?

   B Combine your results with those of your classmates. How does your combined results compare with the theoretical probability of getting at least one ace?

1. The theoretical probability of drawing aces in a 4-card sample is shown in the table. Suppose you carried out the experiment 250 times.

OUTCOME	PROBABILITY
No aces	0.718
Exactly 1 ace	0.256
Exactly 2 aces	0.025
Exactly 3 aces	0.0007
Exactly 4 aces	0.000004

A In how many of the 250 trials would you expect to get no aces?

B In how many of the 250 trials would you expect to get one ace?

C About how many times in 250 trials would you expect to get exactly two aces?

D About how many times in 250 trials would you expect to get exactly three aces?

E Are you very likely to draw all four aces in 250 trials? Explain.

2. In a room of 52 people, four have red hair. If four of the people's names are selected at random to win a door prize, what is the probability that at least one of the red-headed persons will win a prize?

3. A sack contained a mixture of 1200 brown and white beans. Two students performed a sampling experiment with the beans. They withdrew one bean from the sack without looking in the sack, tallied its color, and then put it back into the sack. This experiment was repeated 100 times.

BROWN AND WHITE BEANS

OUTCOME	FREQUENCY
Brown	73
White	27

A About what percent of the 1200 beans were brown?

B About how many of all beans were brown?

C About how many of all beans were white?

4. Carry out an experiment like the one suggested in Exercise 3. Compare the experimental results with the theoretical probability.

SOME SAMPLING EXPERIMENTS

1. Choose a sample of at least 50 students in your school. Find out the favorite kind of television program of each person. What conclusions can you draw from your sample?

Kind of Program	Tally
Musical Variety	𝍸 𝍸
Live Sports	//
Science Fiction	𝍸 ///
Mysteries	𝍸 𝍸 /

2. Choose a sample of 20 girls. Have each girl name her favorite school subject. What conclusions can you draw from your sample?

3. Choose a sample of 20 boys. Have each name the school subject he likes best. Compare the results with those in Exercise 2.

4. Conduct a survey of the students in your class on the question: "How many hours a week do you watch television?"

   Make a record, such as the one shown, of their responses.

Hours	Number of Students
0–3	
4–7	
8–11	
12–15	
16–18	
Over 18	

5. Conduct a survey to determine the number and kind of pets that the students in your school have at home.

6. Conduct a survey to find out what kind of job, business, or profession that the students would like to pursue after they become adults.

7. Mark off a square decimetre on your school grounds or lawn. Count, as carefully as possible, the number of blades of grass in the square decimetre.
   - **A** About how many blades of grass in one square decimetre?
   - **B** Use your sample to estimate the number of blades of grass per square metre.
   - **C** Find the area of the lawn. Estimate the number of blades of grass in the lawn.

The BIRTHDAY Problem

The names and birthdates of 25 well-known persons are given in the list at the right.

Osborne, John; London, Eng., 12/12/1929.
O'Toole, Peter; Connemara, Ireland, 8/2/1933.
Owens, Jesse; Decatur, Ala., 9/12/1913.
Paar, Jack; Canton, Ohio, 5/1/1918.
Page, Geraldine; Kirksville, Mo., 11/22/1924.
Page, Patti (Clara Ann Fowler); Claremore, Okla. 11/8/1927.
Palance, Jack; Latimer, Pa., 2/18/1920.
Paley, William S.; Chicago, Ill., 9/28/1901.
Palmer, Arnold; Youngstown, Pa., 9/10/1929.
Palmer, Lilli; Posen, Germany, 5/27/1917.
Parker, Eleanor; Cedarville, Ohio, 6/26/1922.
Parkinson, C. (Cyril) Northcote; Durham, Eng. 7/30/1909.
Pasternak, Joseph; Silagy-Somlyo, Rum., 9/19/1901.
Patterson, Floyd; Waco, N. C., 1/4/1935.
Paul VI (Giovanni Battista Montini); Concesio, It., 9/26/1897.
Pauling, Linus Carl; Portland, Ore., 2/28/1901.
Peale, Norman Vincent; Bowersville, Ohio, 5/31/1898.
Pearson, Drew; Evanston, Ill., 12/13/1897.
Pearson, Lester B.; Toronto, Ont., Can., 4/23/1897.
Peck, Gregory; La Jolla, Calif., 4/5/1916.
Peerce, Jan; New York City, 1904.
Pegler, Westbrook; Minneapolis, Minn., 8/2/1894.
Perelman, S. J. (Sidney J.); Brooklyn, N. Y., 2/1/1904.
Perkins, Millie; New Jersey, 5/12/1940.
Perkins, Tony (Anthony); New York City, 4/14/1932.

SOURCE: Information Please Almanac

1. Do you think it is very likely that any two of these have the same birthday? (Same month and day.)

2. Check through the list of names. Do any two persons have the same birthday?

3. Do any of the persons listed have the same birthday as someone in your class?

4. Do any two students in your class have the same birthday ?

5. Find a list of famous persons' birthdays. Choose 25 names in a row as a sample. See if two of the persons have the same birthday.

6. Choose several different samples of 25 names and repeat the experiment of Exercise 5. In how many of your samples did you find two people with the same birthday?

7. The probability that two people out of a group of people will have the same birthday is given in the table. Note that for 23 people, the probability of two people having the same birthday is a little more than $\frac{1}{2}$.

Number of People	Probability of at least two people with the same birthday
10	0.117
20	0.411
23	0.507
30	0.706
40	0.891
60	0.994

   A. Choose a sample of 40 famous people from a list and find their birthdays. How many of them have the same birthday?

   B. What is the probability that two of them will have the same birthday?

8. Find a list of presidents, prime ministers, or heads of state and their birthdays. Select some samples of 25 persons. Find which sample contains two persons with the same birthday.

REVIEWING THE IDEAS

1. Georgia earned $10.25, $16.50, and $12.25 in three weeks of part-time work. What was her average (arithmetic mean) earnings per week?

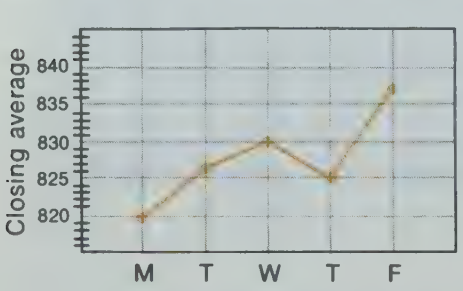
2. Linda had scores on mathematics tests of 89, 83, 94, 81, 78, 85, 90, 96, 88, and 90.
   - A What is the range of the scores?
   - B Find the mean of the scores.
   - C What is the median score?

3. The tally shows the results of a class survey of the number of brothers and/or sisters of each student.
   - A Which number occurred most frequently?
   - B What is the mode of the set of data?

Over 4	||
4	|
3	|||
2	||| |||
1	||| ||| |
0	||| ||

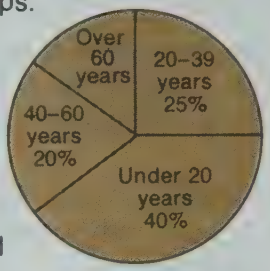
4. Make a histogram of the data given in the table in Exercise 3.

5. The line-segment graph below shows closing averages of the Dow-Jones Industrials. Give the closing average for each day to the nearest whole number.



6. Construct a line-segment graph of the following daily low temperatures: Monday, 12°; Tuesday, 4°; Wednesday, $^{-}3°$; Thursday, $^{-}10°$; Friday, 0°; Saturday, 7°; Sunday, 15°.

7. The circle graph shows the percent of the population of a city in various age groups.
   - A What percent is over 60 years old?
   - B What is the angle of the sector for the 40–60 year old group?



8. Suppose the population of the city in Exercise 7 is 30 000. Determine the approximate number of people in each group.

9. Suppose you ask three of your best friends how they plan to vote in an election for class president. You find that all four of you plan to vote for the same person.
   - A Can you conclude that your candidate will probably be elected?
   - B Is your sample of three friends a random sample of students in your class?

10. In a city of 1 000 000 registered voters, a professional poll of 1000 voters showed that candidate *A* would get 450 votes compared to 325 for candidate *B*. The remaining voters were undecided. If conditions remain as indicated by the poll, what is the maximum number of votes candidate *A* should expect to receive?

11. From the results of several samples, Jeff estimated that about one out of every 15 students was left-handed. If there are 880 students in his school, about how many are left-handed?

TEST YOURSELF

1. In bowling three games, Peggy had scores of 110, 136, and 141. What was her average score?

2. Jack had bowling scores of 108, 124, and 149. What was his average score?

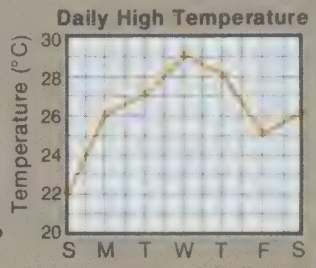
3. Jan's test scores for five tests were 83, 81, 86, 94, and 92. What was the median of these scores?

4. Give the frequency of occurrences of each letter in the word MISSISSAUGA.

5. The price of a certain brand and size of soap flakes was checked in ten different stores. The table shows the results.

Brand	Cost	Brand	Cost
A	72¢	F	78¢
B	69¢	G	72¢
C	66¢	H	69¢
D	72¢	I	68¢
E	69¢	J	69¢

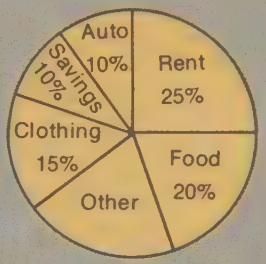
What is the mode (or modes) of the prices?

6. 

   A What is the highest temperature of the week?

   B What is the range of temperatures?

The circle graph shows how Leslie divides her monthly earnings of $800.



7. How much does she spend for food in a month?

8. How much does she spend for clothing?

9. A manufacturer determines from samples that about 3 out of every 800 parts are defective. About how many parts would be expected to be defective out of 100 000?

10. The probability of drawing an ace from a deck of cards is $\frac{1}{13}$. About how many aces would you expect to draw in 500 trials?

RESEARCH PROJECTS

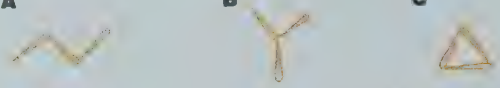
A Carry out a project to determine the "average" person in your class. Use heights, masses, age, waist measurements, foot size, and so on. Determine the mean, median, and mode of each set of data.

B Conduct a survey of the students in your school concerning preferences. Ask questions such as "What is your favorite automobile?" "Who is your favorite TV star?" or "What is your favorite meal?" Select some graphic method of displaying the data. Try to draw conclusions or make predictions from your survey.

TOOTHPICK GRAPHS

If you use three toothpicks, you can connect them at their ends in only three ways.

A B C



Each figure is called a **simple graph**. Two different-looking figures represent the same simple graph if they are connected in the same way. Thus, graph A above could appear in forms such as those below.

Graph B might appear as these shapes:

There is only one possible shape for graph C.

If you use four toothpicks, you can form only five essentially different simple graphs.

D E F G H

Again different shapes represent the same graph. Thus, graph E might have these shapes:

How many different simple graphs can you make using five toothpicks? Record each graph you find.

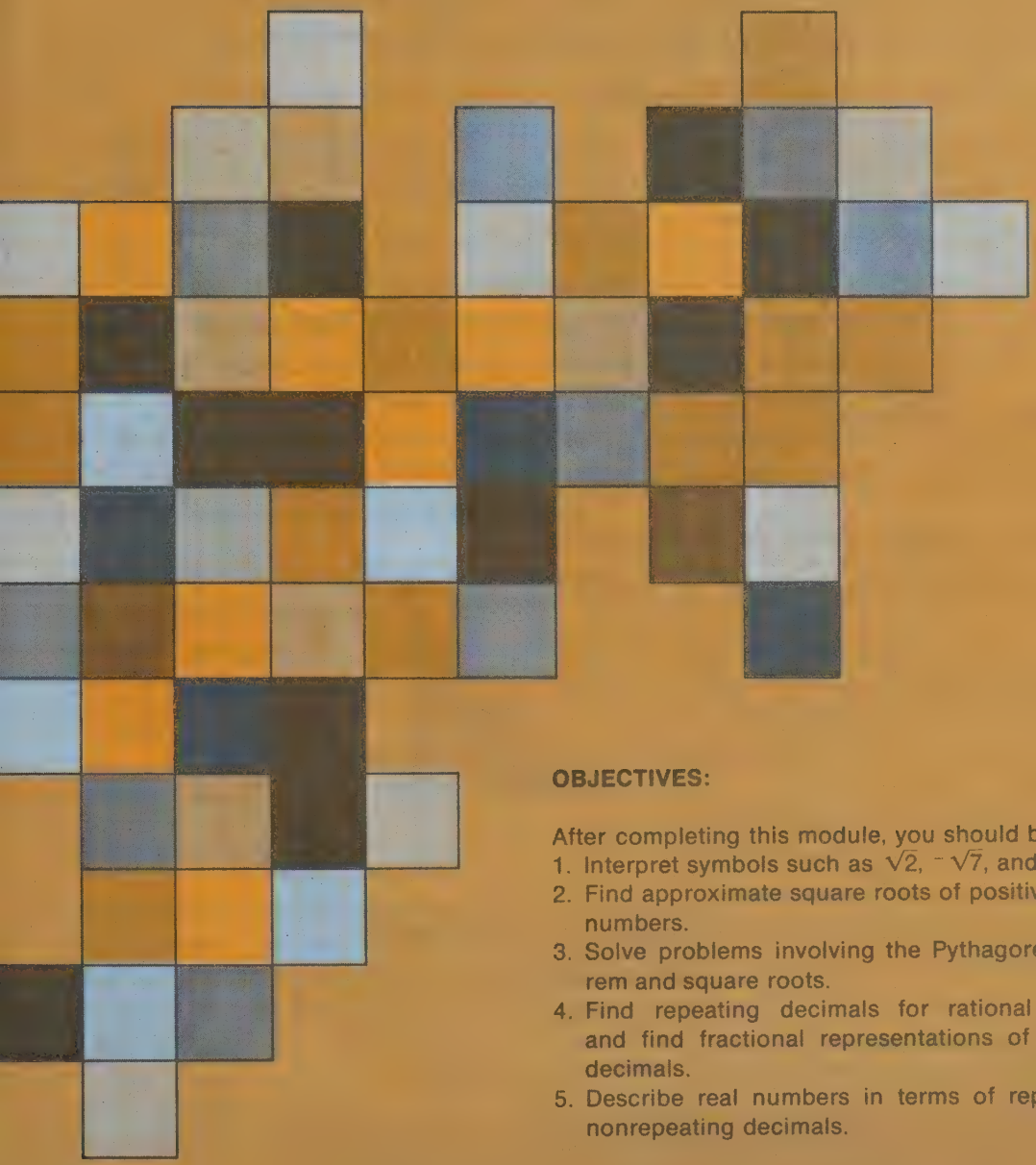
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. 129 **2.** 127 **3.** 86 **4.** M: 1; I: 2; S: 4; A: 2; U: 1; G: 1
5. 69 **6.** A 29°C; B 7° **7.** $160 **8.** $120
9. About 375 **10.** About 38

UNIT H: *Some Special Topics in Mathematics*

MODULE 3: *Real Numbers*



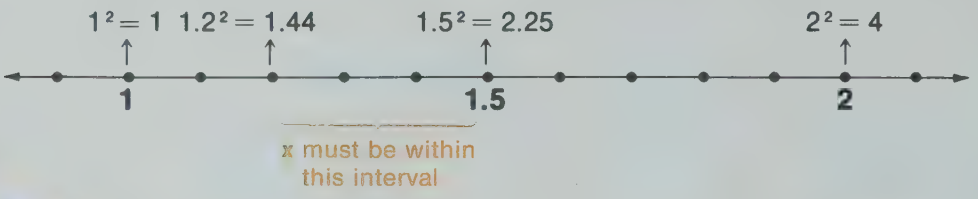
OBJECTIVES:

After completing this module, you should be able to:

1. Interpret symbols such as $\sqrt{2}$, $-\sqrt{7}$, and $\sqrt{11}$.
2. Find approximate square roots of positive rational numbers.
3. Solve problems involving the Pythagorean Theorem and square roots.
4. Find repeating decimals for rational numbers and find fractional representations of repeating decimals.
5. Describe real numbers in terms of repeating or nonrepeating decimals.

Investigating the Ideas

Is there a number whose square is 2? That is, is there a number x such that $x^2 = 2$? Study the number line and the examples below.



Can you find a decimal whose square is a number within 0.02 of 2?

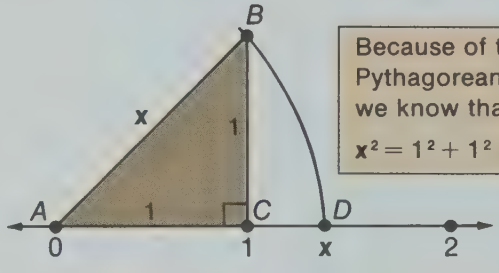
Discussing the Ideas

1. Is $(1.4)^2$ greater than 2 or less than 2?

2. Is $(1.45)^2$ greater than 2 or less than 2?

3. If $x^2 = 2$, is $1.4 < x < 1.45$?

4. The Pythagorean Theorem can help you mark a point on the number line for the number whose square is 2. Make a drawing like this on your paper. Make $\overline{AC}$ and $\overline{BC}$ each 1 decimetre long. Then measure the length of $\overline{AB}$ (distance x).



Because of the Pythagorean Theorem, we know that $x^2 = 1^2 + 1^2$ and $x^2 = 2$.

   A Is the length a number whose square is very nearly 2?

   B What is your estimate of the co-ordinate x?

5. We shall assume that there is a number that multiplies by itself to give 2. We make the following definition:

> The symbol represents the positive number which when multiplied by itself gives 2. We call this number **the positive square root of 2**.

   A Is there a negative number whose square is 2?

   B What symbol would you use for the negative number whose square is 2?

These exercises will help you become familiar with the use of the symbol $\sqrt{\ }$ (called a radical sign) used in writing numerals for square roots. Study the examples.

A $\sqrt{2}$ is the **positive** number which when multiplied by itself gives 2.

B $^{-}\sqrt{2}$ is the **negative** number which when multiplied by itself gives 2.

1. A ▒▒ is the positive number which when multiplied by itself gives **3**.
B ▒▒ is the negative number which when multiplied by itself gives **3**.
C ▒▒ is the positive number which when multiplied by itself gives **17**.
D ▒▒ is the negative number which when multiplied by itself gives **17**.
E $\sqrt{6}$ is the ? number which when multiplied by itself gives ▒▒.
F $^{-}\sqrt{6}$ is the ? number which when multiplied by itself gives ▒▒.
G $\sqrt{7}$ is the ? number which when multiplied by itself gives ▒▒.
H ▒▒ is the negative number which when multiplied by itself gives **47**.
★ I For a positive integer n, $\sqrt{n}$ is the positive number which when multiplied by itself gives ▒▒.

2. Using the definitions that you completed in Exercise 1, give an integer for n in each exercise.

A $\sqrt{2} \cdot \sqrt{2} = n$
B $\sqrt{5} \cdot \sqrt{5} = n$
C $(^{-}\sqrt{6}) \cdot (^{-}\sqrt{6}) = n$
D $(^{-}\sqrt{15}) \cdot (^{-}\sqrt{15}) = n$
E $\sqrt{100} \cdot \sqrt{100} = n$
F $(^{-}\sqrt{225}) \cdot (^{-}\sqrt{225}) = n$

3. Give a positive number that makes each equation true.

A $n \cdot n = 9$
B $n \cdot n = 7$
C $n \cdot n = 11$
D $n \cdot n = 16$
E $n^2 = 17$
F $n^2 = 19$
G $n^2 = 63$
H $n^2 = 64$

4. In the following sequence of square roots, those representing whole numbers are indicated. Use a radical sign to write symbols that represent the whole numbers 6, 7, 8, 9, and 10. EXAMPLE: 6 = $\sqrt{36}$

1 2 3 4 5

$\sqrt{1}$ $\sqrt{2}$ $\sqrt{3}$ $\sqrt{4}$ $\sqrt{5}$ $\sqrt{6}$ $\sqrt{7}$ $\sqrt{8}$ $\sqrt{9}$ $\sqrt{10}$

$\sqrt{11}$ $\sqrt{12}$ $\sqrt{13}$ $\sqrt{14}$ $\sqrt{15}$ $\sqrt{16}$ $\sqrt{17}$ $\sqrt{18}$ $\sqrt{19}$ $\sqrt{20}$

$\sqrt{21}$ $\sqrt{22}$ $\sqrt{23}$ $\sqrt{24}$ $\sqrt{25}$ $\sqrt{26}$ $\sqrt{27}$ $\sqrt{28}$ $\sqrt{29}$ $\sqrt{30}$

Investigating the Ideas

In the previous lesson you learned that certain square roots ($\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$, $\sqrt{25}$, $\sqrt{36}$, and so on) represent whole numbers. Do you think other square roots, such as $\sqrt{2}$, might represent *rational numbers*?

$\sqrt{2}$ is the positive number which when squared gives 2. If there were a positive *rational* number which when squared gave 2, it would be equal to $\sqrt{2}$.

Do any of these rational numbers equal $\sqrt{2}$?

A $\frac{3}{2}$ B $\frac{7}{5}$ C $\frac{141}{100}$

Discussing the Ideas

1. Which of the rational numbers above is closest to $\sqrt{2}$?

2. Do you think it is possible to find a rational number which when squared gives 2?

3. It can be proved that there is no rational number $\frac{a}{b}$ such that $\left(\frac{a}{b}\right)^2 = 2$ and that $\sqrt{2}$ is not a rational number. We shall work with other numbers which, like $\sqrt{2}$, are not rational numbers. We shall call such numbers **irrational numbers** and, as with $\sqrt{2}$, agree that there are points on the number line for these numbers. The Pythagorean Theorem can help us locate such points.

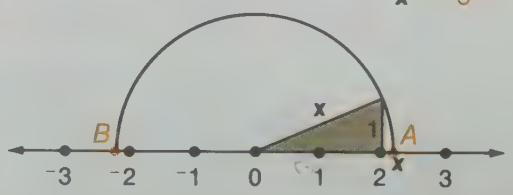
   A Explain why the calculations shown help us to determine the distance **x**.

   $x^2 = 2^2 + 1^2$

   $x^2 = 5$

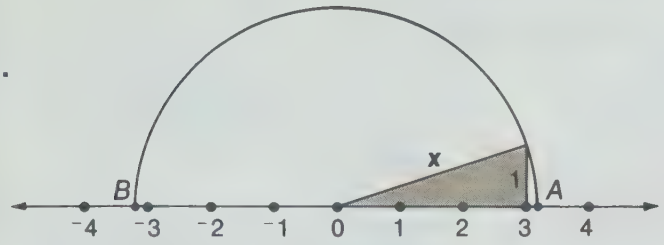
   B How far is the distance **x** from zero to point *A*?

   C What number goes with point *A* on the number line? with point *B*?



For each figure, the Pythagorean Theorem can help you find numbers for points *A* and *B* on the number line. Give the missing numbers for each ▒. Then give the numbers for points *A* and *B*.

1.



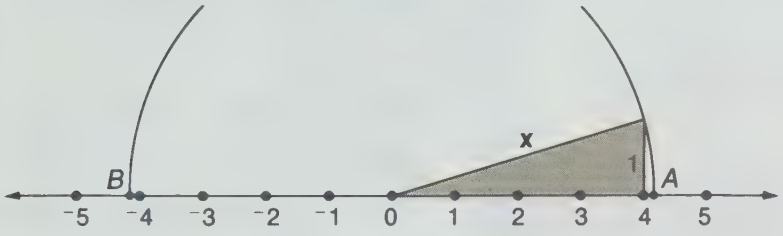
$x^2 = 3^2 + 1^2$

$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

2.



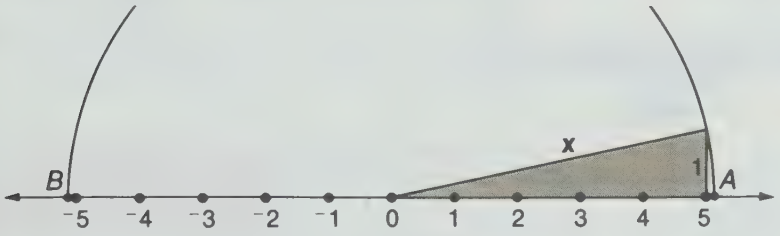
$x^2 = 4^2 + 1^2$

$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

3.



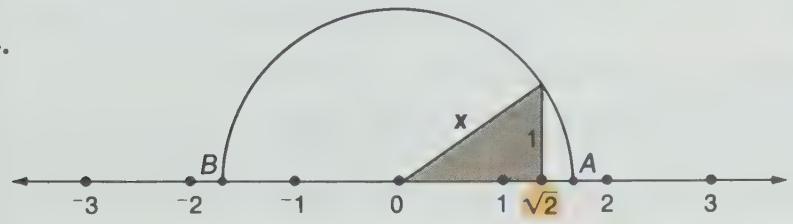
$x^2 = 5^2 + 1^2$

$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

4.



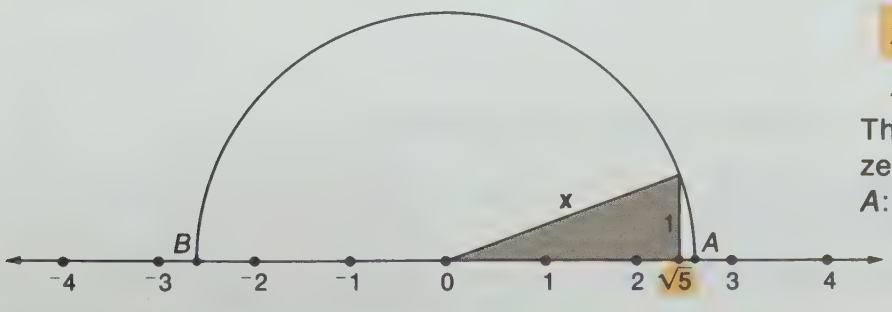
$x^2 = (\sqrt{2})^2 + 1^2$

$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

5.



$x^2 = (\sqrt{5})^2 + 1^2$

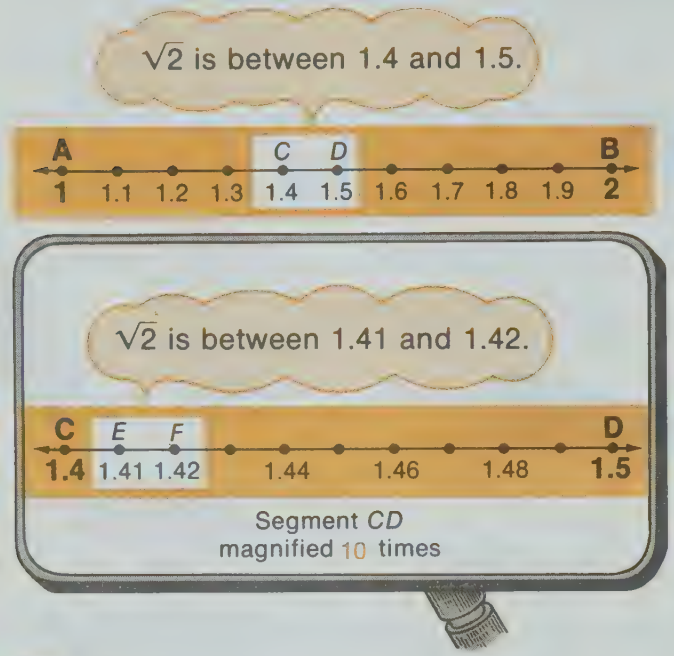
$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

Investigating the Ideas

Although $\sqrt{2}$ is not a rational number, you have seen that there are rational numbers that are "very close" to $\sqrt{2}$. Study the number lines and equations below.



$(1.4)^2 = 1.96$ (less than 2)

$\sqrt{2}^2 = 2$

$(1.5)^2 = 2.25$ (greater than 2)

We write: $1.4 < \sqrt{2} < 1.5$

$(1.41)^2 = 1.9881$ (less than 2)

$\sqrt{2}^2 = 2$

$(1.42)^2 = 2.0164$ (greater than 2)

We write: $1.41 < \sqrt{2} < 1.42$

Can you locate $\sqrt{3}$ between successive hundredths using the method above?

Discussing the Ideas

1. Which set of inequalities is correct? How can you prove it?

   A $1.6 < \sqrt{3} < 1.7$ $\quad (1.6)^2 < 3 < (1.7)^2$

   B $1.7 < \sqrt{3} < 1.8$ $\quad (1.7)^2 < 3 < (1.8)^2$

2. Which inequality is correct? Prove it.

   A $1.71 < \sqrt{3} < 1.72$ B $1.72 < \sqrt{3} < 1.73$ C $1.73 < \sqrt{3} < 1.74$

3. Is $\sqrt{10}$ closer to 3 or to 4? Explain your reasoning.

4. Use the equations to help you complete the sentence.

   To the nearest tenth, $\sqrt{10} \approx$ ▮▮▮.

$(3.1)^2 = 9.61$
$\sqrt{10}^2 = 10$
$(3.2)^2 = 10.24$

1. Give the largest possible integer for ***a***. Give the smallest possible integer for ***b***. Study the examples.

A $\underbrace{2 \cdot 2}_{a \cdot a} \quad \sqrt{7} \cdot \sqrt{7} \quad \underbrace{3 \cdot 3}_{b \cdot b}$

$a \cdot a < 7 < b \cdot b \rightarrow \begin{cases} a = 2 \\ b = 3 \end{cases}$

B $\underbrace{4 \cdot 4}_{a \cdot a} \quad \sqrt{19} \cdot \sqrt{19} \quad \underbrace{5 \cdot 5}_{b \cdot b}$

$a \cdot a < 19 < b \cdot b \rightarrow \begin{cases} a = 4 \\ b = 5 \end{cases}$

A $\sqrt{13} \cdot \sqrt{13}$

$a \cdot a < 13 < b \cdot b$

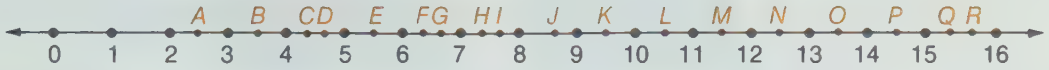
B $\sqrt{27} \cdot \sqrt{27}$

$a \cdot a < 27 < b \cdot b$

C $\sqrt{153} \cdot \sqrt{153}$

$a \cdot a < 153 < b \cdot b$

2. Write the letter for the point on the number line that goes with each number in the exercises.



A $\sqrt{39}$	**D** $\sqrt{6}$	**G** $\sqrt{51}$	**J** $\sqrt{24}$	**M** $\sqrt{129}$	**P** $\sqrt{156}$
B $\sqrt{79}$	**E** $\sqrt{115}$	**H** $\sqrt{12}$	**K** $\sqrt{231}$	**N** $\sqrt{222}$	**Q** $\sqrt{250}$
C $\sqrt{18}$	**F** $\sqrt{33}$	**I** $\sqrt{91}$	**L** $\sqrt{48}$	**O** $\sqrt{174}$	**R** $\sqrt{61}$

3. Give the largest possible one-place decimal for ***a***. Give the smallest possible one-place decimal for ***b***.

A $\sqrt{8} \cdot \sqrt{8}$

$a \cdot a < 8 < b \cdot b$

B $\sqrt{52} \cdot \sqrt{52}$

$a \cdot a < 52 < b \cdot b$

C $\sqrt{113} \cdot \sqrt{113}$

$a \cdot a < 113 < b \cdot b$

4. Use Exercise 3 to give the decimal to the nearest tenth that correctly completes each sentence.

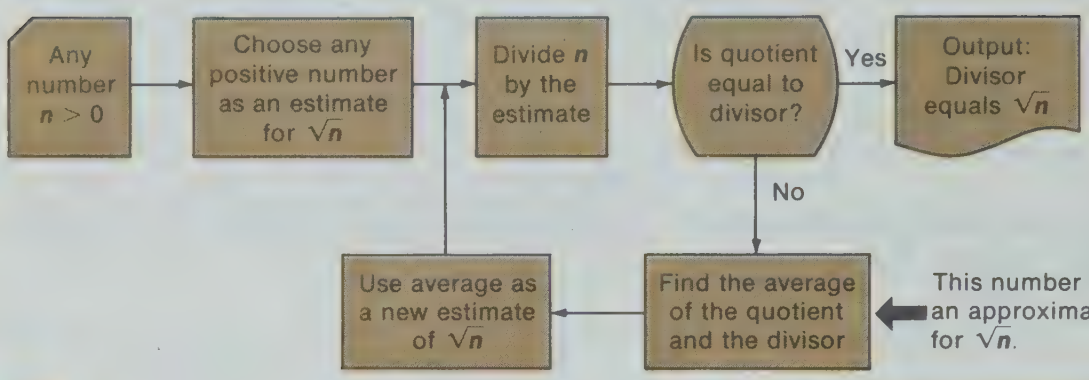
A $\sqrt{8} \approx$ ▒▒▒ **B** $\sqrt{52} \approx$ ▒▒▒ **C** $\sqrt{113} \approx$ ▒▒▒

In Exercises 5 through 8, we are "closing in on" an irrational number. Give the largest possible number for ℓ (lower bound) and the smallest possible number for u (upper bound) in each exercise.

	Between integers	Between tenths	Between hundredths	Between thousandths
5.	$\ell < \sqrt{3} < u$	$\ell < \sqrt{3} < u$	$\ell < \sqrt{3} < u$	$\ell < \sqrt{3} < u$
6.	$\ell < \sqrt{5} < u$	$\ell < \sqrt{5} < u$	$\ell < \sqrt{5} < u$	$\ell < \sqrt{5} < u$
★ **7.**	$\ell < \sqrt{29} < u$	$\ell < \sqrt{29} < u$	$\ell < \sqrt{29} < u$	$\ell < \sqrt{29} < u$
★ **8.**	$\ell < \sqrt{87} < u$	$\ell < \sqrt{87} < u$	$\ell < \sqrt{87} < u$	$\ell < \sqrt{87} < u$

Investigating the Ideas

The flow chart shows instructions for finding the positive square root of a number to any desired accuracy.


Any number $n > 0$
Choose any positive number as an estimate for $\sqrt{n}$
Divide n by the estimate
Is quotient equal to divisor?
Yes
Output: Divisor equals $\sqrt{n}$
No
Find the average of the quotient and the divisor
This number is an approximation for $\sqrt{n}$.
Use average as a new estimate of $\sqrt{n}$

Can you use the flow chart to find $\sqrt{18.49}$?

Discussing the Ideas

1. What number did you first choose for $\sqrt{18.49}$? How did you make your choice?

2. Paul chose 4.5 on his first estimate for $\sqrt{18.49}$.
   - A What should be his next estimate for $\sqrt{18.49}$?
   - B What is $\sqrt{18.49}$ exactly?

Paul

$$4.5\overline{)18.49}\quad 4.1$$
$$180$$
$$49$$
$$45$$
$$4$$

3. Not every number has an exact square root. Study the example below.

EXAMPLE: Find $\sqrt{537}$.

A Estimate of $\sqrt{537}$:
$20^2 = 400$
$25^2 = 625$
Try 22.5

B
$$22.5\overline{)537.000}\quad 23.86$$
$$450$$
$$870$$
$$675$$
$$1950$$
$$1800$$
$$1500$$
$$1350$$
$$150$$

C The average of 22.5 and 23.86 is 23.18. Therefore, $\sqrt{537} \approx 23.2$ to the nearest tenth.

D By carrying out the division to thousandths, we can get better estimates of $\sqrt{537}$. Thus $\sqrt{537} \approx 23.17$

Use this method to find $\sqrt{747}$ to the nearest hundredth.

1. Using the estimation and division method shown in the flow chart, find the positive square root of each of the following numbers. Each can be found exactly.

A 676	C 12.25	E 44.89	G 3249
B 6241	D 0.0324	F 28.09	H 139.24

2. Find an approximation of each number to the nearest tenth.

A $\sqrt{17}$	C $\sqrt{163}$	E $\sqrt{172.89}$	G $\sqrt{69.48}$	I $\sqrt{0.18}$
B $\sqrt{88}$	D $\sqrt{21.4}$	F $\sqrt{435}$	H $\sqrt{200}$	J $\sqrt{139.5}$

3. Find an approximation of each number to the nearest hundredth.

A $\sqrt{86}$	C $\sqrt{13}$	E $\sqrt{31.89}$	G $\sqrt{7.91}$	I $\sqrt{0.394}$
B $\sqrt{623}$	D $\sqrt{18.99}$	F $\sqrt{52.89}$	H $\sqrt{3000}$	J $\sqrt{1.05}$

★ **4.** Which number is the best approximation of $\sqrt{2}$ to the nearest millionth?

[A] 1.414213 [B] 1.414214

★ **5.** The **cube root** of 27 is 3 because $3^3 = 27$.
The symbol for cube root is $\sqrt[3]{\ }$. Thus, $\sqrt[3]{27} = 3$.

What number is represented by each of the symbols?

A $\sqrt[3]{8}$	B $\sqrt[3]{1000}$	C $\sqrt[3]{64}$	D $\sqrt[3]{125}$	E $\sqrt[3]{-1}$

★ **6.** Each number in the sequence below is a whole number.
Can you give the first four whole numbers of the sequence?

$\sqrt{4489}$, $\sqrt{444\,889}$, $\sqrt{44\,448\,889}$, . . .

think

A toy train is made up of an engine, a dining car, and a caboose. The ratio of the length of the dining car to the length of the engine is 3 to 2. The ratio of the caboose to the dining car is 1 to 2.

1. Compare the length of the caboose with the length of the engine.
2. Compare the sum of the lengtns of the engine and the caboose with the dining car.
3. If the whole train is 26 centimetres long, how long is each part?

More practice, page S-32, Set 62

Approximations for Square Roots

Investigating the Ideas

Certain entries are missing in this table of squares and square roots.

Can you give the missing entries to complete the table?

Table of Squares and Square Roots

Number n	Square n^2	Square root $\sqrt{n}$	Square root of 10 times n $\sqrt{10 \times n}$
1	1	1.000	3.162
2	4	1.414	4.472
3	9	1.732	C
4	16	2.000	6.325
5	25	2.236	7.071
6	36	2.449	7.746
7	49	2.646	8.367
8	64	2.828	8.944
9	81	3.000	9.487
10	100	3.162	10.000
11	121	3.317	10.488
12	144	3.464	10.954
13	169	3.606	11.402
14	A	3.742	11.832
15	225	3.875	12.247
16	256	4.000	12.649
17	289	B	13.038
18	324	4.243	13.416
19	361	4.359	13.784
20	400	4.472	14.142

Discussing the Ideas

1. **A** How did you find the number for **A** in the table?
   B Explain your method for finding the number for **B**.
   C The number for **C** is $\sqrt{30}$. Check your estimate by multiplying $\sqrt{3} \cdot \sqrt{10}$.

2. Explain how to use the table to find a decimal approximation of each of the following.

   A $\sqrt{5}$ **B** $\sqrt{8}$ **C** $\sqrt{13}$ **D** $\sqrt{20}$

3. Explain how you can use the table to find a decimal approximation of each of these.

   A $\sqrt{30}$ **B** $\sqrt{90}$ **C** $\sqrt{120}$ **D** $\sqrt{190}$

4. How can you use the table to find each of these?

   A $\sqrt{324}$ **B** $\sqrt{169}$ **C** $\sqrt{256}$ **D** $\sqrt{361}$

5. Close approximations of square roots of numbers not found in the table can be found by a process of **interpolation**. To find $\sqrt{7.5}$, we observe that this number should be between $\sqrt{7}$ and $\sqrt{8}$. The work can be arranged as follows:

   $\sqrt{7} \approx 2.646$ and $\sqrt{8} \approx 2.828$
   difference ≈ 0.182
   $0.5 \times$ difference ≈ 0.091

   $\sqrt{7.5}$ is approximately $2.646 + 0.091$, or 2.737
   $\sqrt{7.5} \approx 2.737$

   Use interpolation to find $\sqrt{17.5}$.

1. Use the table of squares and square roots in the Investigation to find the following.

A $\sqrt{6}$ **C** $\sqrt{13}$ **E** $\sqrt{18}$ **G** $\sqrt{11}$ **I** $\sqrt{196}$

B $\sqrt{60}$ **D** $\sqrt{130}$ **F** $\sqrt{180}$ **H** $\sqrt{110}$ **J** $\sqrt{169}$

2. Using the table, give a decimal approximation for a and b.

A $\sqrt{8} = a$
$\sqrt{4} \cdot \sqrt{2} = b$

B $\sqrt{20} = a$
$\sqrt{4} \cdot \sqrt{5} = b$

C $\sqrt{12} = a$
$\sqrt{4} \cdot \sqrt{3} = b$

3. Use the table to find the number for n.

A $\sqrt{n} = 2.646$

SOLUTION: $n = 7$

B $\sqrt{n} = 3.317$

C $\sqrt{n} = 12.247$

D $\sqrt{n} = 9.487$

E $\sqrt{n} = 3.875$

F $\sqrt{225} = n$

G $\sqrt{10} = n$

H $\sqrt{121} = n$

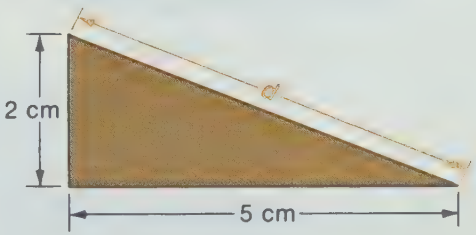
4. Using the table and interpolation, find decimal approximations for each of these.

A $\sqrt{4.5}$ **C** $\sqrt{10.5}$ **E** $\sqrt{8.5}$ **G** $\sqrt{19.4}$ **I** $\sqrt{13.6}$

B $\sqrt{1.5}$ **D** $\sqrt{17.5}$ **F** $\sqrt{18.3}$ **H** $\sqrt{16.2}$ **J** $\sqrt{14.8}$

Complete the solutions in Exercises 5 and 6 by giving the answers as decimal approximations to the nearest hundredth.

5. Find the hypotenuse of a triangle whose legs are 2 centimetres and 5 centimetres respectively.



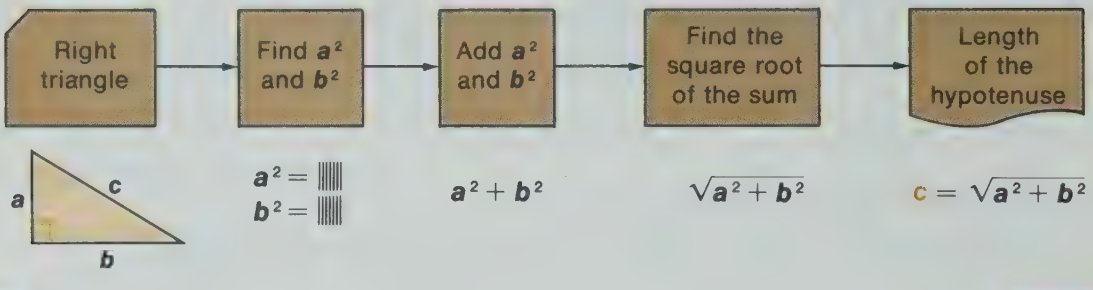
$d^2 = 2^2 + 5^2$
$d^2 = 4 + 25$
$d^2 = 29$
$d =$ ▮

6. It is 4 km along the road from Dock A to Dock C. It is 2 km from Dock C to Dock B. How far is it by boat straight across the lake from Dock A to Dock B?

$= 4^2 + 2^2$,
$=$ ▮

Investigating the Ideas

You can use the Pythagorean Theorem to find the length of the hypotenuse of a right triangle when the lengths of the two legs are known. Study this flow chart.

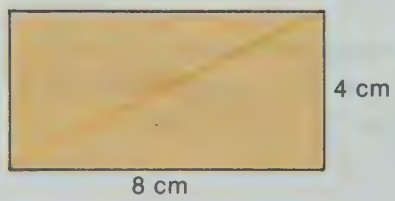


Can you find the length of the hypotenuse (to the nearest tenth) of a right triangle with legs 2 centimetres and 3 centimetres?

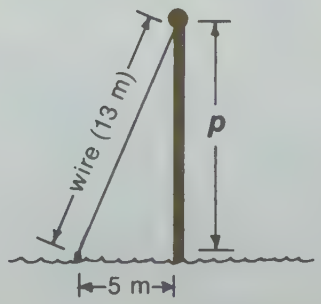
Discussing the Ideas

1. Draw a right triangle with legs 2 centimetres and 3 centimetres in length. Measure the length of the hypotenuse. Is this length about the same as the length you found using the Pythagorean Theorem?

2. A draftsman wanted to give the diagonal distance (to the nearest hundredth) across this rectangle. How can he find this distance?



3. A 13-metre wire supported a pole. It was fastened 5 metres from the base of the pole. How long was the pole?



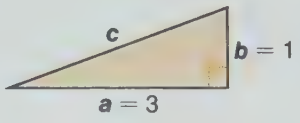
   A A student started to solve the problem this way: $p^2 = 5^2 + 13^2$
   $p^2 = 25 + 169$
   $p^2 = 194$

   What is wrong with this solution?

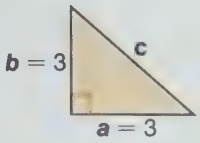
   B How would you solve this problem?

1. Find the length of the hypotenuse of each right triangle.

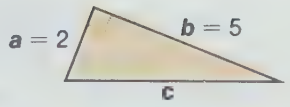
A



B

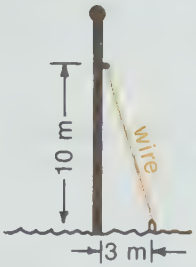


C

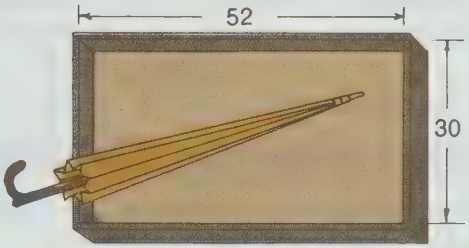


2. A rectangle is 9 units long and 3 units wide. How long is the diagonal of the rectangle, to the nearest tenth of a unit?

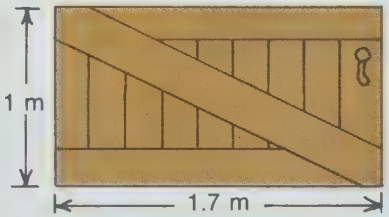
3. A guy wire is attached to an upright light pole 10 metres above the ground. The wire is anchored to the ground 3 metres from the base of the pole. How long is the wire to the nearest tenth of a metre?



4. Can an umbrella 60 centimetres long be packed to lie flat in a box 52 centimetres long and 30 centimetres wide?



5. What is the length of the diagonal brace for the gate in the figure?

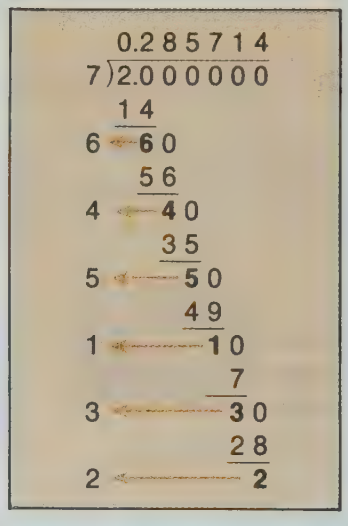


6. It is 10 kilometres from Fisherman's Point to the Lodge and 8 kilometres from Fisherman's Point to Townsville. How far is it from the Lodge across the lake to Townsville?

Repeating Decimals Again

Investigating the Ideas

In computing the decimal for the fraction $\frac{2}{7}$, a student found that six different remainders occurred before the decimal started to repeat.



Can you find all the different remainders when the decimal for $\frac{3}{17}$ is computed?

Discussing the Ideas

0.17
17)3.0000000000
17
130
119
11

1. Are you sure that $\frac{3}{17}$ can be represented by a repeating decimal? Why?

2. It can be proved that:

   Every rational number has a repeating decimal representation.

   How would you convince someone that this is true?

3. It can be proved that:

   Every repeating decimal represents a rational number.

   The examples below illustrate a way to prove the statement above for some particular cases.

1 To find a fraction q for $0.\overline{7}$

$q = 0.777\ldots \rightarrow 10q = 7.777\ldots$

$10q = 7.777\ldots$
$1q = 0.777\ldots$
$9q = 7$

$q = \frac{7}{9}$

2 To find a fraction q for $0.\overline{45}$

$q = 0.4545\ldots \rightarrow 100q = 45.4545\ldots$

$100q = 45.4545\ldots$
$1q = 0.4545\ldots$
$99q = 45$

$q = \frac{45}{99} = \frac{5}{11}$

Use this method to find a fraction for these repeating decimals.

A $0.\overline{5}$ **B** $0.\overline{73}$ **C** $0.\overline{129}$

1. The **period** of a repeating decimal is the number of digits in the entire group of digits that repeat over and over again. For example, the period of the repeating decimal $.\overline{54}$ is 2. Write a repeating decimal for each rational number and give its period.

A $\frac{2}{3}$ **B** $\frac{2}{7}$ **C** $\frac{5}{8}$ **D** $\frac{5}{13}$ **E** $\frac{5}{6}$ **F** $\frac{1}{27}$ **G** $\frac{9}{11}$

2. Find the products.

A $10 \cdot 0.\overline{6}$
Answer: $6.\overline{6}$

B $10 \cdot 0.\overline{4}$

C $10 \cdot 0.\overline{38}$

D $100 \cdot 0.\overline{07}$

E $1000 \cdot 0.\overline{842}$

In Exercises 3, 4, and 5, copy the exercises, giving the correct decimal for ***a***, the number for ***b***, and the fraction for ***c***.

3. $q = 0.4444\ldots$

$10\,q = a$

$1\,q = 0.4444\ldots$

$9\,q = b$

$q = c$

4. $q = 0.\overline{27}$

$100\,q = a$

$1\,q = 0.272727\ldots$

$99\,q = b$

$q = c$

5. $q = 0.\overline{074}$

$1000\,q = a$

$1\,q = 0.074074\ldots$

$999\,q = b$

$q = c$

6. Using the method illustrated in Exercises 3, 4, and 5, find a fraction for each repeating decimal.

A $0.555\ldots$ **B** $0.\overline{37}$ **C** $2.\overline{27}$ **D** $0.\overline{015}$ **E** $0.\overline{123}$ **F** $0.0111\ldots$ **G** $0.0\overline{5}$ **H** $4.0\overline{27}$ **I** $3.\overline{2}$ **J** $0.001\overline{001}$

7. Find the repeating decimal for each fraction.

A $\frac{1}{9}$ **B** $\frac{2}{9}$ **C** $\frac{1}{99}$ **D** $\frac{2}{99}$ **E** $\frac{1}{999}$ **F** $\frac{2}{999}$

8. These exercises suggest another way to find a fraction for a repeating decimal. Give the number for ***a***. Then give the number for ***b***.

A $0.11\overline{1} = a \rightarrow 0.22\overline{2} = b$

B $0.11\overline{1} = a \rightarrow 0.33\overline{3} = b$

C $0.11\overline{1} = a \rightarrow 0.44\overline{4} = b$

D $0.0\overline{101} = a \rightarrow 0.0\overline{202} = b$

E $0.0\overline{101} = a \rightarrow 0.0\overline{909} = b$

F $0.0\overline{101} = a \rightarrow 0.10\overline{10} = b$

G $0.0\overline{101} = a \rightarrow 0.11\overline{11} = b$

H $0.0\overline{101} = a \rightarrow 0.12\overline{12} = b$

I $0.0\overline{101} = a \rightarrow 0.23\overline{23} = b$

J $0.0\overline{101} = a \rightarrow 0.68\overline{68} = b$

K $0.001\overline{001} = a \rightarrow 0.002\overline{002} = b$

L $0.001\overline{001} = a \rightarrow 0.009\overline{009} = b$

M $0.001\overline{001} = a \rightarrow 0.023\overline{023} = b$

N $0.001\overline{001} = a \rightarrow 0.467\overline{467} = b$

Repeating and Nonrepeating Decimals

Investigating the Ideas

The digits in each decimal below form a pattern.

Can you discover the pattern and extend each decimal five more places?

- **A** 0.233233 . . .
- **B** 0.050050005 . . .
- **C** 0.1234567891011 . . .
- **D** 0.1428571428571 . . .
- **E** 0.2468101214 . . .
- **F** 6.262626 . . .
- **G** 0.1001002003 . . .
- **H** 1.41425314253142 . . .

Discussing the Ideas

1. All decimals can be considered to be "unending." Certain decimals, in which the same sequence of digits repeats again and again, are called **repeating decimals**. Every repeating decimal represents a rational number and every rational number can be represented by a repeating decimal. Which decimals in the Investigation represent rational numbers?

2. A **nonrepeating decimal** is "unending," but the same sequence of digits is not repeated over and over again. An **irrational number** is any number that is represented by a nonrepeating decimal. Which decimals in the Investigation represent irrational numbers?

3. Can you create your own nonrepeating decimal? What kind of a number does it represent?

4. The number $\sqrt{10}$ is irrational. What kind of decimal would represent $\sqrt{10}$?

5. Many irrational numbers do not have patterns in their decimal representations. The ratio of the circumference to the diameter of any circle is an irrational number denoted by the symbol π. $\pi \approx 3.141592653589793 \ldots$

   What must be true of the decimal for π?

6. **A** Does $0.\overline{72}$ represent a rational number? Explain.
   B Does 0.9217 represent a rational number? Explain.

1. Tell whether each decimal represents a rational number or an irrational number.

A 0.127127 . . .
B 0.4
C 3.17117111711117 . . .
D 0.142857142857 . . .
E 0.0369121518 . . .
F 3.249999999 . . .
G 189.647764776 . . .
H 0.01020304050 . . .

2. Find a decimal approximation to the nearest hundredth for each of the following expressions. Use $\sqrt{2} \approx 1.414$.

A $3 + \sqrt{2}$ **B** $2 \cdot \sqrt{2}$ **C** $\sqrt{2} - 1$ **D** $\sqrt{2} + \sqrt{2}$

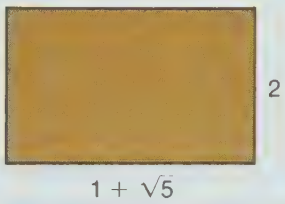
E $\frac{\sqrt{2}}{5}$ **F** $\frac{4 + \sqrt{2}}{6}$ **G** $\frac{1}{\sqrt{2}}$ **H** $\frac{\sqrt{2}}{2}$

3. Use $\pi \approx 3.14$ to find a decimal approximation to the nearest hundredth for each of the following.

A $10 \cdot \pi$ **B** $4 \cdot \pi$ **C** $\pi \cdot 3^2$ **D** $\frac{1}{\pi}$

E $\frac{\pi}{180}$ **F** π^2 **G** $5.6 \cdot \pi$ **H** $\frac{25\pi}{4}$

4. The ratio $\frac{1 + \sqrt{5}}{2}$ is called the **Golden Ratio**. A rectangle whose length and width are in this ratio is said to be the most pleasing rectangle to the artistic eye. Find a decimal approximation of this ratio to the nearest hundredth. Use $\sqrt{5} \approx 2.236$.



think

π is an interesting irrational number. The decimal approximation of π has been calculated to more than 100 000 decimal places. An approximation to π is given below to 45 places.

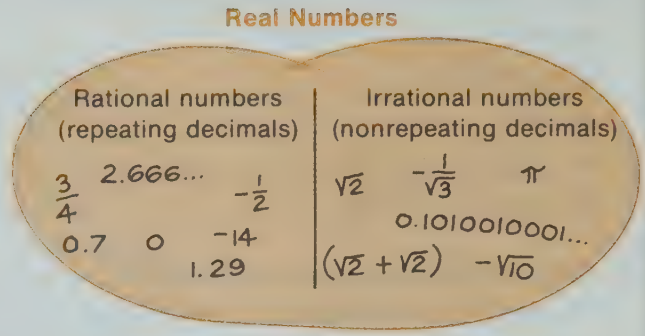
3.14159265358979323846264338327950288419716939 9

A Which of the following rational numbers is closest to π?

$\frac{22}{7}$, $\frac{179}{57}$, $\frac{355}{113}$

B Give the decimal for a rational number between $\frac{355}{113}$ and π.

Discussing the Ideas



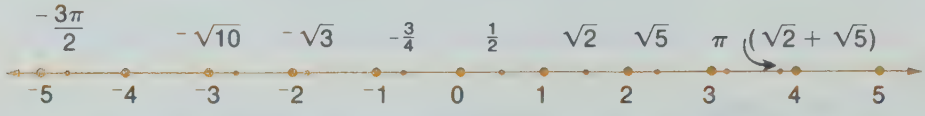
1. The union of the set of rational numbers and the set of irrational numbers is a set of numbers called the **real numbers**.

   A Is an integer a real number? Is an integer rational or irrational?

   B A certain real number is not a rational number. What kind of number must it be?

2. Since every decimal (repeating or nonrepeating) determines a certain point on the number line and every point on the number lines goes with a certain decimal, we say that the number line is **complete**, and call it the **real number line**.

   Some real numbers matched with their points on the real number line are shown below.



   A Name a rational number between $\sqrt{2}$ and $\sqrt{5}$.

   B Can you name an irrational number between $\sqrt{2}$ and $\sqrt{5}$?

3. You are familiar with the basic principles for addition and multiplication of rational numbers. Because the set of real numbers contains the set of rational numbers, we accept the same basic principles for addition and multiplication of real numbers.

Commutative Principle	Zero Principle
Associative Principle	One Principle
Distributive Principle	

   Use the commutative and associative principles to find the products. Study the examples.

   A Multiply these factors first.

$$(3 \cdot \sqrt{2}) \cdot \sqrt{2} = 3 \cdot 2 = \text{[redacted]}$$

   B Multiply these factors first.

$$(\sqrt{5} \cdot 4) \cdot \sqrt{5} = 4 \cdot 5 = \text{[redacted]}$$

   C $(\sqrt{3} \cdot \sqrt{5}) \cdot (\sqrt{3} \cdot \sqrt{5}) = \text{[redacted]}$

   D $(8 \cdot \sqrt{7}) \cdot \sqrt{7} = \text{[redacted]}$

1. Give the word (**rational** or **irrational**) that best describes the number n.

- **A** n is a real number represented by a repeating decimal.
- **B** n is a real number that is not represented by a repeating decimal.
- **C** n is a real number that is not a rational number.
- **D** n can be represented by a fraction.

2. Find the products.

- **A** $(2 \cdot \sqrt{3}) \cdot \sqrt{3}$
- **B** $\sqrt{2} \cdot (\sqrt{2} \cdot 9)$
- **C** $\sqrt{7} \cdot (3 \cdot \sqrt{7})$
- **D** $\sqrt{6} \cdot (\sqrt{6} \cdot 4)$
- **E** $(2 \cdot \sqrt{2}) \cdot (3 \cdot \sqrt{2})$
- **F** $(5 \cdot \sqrt{3}) \cdot (7 \cdot \sqrt{3})$

3. Give the integers for a and b. Then give the final product for c.

- **A** $(\sqrt{2} \cdot \sqrt{3})^2 = (\sqrt{2} \cdot \sqrt{3}) \cdot (\sqrt{2} \cdot \sqrt{3}) = a \cdot b = c$ (Multiply; Multiply)
- **B** $(\sqrt{3} \cdot \sqrt{5})^2 = (\sqrt{3} \cdot \sqrt{5}) \cdot (\sqrt{3} \cdot \sqrt{5}) = a \cdot b = c$
- **C** $(\sqrt{2} \cdot \sqrt{7})^2 = (\sqrt{2} \cdot \sqrt{7}) \cdot (\sqrt{2} \cdot \sqrt{7}) = a \cdot b = c$
- **D** $(\sqrt{7} \cdot \sqrt{5})^2 = (\sqrt{7} \cdot \sqrt{5}) \cdot (\sqrt{7} \cdot \sqrt{5}) = a \cdot b = c$

4. In Exercise **3A**, $\sqrt{2} \cdot \sqrt{3}$ gave 6 when squared. What other symbol represents the number that gives 6 when squared?

5. **A** Using the facts that $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$, find a decimal approximation for $\sqrt{2} \cdot \sqrt{3}$.

B What is a decimal approximation for $\sqrt{6}$?

6. Give a simple square root for the product.

A $\sqrt{3} \cdot \sqrt{5}$ Answer: $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$

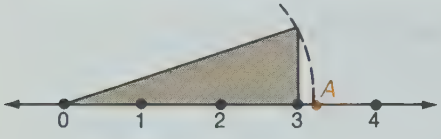
- **B** $\sqrt{2} \cdot \sqrt{7}$
- **C** $\sqrt{7} \cdot \sqrt{5}$
- **D** $\sqrt{3} \cdot \sqrt{7}$
- **E** $\sqrt{5} \cdot \sqrt{11}$
- **F** $\sqrt{13} \cdot \sqrt{11}$
- **G** $\sqrt{11} \cdot \sqrt{17}$
- **H** $\sqrt{3} \cdot \sqrt{17}$
- **I** $\sqrt{13} \cdot \sqrt{13}$

7. Use the distributive principle to give a rational number for the product $\sqrt{2} \cdot (\sqrt{2} + \sqrt{8})$.

REVIEWING THE IDEAS

1. **A** ▒▒ is the **positive** number that multiplied by itself gives 5.
   B ▒▒ is the **negative** number that multiplied by itself gives 5.
   C $\sqrt{13}$ is the ? number that multiplied by itself gives ▒▒.
   D $^{-}\sqrt{13}$ is the ? number that multiplied by itself gives ▒▒.

2. Give the number for point *A* on the number line.



3. Between what two integers is the point for $^{-}\sqrt{37}$?

4. Give the largest possible integer for ***a***. Give the smallest possible integer for ***b***.

   A $\sqrt{12} \cdot \sqrt{12}$

   $a \cdot a < 12 < b \cdot b$

   B $\sqrt{29} \cdot \sqrt{29}$

   $a \cdot a < 29 < b \cdot b$

   C $\sqrt{333} \cdot \sqrt{333}$

   $a \cdot a < 333 < b \cdot b$

5. Give each square root to the nearest integer.

   A $\sqrt{12}$ **C** $\sqrt{333}$ **E** $\sqrt{250}$

   B $\sqrt{29}$ **D** $\sqrt{19}$ **F** $\sqrt{163}$

6. **A** Which number, 3.6 or 3.7, is closer to $\sqrt{13}$?
   B Which number, 7.28 or 7.29, is closer to $\sqrt{53}$?

7. Find a decimal approximation of $\sqrt{18.44}$ to the nearest tenth.

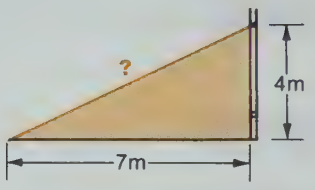
8. Find a decimal approximation of $\sqrt{163}$ to the nearest hundredth.

9. These numbers are approximations of the square roots of certain whole numbers. Find these numbers without using a table of square roots.

   A 1.7 **C** 7.21

   B 3.16 **D** 2.646

10. The wire braces the post. Find the length of the wire to the nearest tenth of a metre.



11. Write a repeating decimal for $\frac{6}{7}$.

12. Find a fraction for the number $.23\overline{23}$.

13. **A** Write a nonrepeating decimal.
    B Does it represent a rational number or an irrational number?

14. Find the product $(2 \cdot \sqrt{7}) \cdot \sqrt{7}$.

15. Give a decimal approximation for $3 \cdot \sqrt{5}$.

16. The product $\sqrt{3} \cdot \sqrt{11}$ is the square root of what number?

17. Write the decimal for a real number between 4.8629 . . . and 4.8643 . . . .

18. Find the length of the diagonal of a square whose sides are 12 centimetres long.

TEST YOURSELF

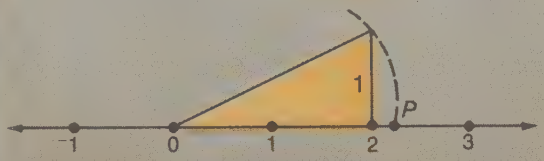
1. What positive number squared is equal to 5?

2. The symbol $-\sqrt{2}$ represents a number whose square is ▨.

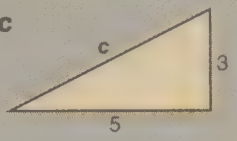
3. Which number is the best approximation of the positive square root of 5?
   A 2.1 B 2.2 C 2.3

4. Find $\sqrt{34.76}$ to the nearest tenth.

5. Give the co-ordinate of point *P* on the number line.



6. Find the length of the hypotenuse **c** of the right triangle.



7. What is the repeating decimal for $\frac{5}{6}$?

8. What rational number in fractional form is represented by $0.\overline{4}$?

9. The union of the set of rational numbers and the set of irrational numbers is the set of ? numbers.

10. The number **x** has a nonrepeating decimal representation. Is **x** a rational or an irrational number?

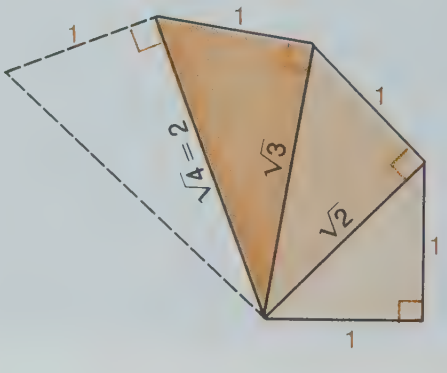
RESEARCH PROJECTS

A Measure the length and width of your mathematics classroom. Then, by using the Pythagorean Theorem, determine the length of the diagonal of the room. Check your computation by measuring the diagonal length of your classroom. Find the height of your classroom. Use the height and the diagonal length of the floor to compute the diagonal length from one corner of the floor of your classroom to the diagonally opposite corner of the ceiling.

B Do you think there are other kinds of numbers that are not real numbers? You may be surprised to find that there is no real number **x** such that $x^2 = -1$. Mathematicians have invented complex numbers or imaginary numbers so that such equations can be solved. Find out some things about imaginary numbers and give a report to the class or write a paper about them. (See *Numbers: Fun and Facts* by John Newton Friend; New York: Charles Scribner's Sons, 1954, pp. 34-41; John Wiley.)

A SQUARE ROOT SPIRAL

How many right triangles like these can you draw before they start to overlap? The outer leg is always 1 unit long and the other leg is the hypotenuse of the preceding right triangle.



1. Try to find the length of the hypotenuse of each triangle you drew.

2. Can you find the total perimeter of the polygon you constructed?

3. Can you find the total area of the polygon?

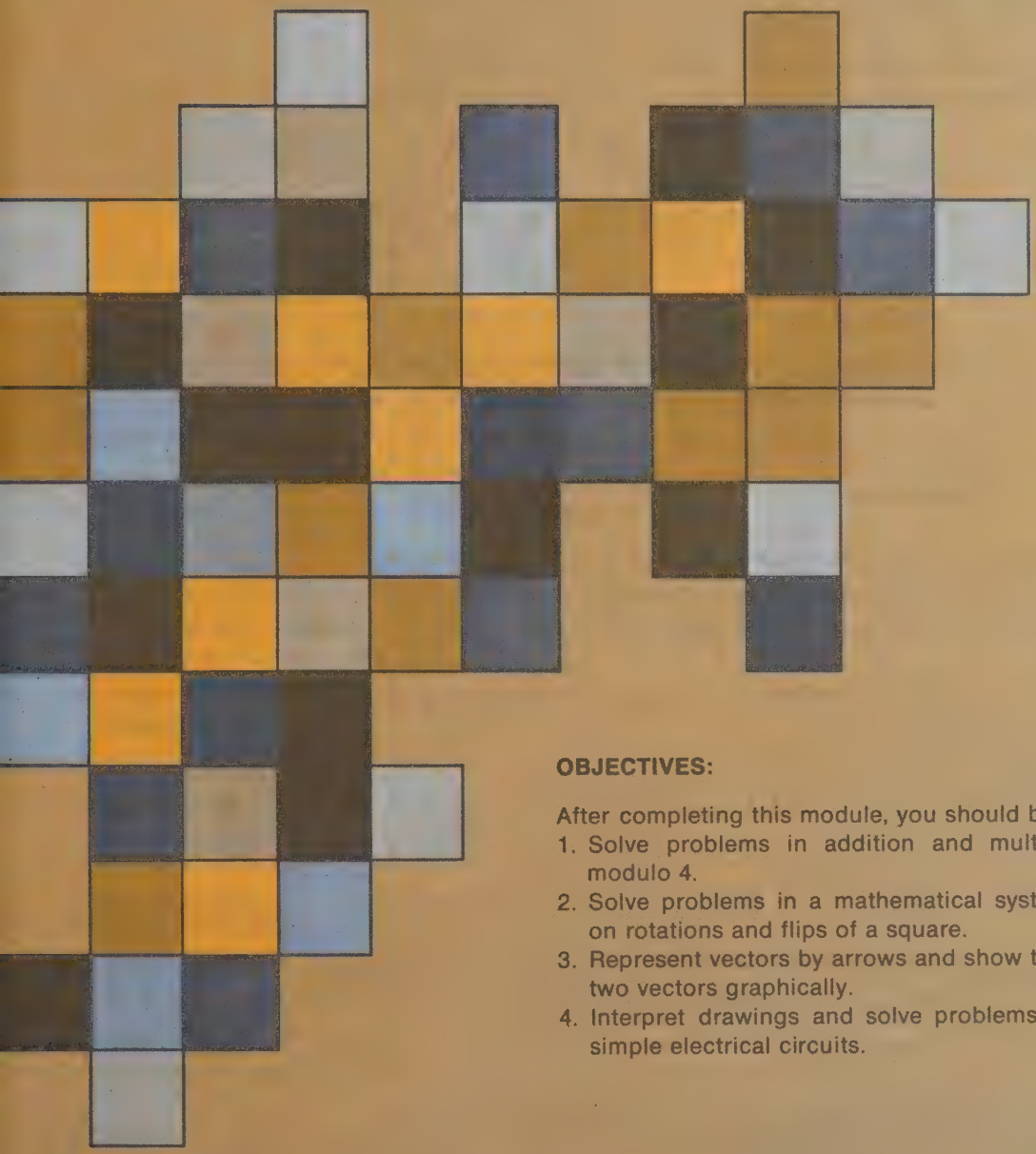
MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. $\sqrt{5}$ 2. 2 3. B 4. 5.9 5. $\sqrt{5} \approx 2.236$
6. $\sqrt{34} \approx 5.8$ 7. $0.8\overline{3}$ 8. $\frac{4}{9}$ 9. real
10. irrational

UNIT H: *Some Special Topics in Mathematics*

MODULE 4: *Other Topics in Mathematics*



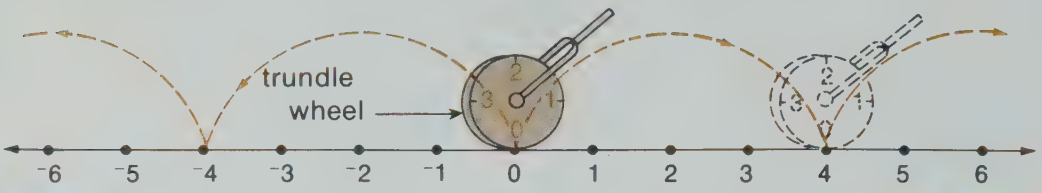
OBJECTIVES:

After completing this module, you should be able to:

1. Solve problems in addition and multiplication, modulo 4.
2. Solve problems in a mathematical system based on rotations and flips of a square.
3. Represent vectors by arrows and show the sum of two vectors graphically.
4. Interpret drawings and solve problems involving simple electrical circuits.

Investigating the Ideas

The trundle wheel starts on the number line with the zeros aligned and rolls back and forth. The wheel is designed so that 0 on the trundle wheel will touch the points for each integer in the set $\{\ldots, {}^{-}16, {}^{-}12, {}^{-}8, {}^{-}4, 0, 4, 8, 12, 16, \ldots\}$.



Can you list several positive and negative integers in the sets that match with 1, 2, and 3 on the trundle wheel?

Discussing the Ideas

1. The numbers in the set that match with 0 on the trundle wheel are the integers that are multiples of 4. Describe the pattern for the numbers in the sets for 1, 2, and 3.

2. Suppose we designate the four sets above by the four trundle wheel numbers.

   $0 \leftrightarrow \{\ldots, {}^{-}8, {}^{-}4, 0, 4, 8, \ldots\}$

   $1 \leftrightarrow \{\ldots, {}^{-}7, {}^{-}3, 1, 5, 9, \ldots\}$

   $2 \leftrightarrow \{\ldots, {}^{-}6, {}^{-}2, 2, 6, 10, \ldots\}$

   $3 \leftrightarrow \{\ldots, {}^{-}5, {}^{-}1, 3, 7, 11, \ldots\}$

   A Is every integer in exactly one of the four sets?

   B Is the difference between any two numbers in the same set a multiple of 4?

3. The integers $^{-}15$ and 5 are in the same set, the 1 set, because $5 - {}^{-}15 = 20$ and 20 is a multiple of 4. Which of these pairs are in the same set?

   A 21, 1 **B** 3, $^{-}14$ **C** $^{-}2$, $^{-}34$ **D** 105, $^{-}7$

4. We say that 18 **is congruent to** $^{-}6$, **modulo 4** because their difference is a multiple of 4: $18 - {}^{-}6 = 24$.
   We write: $18 \equiv {}^{-}6 \pmod 4$
   Similarly, $^{-}11 \equiv {}^{-}3 \pmod 4$ because $^{-}11 - {}^{-}3 = {}^{-}8$ and $^{-}8$ is a multiple of 4.

   Which of these statements are true and which are false?

   A $1 \equiv {}^{-}7 \pmod 4$ **C** $16 \equiv {}^{-}8 \pmod 4$ **E** $^{-}47 \equiv 1 \pmod 4$

   B $15 \equiv {}^{-}9 \pmod 4$ **D** $0 \equiv 100 \pmod 4$ **F** $23 \equiv {}^{-}51 \pmod 4$

The exercises on this page refer to these four sets.

0 ↔ {. . ., $^{-}12$, $^{-}8$, $^{-}4$, 0, 4, 8, 12, . . .}

1 ↔ {. . ., $^{-}11$, $^{-}7$, $^{-}3$, 1, 5, 9, 13, . . .}

2 ↔ {. . ., $^{-}10$, $^{-}6$, $^{-}2$, 2, 6, 10, 14, . . .}

3 ↔ {. . ., $^{-}9$, $^{-}5$, $^{-}1$, 3, 7, 11, 15, . . .}

1. In which set is the given integer?

A 19	**C** 26	**E** $^{-}101$	**G** $^{-}16$	**I** $^{-}1002$
B $^{-}14$	**D** 44	**F** 43	**H** 201	**J** 77

2. Find the differences.

A $6 - {}^{-}2$	**C** $17 - 8$	**E** $13 - 53$	**G** $^{-}42 - 2$
B $^{-}3 - 21$	**D** $^{-}1 - 27$	**F** $10 - {}^{-}20$	**H** $^{-}53 - 7$

3. Tell whether each statement is true or false. Use Exercise 2 to help you decide.

A $6 \equiv {}^{-}2 \pmod 4$	**C** $17 \equiv 8 \pmod 4$	**E** $13 \equiv 53 \pmod 4$
B $^{-}3 \equiv 21 \pmod 4$	**D** $^{-}1 \equiv 27 \pmod 4$	**F** $10 \equiv {}^{-}20 \pmod 4$

4. Copy and complete each statement with one of the integers 0, 1, 2, or 3.

A $9 \equiv$ ▮ (mod 4)	**D** $27 \equiv$ ▮ (mod 4)	**G** $^{-}39 \equiv$ ▮ (mod 4)
B $^{-}6 \equiv$ ▮ (mod 4)	**E** $^{-}19 \equiv$ ▮ (mod 4)	**H** $^{-}40 \equiv$ ▮ (mod 4)
C $^{-}16 \equiv$ ▮ (mod 4)	**F** $76 \equiv$ ▮ (mod 4)	**I** $^{-}41 \equiv$ ▮ (mod 4)

5. Complete each sentence.

A Every number in set ? is congruent to 0, modulo 4.

B Every number in set ? is congruent to 1, modulo 4.

C Every number in set ? is congruent to 2, modulo 4.

D Every number in set ? is congruent to 3, modulo 4.

★**6.** Find the sum of the two integers. Then tell whether the sum is congruent to **0**, **1**, **2**, or **3**.

EXAMPLE 1: $7 + {}^{-}16 = {}^{-}9$, $^{-}9 \equiv$ **3** (mod 4)

EXAMPLE 2: $^{-}14 + {}^{-}8 = {}^{-}22$, $^{-}22 \equiv$ **2** (mod 4)

A $12 + {}^{-}5$	**C** $8 + 7$	**E** $23 + {}^{-}47$	**G** $38 + 52$
B $^{-}10 + {}^{-}6$	**D** $^{-}5 + 15$	**F** $^{-}19 + {}^{-}2$	**H** $64 + {}^{-}32$

Investigating the Ideas

$0 \leftrightarrow \{\ldots, {}^{-}8, {}^{-}4, 0, 4, 8, \ldots\}$ $\quad$ $2 \leftrightarrow \{\ldots, {}^{-}6, {}^{-}2, 2, 6, 10, \ldots\}$

$1 \leftrightarrow \{\ldots, {}^{-}7, {}^{-}3, 1, 5, 9, \ldots\}$ $\quad$ $3 \leftrightarrow \{\ldots, {}^{-}5, {}^{-}1, 3, 7, 11, \ldots\}$

Can you complete each equation below by using only the four integers 0, 1, 2, or 3?

1. $({}^{-}3 + 9) \equiv \underline{?} \pmod 4$
2. $(5 + {}^{-}2) \equiv \underline{?} \pmod 4$
3. $(32 + {}^{-}20) \equiv \underline{?} \pmod 4$
4. $(7 + 10) \equiv \underline{?} \pmod 4$
5. $(11 + {}^{-}5) \equiv \underline{?} \pmod 4$
6. $({}^{-}7 + {}^{-}1) \equiv \underline{?} \pmod 4$
7. $(14 + {}^{-}2) \equiv \underline{?} \pmod 4$
8. $({}^{-}4 + 14) \equiv \underline{?} \pmod 4$
9. $(13 + {}^{-}4) \equiv \underline{?} \pmod 4$
10. $({}^{-}5 + {}^{-}8) \equiv \underline{?} \pmod 4$

Discussing the Ideas

1. A In Equation 1 above, in which set was $^{-}3$?
   B In which set was 9?
   C In which set was $({}^{-}3 + 9)$?
   D If any two numbers from set 1 are chosen, in which set is their sum?

2. A In Equation 2, in which sets were the two addends 5 and $^{-}2$?
   B In which set is their sum?

3. If you know one integer is in 1 and a second integer is in 2, in which of the four sets is the sum of the integers?

4. Suppose that an operation of **addition**, **modulo 4**, is defined on the set $\{0, 1, 2, 3\}$ and that $\oplus_4$ designates the operation.

   We write $2 \oplus_4 1 = 3$ because the sum of any number in set 2 and any number in set 1 is a number in set 3.

   A What does this equation mean? $1 \oplus_4 3 = 0$
   B What is $3 \oplus_4 2$?
   C What is $2 \oplus_4 0$?

5. How is modulo 4 addition like ordinary addition? How is it different?

1. Copy and complete the addition table for modulo 4.

$\oplus_4$				
			2	
				0
		3		
			1	

2. Use the table to solve each equation in modulo 4.

A $1 \oplus_4 2 = x$ **D** $r \oplus_4 3 = 1$

B $2 \oplus_4 2 = y$ **E** $2 \oplus_4 n = 2$

C $3 \oplus_4 z = 0$ **F** $3 \oplus_4 t = 2$

3. Is the operation $\oplus_4$ commutative?

4. Does 0 in modulo 4 have the same property as 0 in ordinary addition? That is, is 0 the identity element for addition?

5. Find the modulo 4 solution for each equation.

A $0 \oplus_4 p = 0$ **C** $2 \oplus_4 r = 0$

B $1 \oplus_4 q = 0$ **D** $3 \oplus_4 s = 0$

6. In modulo 4, 1 and 3 are additive inverses because $1 \oplus_4 3 = 0$.

A Give the additive inverse of 2.

B What is the inverse of 0?

7. One interpretation of modulo 4 addition can be made by using "clock arithmetic." Using a clock with the four numbers as shown, you can write $3 \oplus_4 2 = 1$ because 2 hours after 3 o'clock would be 1 o'clock.

Complete each statement using the modulo clock.

A 1 hour after 2 o'clock will be ▒ o'clock.

B 2 hours after 1 o'clock will be ▒ o'clock.

C 3 hours after 3 o'clock will be ▒ o'clock.

D 2 hours after 2 o'clock will be ▒ o'clock.

E 3 hours after 0 o'clock will be ▒ o'clock.

8. Write equations using modulo 4 addition for each statement in Exercise 7.

Investigating the Ideas

How many of these equations can you complete using only the integers 0, 1, 2, or 3?

1. $2 \times 3 \equiv \underline{?} \pmod 4$
2. $^{-}2 \times 2 \equiv \underline{?} \pmod 4$
3. $^{-}3 \times 3 \equiv \underline{?} \pmod 4$
4. $4 \times 3 \equiv \underline{?} \pmod 4$
5. $6 \times {}^{-}4 \equiv \underline{?} \pmod 4$
6. $5 \times {}^{-}3 \equiv \underline{?} \pmod 4$
7. $8 \times 12 \equiv \underline{?} \pmod 4$
8. $7 \times {}^{-}1 \equiv \underline{?} \pmod 4$

Discussing the Ideas

1. **A** In Equation 1 above, in which of the four sets 0, 1, 2, or 3 is 2?
   B In which set is 3?
   C In which set is 2×3?

   $0 \leftrightarrow \{\ldots, {}^{-}8, {}^{-}4, 0, 4, 8, \ldots\}$
   $1 \leftrightarrow \{\ldots, {}^{-}7, {}^{-}3, 1, 5, 9, \ldots\}$
   $2 \leftrightarrow \{\ldots, {}^{-}6, {}^{-}2, 2, 6, 10, \ldots\}$
   $3 \leftrightarrow \{\ldots, {}^{-}5, {}^{-}1, 3, 7, 11, \ldots\}$

2. Repeat question 1**A–C** using the factors in the other equations above.

3. If an integer is from set 2 and another integer is from set 1, in which set is the product?

4. If two integers are both from set 2, in which set is the product?

5. The examples above suggest a kind of "multiplication" for integers, modulo 4. The operation is designated by $\otimes_4$.

Any integer in 3	$\times$	Any integer in 2	$=$	An integer in 2
3	$\otimes_4$	2	$=$	2

Explain these products.

A $2 \otimes_4 2 = 0$ **B** $3 \otimes_4 1 = 3$ **C** $0 \otimes_4 3 = 0$

1. Copy and complete the multiplication table for modulo 4.

$\otimes_4$				
				0
			0	
			2	

2. Use the table to solve each equation in modulo 4.

A $1 \otimes_4 2 = x$

B $2 \otimes_4 2 = y$

C $1 \otimes_4 z = 3$

D $3 \otimes_4 t = 1$

E $s \otimes_4 1 = 0$

F $2 \otimes_4 3 = n$

3. Is the operation $\otimes_4$ a commutative operation?

4. The equation $2 \otimes_4 k = 0$ has **two** solutions in modulo 4: $k = 2$ or $k = 0$. What are the solutions for $2 \otimes_4 m = 2$?

5. In ordinary arithmetic, the identity element for multiplication is 1 because for any real number n, $n \cdot 1 = n$. What is the identity element for modulo 4 multiplication?

6. The rational numbers $\frac{3}{2}$ and $\frac{2}{3}$ are multiplicative inverses because $\frac{3}{2} \cdot \frac{2}{3} = 1$.

A Does each modulo 4 integer 0, 1, 2, and 3, have a multiplicative inverse?

B Give the multiplicative inverses of the modulo 4 integers which have inverses.

7. Perform the indicated operations.

A $2 \otimes_4 (3 \oplus_4 2)$

B $(2 \otimes_4 3) \oplus_4 (2 \otimes_4 2)$

C $3 \otimes_4 (2 \oplus_4 1)$

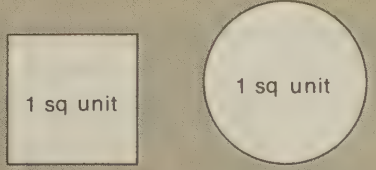
D $(2 \otimes_4 3) \oplus_4 (1 \otimes_4 3)$

E $1 \otimes_4 (0 \oplus_4 2)$

F $(1 \otimes_4 0) \oplus_4 (1 \otimes_4 2)$

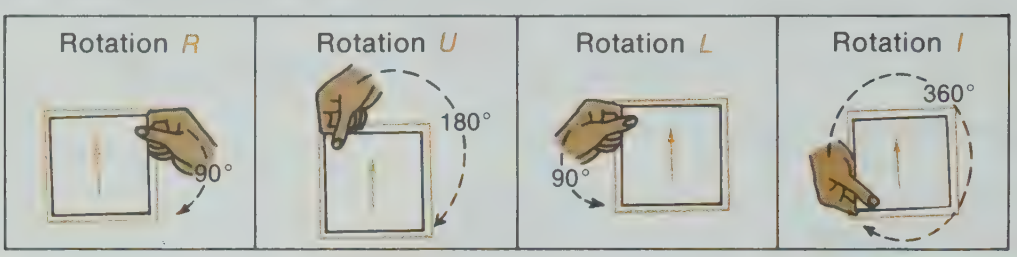
Think

Which is larger, the ratio of the circumference of the circle to the perimeter of the square or the ratio of the area of the circle to the area of the square?



Investigating the Ideas

Cut out a small cardboard square. Draw an outline of the square on a piece of paper. The display below shows four rotations in which the square is turned so that it fits back into the outline again. The arrow on the square shows its starting position.



Can you show that a combination of any two of these rotations will give the same result as one of the four rotations?

Discussing the Ideas

1. Rotation *I* is so named because it is the **identity** rotation. It leaves the square in its original position. How can the other letters *U*, *R*, and *L* help you to remember these rotations?

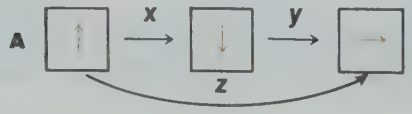
2. What single rotation produces the same result as a combination of these two rotations?

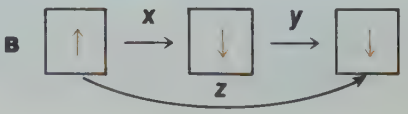
   A *R* followed by *U*
   B *R* followed by *L*
   C *R* followed by *R*
   D *U* followed by *U*
   E *L* followed by *L*
   F *I* followed by *U*

3. Suppose the operation "followed by" is denoted by a star (★). Copy and complete the "followed by" table shown at the right. It has been completed for $U \star R = L$.

★	*I*	*R*	*U*	*L*
I				
R				
U				*L*
L				

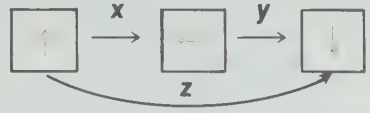
4. For each part, give the rotations for **x** and **y**. Then give the single rotation for **z** that is the same as the combined rotations.



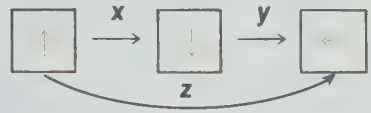


1. Give the rotations for **x** and **y**. Then give the combined rotation **z**.

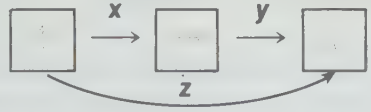
A



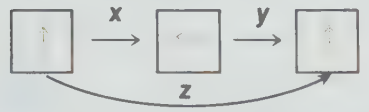
B



C



D



2. Using elements of the set $\{I, R, U, L\}$ and the operation ★, write an equation for each part of Exercise 1. The equation for part **A** of Exercise 1 is $L \star L = U$

3. Solve each equation. The replacement set is $\{I, R, U, L\}$. Use the table completed in the Discussion or actually use a square to help you find the solution.

A $L \star x = R$
B $R \star x = U$
C $x \star R = I$
D $x \star U = I$
E $U \star x = U$
F $I \star x = R$
G $L \star L = x$
H $L \star x = I$
I $x \star L = R$

4. Solve each equation for **x** in the *IRUL* system. Use the table of Discussion Exercise 3 to help you.

A $(R \star L) \star U = x$
B $R \star (L \star U) = x$
C $(I \star R) \star R = x$
D $I \star (R \star R) = x$
E $(U \star R) \star U = x$
F $U \star (R \star U) = x$
G $(L \star R) \star L = x$
H $L \star (R \star L) = x$

5. What basic principle for the operation ★ is suggested by the equations of Exercise 4?

6. In what way is the identity element I for the operation ★ like an identity element for arithmetic operations?

7. For each equation, give the rotation for **y**.

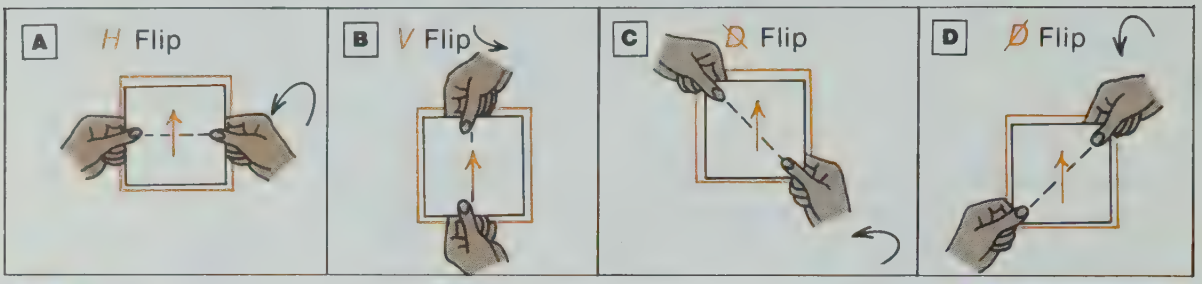
A $I \star y = I$
B $R \star y = I$
C $U \star y = I$
D $L \star y = I$

8. In the set of integers, each integer has an **additive inverse**. That is, if **x** is any integer, $x + (-x) = 0$.

Does each rotation I, R, U, and L have an **inverse** with respect to the operation ★? Give the inverse of each rotation. (See Exercise 7.)

Investigating the Ideas

You can get some new moves of a square by turning, or flipping, a square around its four lines of symmetry.



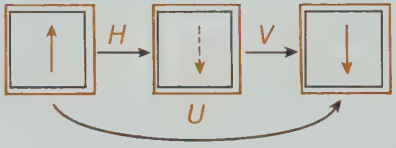
Can you find a combination of two of the flips that will give the same result as one of the four rotations (*I*, *R*, *U*, or *L*)?

Discussing the Ideas

1. The ⧹ flip is called the "left diagonal" flip and ⧸ is the "right diagonal" flip. How would you name the flips *H* and *V*?

2. Will the combination of any two flips give the same result as any single flip?

3. The figure shows that an *H* flip followed by a *V* flip is the same as a *U* rotation alone. (The dotted arrow shows the reverse side of the card.)

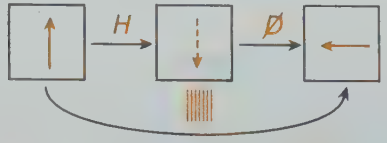


What single rotation could replace these flips?

A *H* followed by ⧹
B *V* followed by *V*
C *D* followed by *V*
D *D* followed by ⧸
E *H* followed by ⧸
F *V* followed by ⧸

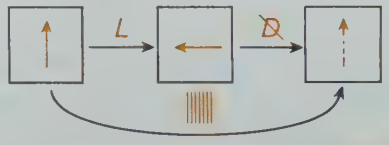
4. Suppose ★ denotes the operation "followed by" for both rotations and flips. Complete each statement and equation.

A



An *H* flip followed by a ⧸ flip is the same as a ▒ rotation alone.
We write: *H* ★ ⧸ = ▒

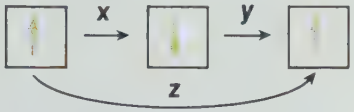
B



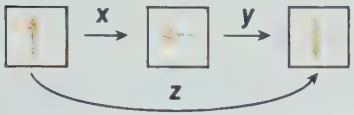
An *L* rotation followed by a ⧹ flip is the same as a ▒ flip alone.
We write: *L* ★ ⧹ = ▒

1. Give the rotation or flip for **x** and **y**. Then give the rotation or flip for **z** that gives the same result as **x** followed by **y**.

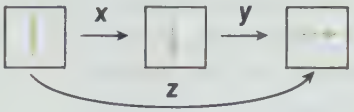
A



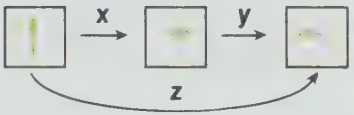
B



C



D



2. Write equations for each part of Exercise 1 using flip and rotation symbols. The solution for Part **A** is $\not{\Omega} \star L = H$.

3. Copy and complete the table for the operation "followed by." In the table, the *IRUL* system has already been completed for you.

★	I	R	U	L	H	V	$\not{\Omega}$	$\not{\emptyset}$
I	I	R	U	L				
R	R	U	L	I				
U	U	L	I	R				
L	L	I	R	U				
H								
V								
$\not{\Omega}$								
$\not{\emptyset}$								

4. Suppose we call the system consisting of the set of eight elements, $\{I, R, U, L, H, V, \not{\Omega}, \not{\emptyset}\}$, and the binary operation ★, the **octic mathematical system**. Which element in the octic system is the **identity element**?

5. Use the table in Exercise 3 to show that ★ is **not** a commutative operation for every pair of elements in the octic system.

6. In the octic system, the element I is the identity element. Show that each element in the system has an inverse element.

EXAMPLES: Since $\not{\Omega} \star \not{\Omega} = I$, the inverse of $\not{\Omega}$ is $\not{\Omega}$.
Since $R \star L = I$, the inverse of R is L.

7. Solve each equation in the octic system.

A $(U \star H) \star V = x$

B $U \star (H \star V) = x$

C $\not{\Omega} \star (V \star \not{\emptyset}) = x$

D $(\not{\Omega} \star V) \star \not{\emptyset} = x$

E $(V \star V) \star R = x$

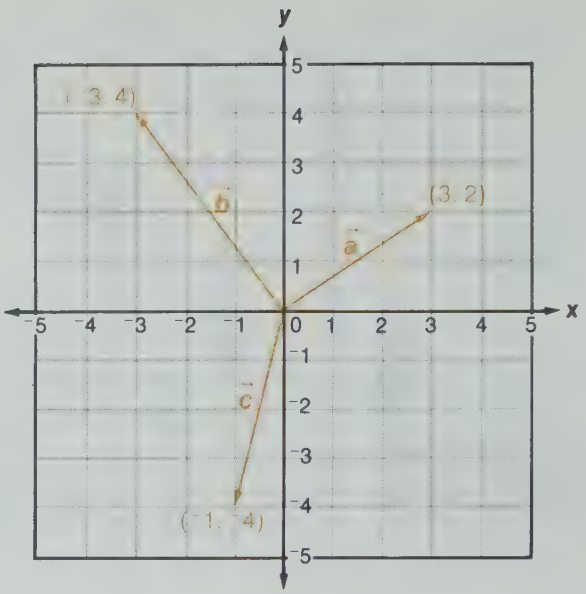
F $V \star (V \star R) = x$

8. Do you think that the operation ★ is associative in the octic system? (HINT: Look at the results of Exercise 7.)

★ **9.** Solve the equation $I \star R \star L \star U \star V \star H \star \not{\Omega} \star \not{\emptyset} = x$.

Investigating the Ideas

Three arrows are drawn from the origin to the points whose co-ordinates are (3, 2), (⁻3, 4), and (⁻1, ⁻4). The arrows are denoted by $\vec{a}$, $\vec{b}$, and $\vec{c}$ respectively.



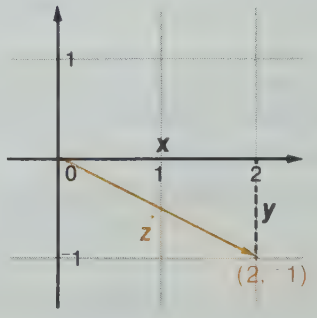
Can you show three more arrows on graph paper that are the same length as these arrows but point in the opposite direction?

Discussing the Ideas

1. What are the co-ordinates of the tips of the arrows you drew? Compare these co-ordinates with the original co-ordinates of ***a, b,*** and ***c.***

2. The mathematical term for an arrow is **vector.** Each vector can be named by giving the co-ordinates of its endpoint (the arrow tip). Thus, $\boldsymbol{a} = (3, 2)$. How would $\vec{b}$ and $\vec{c}$ be designated?

3. The **length** or **magnitude** of a vector can be found by applying the Pythagorean Theorem. The length of $\vec{z} = (2, ^-1)$ is found as follows:



(length $\vec{z}$)$^2 = x^2 + y^2 = 2^2 + (^-1)^2 = 4 + 1 = 5$;

length $\vec{z} = \sqrt{5}$.

What is the length of each of these vectors?

A $\vec{a} = (3, 4)$ B $\vec{b} = (^-2, ^-3)$ C $\vec{c} = (^-5, ^-2)$

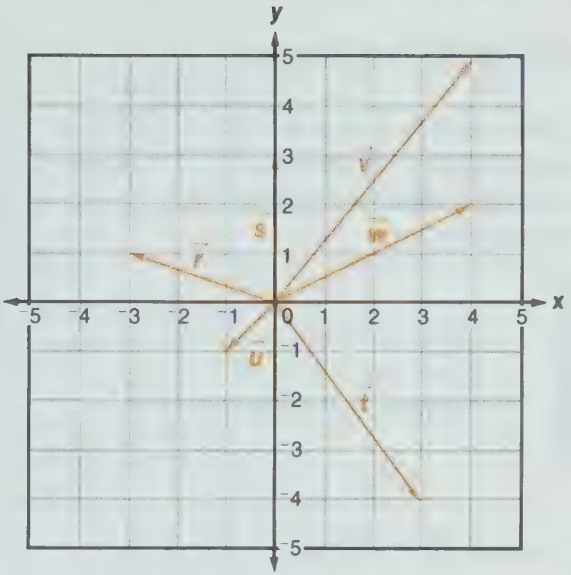
4. What vector would have the same length as $\vec{z}$ in Exercise 3 but point in an opposite direction?

5. The special vector (0, 0) is called the **zero vector** or **null vector**. What is its length?

1. Give the ordered pair that names each vector.

A $\vec{r}$ C $\vec{s}$ E $\vec{t}$

B $\vec{u}$ D $\vec{v}$ F $\vec{w}$



2. Find the length of each vector in Exercise 1.

3. Show the arrow for each vector on graph paper.

A $\vec{p} = (^{-}4, ^{-}1)$ E $\vec{r} = (0, ^{-}2)$

B $\vec{t} = (1, ^{-}1)$ F $\vec{n} = (^{-}2, 5)$

C $\vec{q} = (4, 0)$ G $\vec{s} = (3, 3)$

D $\vec{k} = (3, ^{-}4)$ H $\vec{r} = (^{-}5, 0)$

4. Find the length of each vector in Exercise 3.

5. Which of the two vectors is longer? Compute their lengths, then check by drawing arrows.

A $\vec{a} = (2, 3)$ or $\vec{b} = (1, 4)$

B $\vec{c} = (1, 1)$ or $\vec{d} = (2, 0)$

C $\vec{e} = (6, 8)$ or $\vec{f} = (^{-}7, 7)$

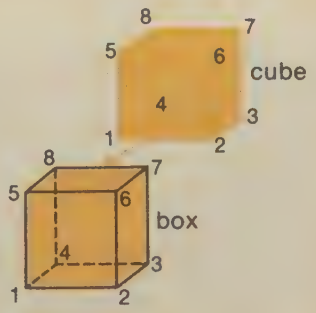
D $\vec{g} = (5, 0)$ or $\vec{h} = (0, ^{-}4)$

★**6.** Prove that $\vec{v} = (^{-}8, 6)$ is twice as long as $\vec{u} = (4, ^{-}3)$.

★**7.** What vector is three times as long as $\vec{b} = (2, 1)$ and points in the same direction?

think

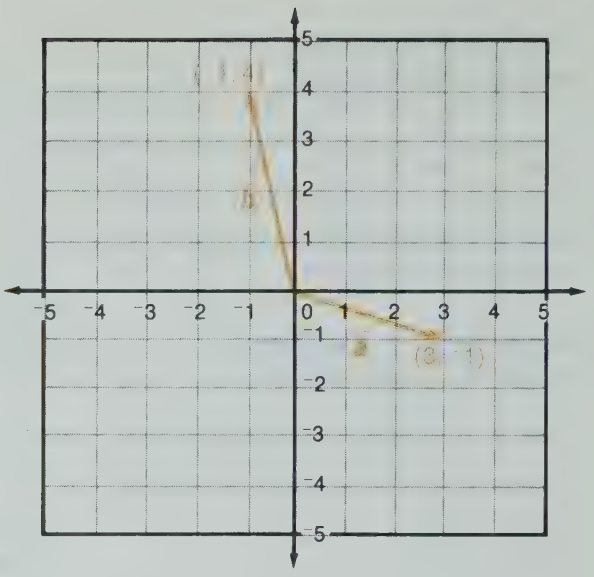
Suppose a cube and an open-top box, into which the cube will just fit, have their vertices numbered as shown. How many different ways can the cube be placed into the box?



Investigating the Ideas

Draw vectors $\vec{a}$ and $\vec{b}$ on graph paper. Suppose $\vec{a}$ and $\vec{b}$ are two sides of a parallelogram.

Can you draw the other two sides of the parallelogram?



Discussing the Ideas

1. What are the co-ordinates of the other vertex of the parallelogram in the Investigation?

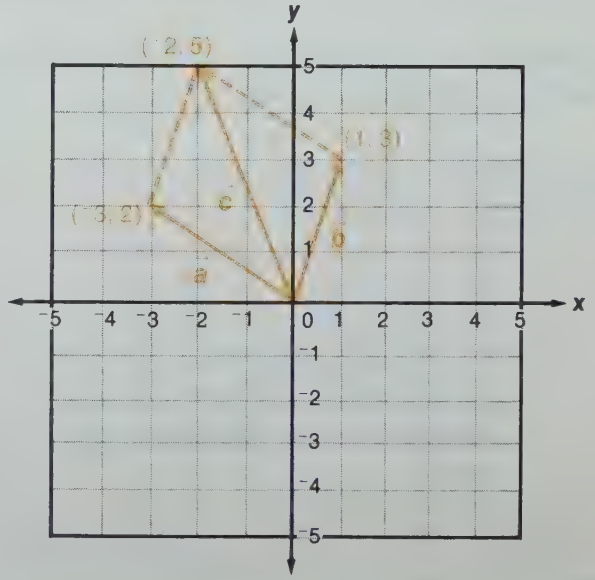
2. Draw an arrow $\vec{c}$ to the vertex of the parallelogram you found. The vector $\vec{c}$ is called the **sum** of vectors $\vec{a}$ and $\vec{b}$. We denote this as $\vec{a} \oplus \vec{b} = \vec{c}$. The symbol $\oplus$ is used to distinguish it from the addition symbol for numbers. Do you see any numerical relation between $\vec{a} = (3, -1)$, $\vec{b} = (-1, 4)$, and the ordered pair for $\vec{c}$? Describe the relationship.

3. Lisa said that she found the sum of two vectors by adding the numbers in the ordered pairs together.

$\vec{a} = (-3 \ , \ 2)$

$\vec{b} = (1 \ , \ 3)$

$\vec{c} = (-3 + 1, 2 + 3) = (-2, 5)$



Try Lisa's method for these sums.

A $\vec{r} = (4, -3)$, $\vec{s} = (1, 5)$. Find $\vec{r} \oplus \vec{s}$.

B $\vec{j} = (1, -1)$, $\vec{k} = (-2, -4)$. Find $\vec{j} \oplus \vec{k}$.

1. Sketch the arrow for each vector.
   - A $\vec{r} = (2, 2)$
   - B $\vec{s} = (3, ^-1)$
   - C $\vec{t} = (^-2, 2)$
   - D $\vec{u} = (^-1, ^-3)$

2. Use the vectors in Exercise 1 to find these sums.
   - A $\vec{r} \oplus \vec{s}$
   - B $\vec{s} \oplus \vec{t}$
   - C $\vec{t} \oplus \vec{u}$
   - D $\vec{s} \oplus \vec{u}$
   - E $\vec{u} \oplus \vec{r}$
   - F $\vec{t} \oplus \vec{r}$

3. The **null vector** or **zero vector** is $(0, 0)$. If $\vec{p} = (^-7, 3)$, what is the sum of $\vec{p}$ and the zero vector?

4. Is there an identity element for addition of vectors? That is, for every vector $\vec{j}$, is there a vector whose sum with $\vec{j}$ is vector $\vec{j}$? What is the vector?

5. A What is the sum $\vec{a} \oplus \vec{b}$ where $\vec{a} = (7, ^-3)$ and $\vec{b} = (^-7, 3)$?
   - B What is the sum $\vec{b} \oplus \vec{a}$?
   - C Do you think addition of vectors is commutative?

6. If the sum of two vectors is the null vector, each vector is called the **additive inverse** of the other.
   - A What is the additive inverse of $\vec{t} = (6, ^-5)$?
   - B Does every vector have an additive inverse vector?

7. Let $\vec{d} = (2, 6)$, $\vec{e} = (^-3, ^-2)$, and $\vec{f} = (4, ^-3)$.
   - A Find $\vec{d} \oplus \vec{e}$.
   - B Find $\vec{e} \oplus \vec{f}$.
   - C Find $(\vec{d} \oplus \vec{e}) \oplus \vec{f}$.
   - D Find $\vec{d} \oplus (\vec{e} \oplus \vec{f})$.

8. Do you think addition of vectors is associative?

9. ☆ Let $\vec{n} = (^-3, ^-7)$ and $\vec{m} = (5, 4)$. Find the ordered pair for $\vec{x}$ if $\vec{m} \oplus \vec{x} = \vec{n}$.

think

The number 145 can be expressed as the sum of two different squares in two different ways.

$$145 = 9^2 + 8^2 = 12^2 + 1^2$$

Can you find a smaller number with the same property?

What is the **smallest** number that can be expressed as the sum of two different squares in two different ways?

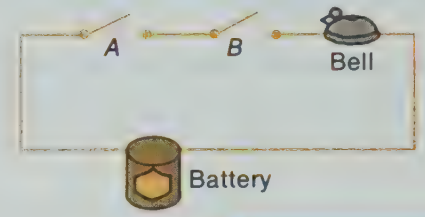
Investigating the Ideas

Electrical circuits have **switches** which can be opened or closed to determine whether or not current will flow through the circuit.

AN **OPEN** SWITCH
No current flows through the switch.

A **CLOSED** SWITCH
Current will flow through the switch.

Switches *A* and *B* are connected in **series**. The bell will not ring if both switches are in open positions. Show the circuits for other possible positions for switches *A* and *B*.

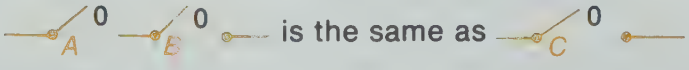


Can you decide for which circuits the bell will ring?

Discussing the Ideas

1. **A** Will the bell ring in the series circuit if only switch *A* is closed?
   B Will the bell ring if only switch *B* is closed?
   C Will the bell ring if both switches *A* and *B* are closed?

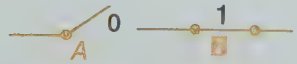
2. Suppose an **open switch** is designated by **0** and a **closed switch** is designated by **1**.



A is open **and** *B* is open. One open switch

We can describe this by the equation $0 \otimes 0 = 0$ where $\otimes$ means **and**.

What equations can you write using the symbols 0, 1, and $\otimes$ for *A*, *B*, and *C*?

A 

is the same as

A open **and** *B* closed — One open switch

B

is the same as

A closed **and** *B* open — One open switch

C

is the same as

A closed **and** *B* closed — One closed switch

1. Use the results of Discussion Exercise 2 to complete the table for the circuits.

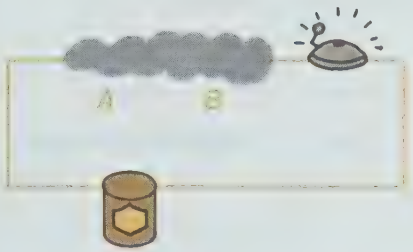
⊗	0	1
0		
1		

2. Use the table in Exercise 1 to solve the equations.

   A $0 \otimes 1 = x$ C $1 \otimes z = 0$

   B $1 \otimes y = 0$ D $t \otimes 1 = 1$

3. The circuit has two switches, A and B, connected in series. The bell is ringing.



   A In what position are switches A and B?

   B Write an equation for the circuit.

4. The bell is not ringing in the circuit. Switch A is closed.

   A In what position is B?

   B Write an equation for the circuit.

5. A, B, and C are three switches in series.

   A Will current flow through the circuit?

   B Which single switch could replace these three switches, an open switch or a closed switch?

   C Solve the equation $(0 \otimes 1) \otimes 0 =$ ▮.

6. Solve the equations.

   A $(0 \otimes 0) \otimes 1 =$ ▮ C $(0 \otimes 1) \otimes 1 =$ ▮

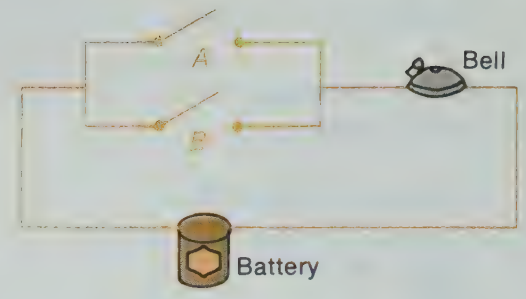
   B $(1 \otimes 0) \otimes 1 =$ ▮ D $(1 \otimes 1) \otimes 1 =$ ▮

7. Can you show that the operation ⊗ is commutative?

8. Is ⊗ an associative operation?

Investigating the Ideas

Two electrical switches A and B are shown in **parallel**. Current will flow through the circuit and the bell will ring when **at least one** of the switches, A or B, is closed. Describe all the possible positions for switches A and B.



Can you decide in which positions the switches must be in order for the bell to ring?

Discussing the Ideas

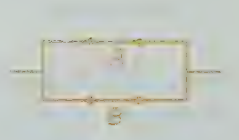
1. What single switch, open or closed, is equivalent to each of these parallel switches?

A

B

C

D



2. Suppose that 0 denotes an open switch, 1 denotes a closed switch, and $\oplus$ denotes switches in parallel. Exercise 1**A** can be described by the equation $0 \oplus 1 = 1$ because if A is open and B is closed, current will flow through the circuit.

   Write equations for the switches in parts **B**, **C**, and **D** of Exercise 1.

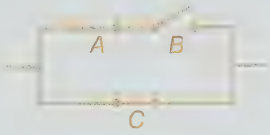
3. Use the results of Exercise 2 to copy and complete the table for parallel circuits.

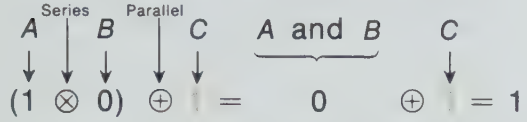
1. Use the table in Discussion Exercise 3 to solve these equations for parallel switches.

A $0 \oplus 1 = x$ **C** $1 \oplus 1 = z$ **E** $0 \oplus 0 = t$

B $y \oplus 0 = 0$ **D** $1 \oplus 0 = r$ **F** $0 \oplus n = 1$

2. A circuit contains two switches in series which are connected in parallel with a third switch. To determine if a current will flow through the circuit, we can write

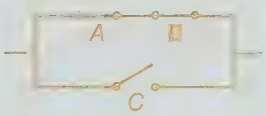




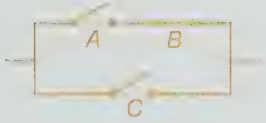
Therefore, the circuit is equivalent to one closed switch and current will flow through the circuit.

Write and solve similar equations for the switches below.

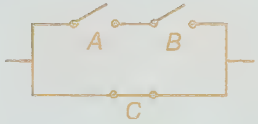
A



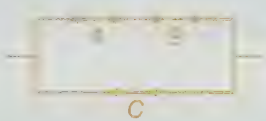
B



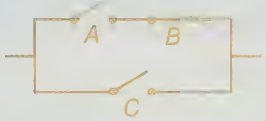
C



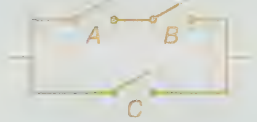
D



E



F



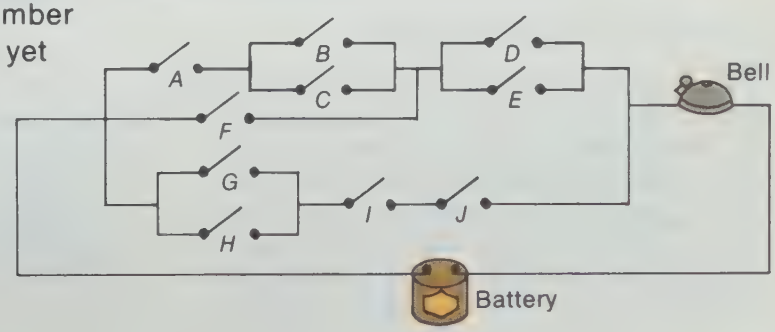
3. Solve the equations. Each contains ⊗ and ⊕.

A $(1 \otimes 0) \oplus 1 =$ **C** $(0 \otimes 0) \oplus 1 =$ **E** $(1 \oplus 1) \otimes 0 =$

B $(1 \oplus 0) \otimes 1 =$ **D** $(0 \otimes 1) \oplus 1 =$ **F** $(1 \otimes 1) \oplus 0 =$

★ **4. A** For the circuit below, what is the fewest number of switches that can be closed to make the bell ring?

B What is the greatest number that can be closed and yet the bell will not ring?



REVIEWING THE IDEAS

1. Complete the addition table for the modulo 4 system.

$\oplus_4$				
			0	1

2. Solve the equations in the modulo 4 system.

A $3 \oplus_4 1 = x$ **C** $0 \oplus_4 1 = z$

B $2 \oplus_4 3 = y$ **D** $2 \oplus_4 2 = t$

3. Complete the multiplication table for the modulo 4 system.

$\otimes_4$				
				2
				1

4. Using the table shown, solve the equations in the *IRUL* system.

★	*I*	*R*	*U*	*L*
I	*I*	*R*	*U*	*L*
R	*R*	*U*	*L*	*I*
U	*U*	*L*	*I*	*R*
L	*L*	*I*	*R*	*U*

A $R \star x = U$

B $I \star x = L$

C $R \star U = x$

D $x \star L = I$

E $L \star L = x$

F $L \star x = U$

5. Solve the equations in the octic system.

A $\not{\Delta} \star x = I$

B $x \star V = R$

C $(U \star x) \star V = V$

D $(V \star L) \star \not{\Delta} = x$

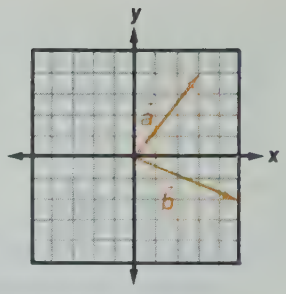
E $(R \star \not{\Delta}) \star x = H$

F $(\not{\Delta} \star \not{\Delta}) \star x = I$

6. **A** Give the co-ordinates of $\vec{a}$.

B Find the length of $\vec{a}$.

C What vector has the same length as $\vec{a}$ but is opposite in direction?



7. What vector is $\vec{a} \oplus \vec{b}$? (See drawing for Exercise 6.)

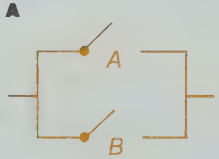
8. If $\vec{r} = (^-2, 3)$ and $\vec{s} = (7, ^-1)$, what vector is $\vec{r} \oplus \vec{s}$?

9. What vector is the additive inverse of $\vec{v} = (6, ^-2)$?

10. If $\vec{j} = (5, 6)$, $\vec{k} = (3, ^-2)$, and $\vec{m} = (^-4, 0)$, what is $(\vec{j} \oplus \vec{k}) \oplus \vec{m}$?

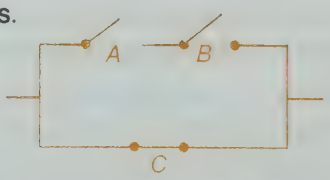
11. Tell whether the switches are in series or in parallel.

A



B

12. Will current flow through this circuit? A, B, and C are switches.



13. Solve the equations for series $\otimes$ and parallel $\oplus$ circuits.

A $(1 \otimes 0) = x$ **E** $(1 \otimes 0) \oplus 0 = x$

B $(1 \oplus 0) = x$ **F** $(1 \oplus 1) \otimes 1 = x$

C $(1 \otimes 1) = x$ **G** $(0 \oplus 1) \otimes 1 = x$

D $(1 \oplus 1) = x$ **H** $0 \oplus (1 \otimes 1) = x$

TEST YOURSELF

1. Solve the modulo 4 addition equations.

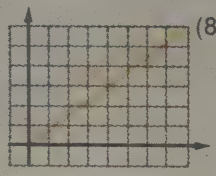
   A $3 \oplus_4 3 = x$

   B $2 \oplus_4 1 = y$

2. Solve the modulo 4 multiplication equations.

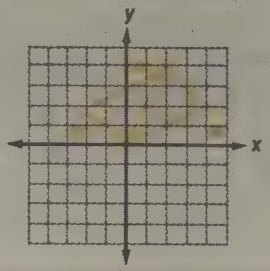
   A $2 \otimes_4 3 = x$ B $3 \otimes_4 3 = y$

3. Find the length of $\vec{v}$.



4. What vector is the additive inverse of $\vec{r} = (1, -3)$?

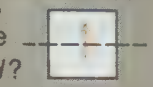
5. What addition equation for vectors does the drawing suggest?



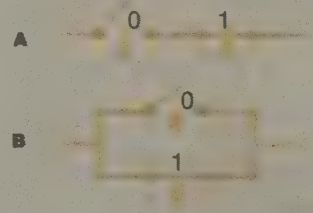
6. If $\vec{c} = (9, 9)$ *and* $\vec{d} = (-1, 4)$, what is $\vec{c} \oplus \vec{d}$?

7. In the octic system, $R \star R =$ ▨.

8. In the octic system, which motion is the inverse of the flip H?



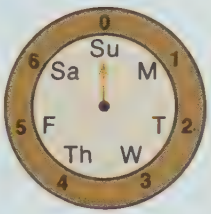
9. What single switch, **open** or **closed**, could replace the two switches in each circuit?



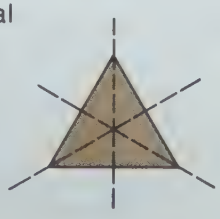
10. Solve the circuit equation. $(1 \otimes 1) \oplus 0 = x$

RESEARCH PROJECTS

A Invent a modular arithmetic based on the days of the week. Make an "addition" table and a "multiplication" table for the system.



B Invent a mathematical system based on the rotations and flips of an equilateral triangle around its centre and its lines of symmetry.



C There is an algebra of "tire switching" on an automobile. Find out about this nonmathematical system. (See *Algebra: A New Way of Looking at* Weybright and Talley, 1968, pp. 32-54; Musson Book Company.)

D Invent more complicated circuits. Analyze the circuits using the symbols 0, 1, $\otimes$, and $\oplus$ as developed in this module. (See *Mathematics and the Modern World* by Mario F. Triola; Menlo Park, California: Cummings Publishing Company, 1973, pp. 39-48; Addison-Wesley.)

CUMULATIVE REVIEW

1. Six names are placed in a hat. Two of the names are boys' names. What are the chances of drawing a girl's name from the hat?

2. A probability experiment has six equally likely outcomes. What is the probability of each outcome?

3. A probability experiment has only two outcomes, A and B. If $P(A) = \frac{3}{8}$, what is $P(B)$?

4. A penny and a nickel are tossed. Give the probability of each event.
   - **A** Two heads
   - **B** Two tails
   - **C** A head and a tail
   - **D** At least one tail

5. Two black marbles and two white marbles are placed in a box. The marbles are withdrawn one at a time until all have been withdrawn. Draw a tree diagram that will show all possible outcomes for this experiment.

6. How many different arrangements (permutations) are there for the letters A, B, C, and D?

7. How many combinations can be made with the letters A, B, C, D, E, and F if three letters are taken at a time?

8. Compute. $\dfrac{7!}{5! \cdot 2!}$

9. A die is tossed. Let E be the event of getting a 3 or more and let F be the event of getting a 1 or 2.
   - **A** What is $P(E)$?
   - **B** What is $P(F)$?
   - **C** What is $P(E \cap F)$?
   - **D** What is $P(E \cup F)$?

10. **A** What is the range of the heights?
    B What is the arithmetic mean of the heights?

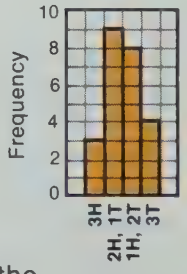
Name	Height
Ann	141 cm
Dave	158 cm
Ruth	173 cm
Paul	180 cm

11. On five tests Kitty had scores of 91, 82, 88, 94, and 85. What was her median score?

12. The table shows the number of points scored by each player on a basketball team.
    - **A** What is the median score?
    - **B** What is the mean?
    - **C** What is the mode?

Name	Points
Jones	21
Willis	10
Marto	8
Mills	0
Ford	2
Baker	6
Evans	0
Leeds	1

13. The histogram shows the results of an experiment of tossing three coins.
    - **A** What was the most frequent outcome?
    - **B** What outcome occurred the fewest number of times?
    - **C** How many trials of the experiment does the graph show?

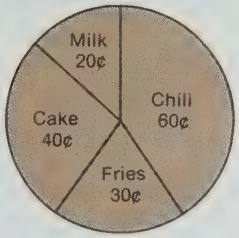


14. Construct a line-segment graph of these hourly temperatures.

9 A.M.	−2°C	1 P.M.	7°C
10 A.M.	0°C	2 P.M.	9°C
11 A.M.	3°C	3 P.M.	11°C
12 Noon	5°C	4 P.M.	10°C

15. **A** What percent of the lunch was the chili?

B How large is the central angle for the fries?



Lunch $1.50

16. A sample of voters in a city shows that 53% favor candidate A and 47% favor candidate B. If 20 000 people vote in the election, about how many votes can each candidate expect to get based on this sample?

17. If $x^2 = 2$, then $x = \sqrt{}$ or $x = {}^{-}\sqrt{}$.

18. Which decimal is the best approximation for $\sqrt{2}$? 1.4 or 1.5?

19. Locate approximately on a number line the points which correspond to $\sqrt{3}$, $\sqrt{21}$, and $\sqrt{20}$.

20. Give the best choices of integers for a and b.

A $a < \sqrt{10} < b$ **C** $a < \sqrt{95} < b$

B $a < \sqrt{40} < b$ **D** $a < \sqrt{601} < b$

21. Find the decimal approximation for the following numbers to the nearest tenth.

A $\sqrt{48}$ **B** $\sqrt{930}$ **C** $\sqrt{5625}$

22. Write a repeating decimal for $\frac{5}{9}$.

23. Write a fraction for each decimal.

A 0.32 **B** 0.071 **C** $0.7\overline{7}$

24. Complete each sentence with **repeating** or **nonrepeating.**

A Every rational number has a ? decimal representation.

B Every irrational number has a ? decimal representation.

25. Find each product.

A $\sqrt{15} \cdot \sqrt{15}$ **C** $5\sqrt{7} \cdot 3 \cdot \sqrt{7}$

B $\sqrt{7} \cdot 2 \cdot \sqrt{7}$ **D** $\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{2} \cdot \sqrt{3}$

26. Give a decimal approximation (to the nearest hundredth) for each expression. Use $\sqrt{5} \approx 2.236$ and $\sqrt{2} \approx 1.414$.

A $\sqrt{5} + \sqrt{2}$ **C** $\sqrt{5} - \sqrt{2}$

B $\sqrt{5} \cdot \sqrt{2}$ **D** $\sqrt{5} \div \sqrt{2}$

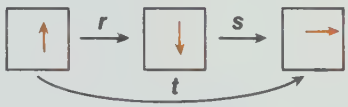
27. Solve the addition equations in the modulo 4 system.

A $1 \oplus_4 3 = x$ **B** $2 \oplus_4 3 = y$

28. Solve the multiplication equations in the modulo 4 system.

A $0 \otimes_4 3 = x$ **B** $2 \otimes_4 2 = y$

29. Give the rotations for r and s and then give the combined rotation for t.



30. In the octic system, $R \bigstar R =$ ▒.

31. If $\vec{v} = (7, 4)$ and $\vec{t} = (2, 2)$, what is $\vec{v} \oplus \vec{t}$?

32. What is the inverse vector for the vector $\vec{r} = (^{-}2, 6)$?

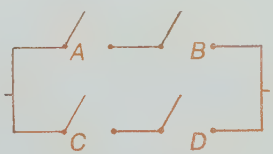
33. Which switches must be closed before current will flow through the circuit?

A Only A and C.

B Either A or C.

C Both A and B.

D Both A and D.



GAME THEORY

A recent and interesting branch of mathematics is known as "Game Theory." The theory of games began with the work of the mathematician John von Neumann (1903–1957). Game theory not only applies to simple games but also has important applications in business, psychology, and the social sciences. The game described below is a simple application of game theory.

Try playing this game with a classmate.

Each player starts a game with 20 counters.
Each player has two cards, one labelled X and the other Y.
Each player chooses one of his cards and both players turn up the cards simultaneously.

The table below shows how the players win counters according to their choices.

		Player B	
		Card X	Card Y
Player A	Card X	A wins 5 counters	B wins 5 counters
	Card Y	B wins 1 counter	A wins 1 counter

The game is completed when one player wins all of the other player's counters.

Try to develop some strategies that may help you win the game.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A $x = 2$; B $y = 3$ 2. A $x = 2$; B $y = 1$ 3. 10 units
4. $(^{-}1, 3)$ 5. $\vec{a} \oplus \vec{b} = \vec{c}$ 6. $\vec{c} \oplus \vec{d} = (8, 13)$
7. U 8. H 9. A open; B closed 10. $x = 1$

Supplementary Exercises

SET 1

1. Write each product using an exponent.

A $5 \cdot 5$ **C** $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ **E** $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

B $8 \cdot 8 \cdot 8$ **D** $95 \cdot 95 \cdot 95 \cdot 95$ **F** $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

2. Write each product as a power of ten.

A $10^2 \cdot 10^1$ **C** $10^3 \cdot 10^2$ **E** $10^2 \cdot 10^4$ **G** $10^5 \cdot 10^0$

B $10^2 \cdot 10^2$ **D** $10^4 \cdot 10^1$ **F** $10^4 \cdot 10^3$ **H** $10^7 \cdot 10^2$

3. Write each quotient as a power of ten.

A $10^3 \div 10^1$ **C** $10^5 \div 10^2$ **E** $10^9 \div 10^3$ **G** $10^6 \div 10^0$

B $10^2 \div 10^1$ **D** $10^6 \div 10^1$ **F** $10^6 \div 10^4$ **H** $10^4 \div 10^4$

4. Write each base-ten numeral in expanded notation.

A 38 **C** 802 **E** 2035 **G** 14 035 **I** 687 524

B 621 **D** 1259 **F** 8001 **H** 81 253 **J** 300 050

SET 2

1. Round each number to the nearest ten.

A 72 **C** 129 **E** 1253 **G** 15 633 **I** 200 008

B 85 **D** 635 **F** 7507 **H** 48 106 **J** 555 555

2. Round each number to the nearest hundred.

A 385 **C** 1258 **E** 9050 **G** 23 975 **I** 843 705

B 649 **D** 6845 **F** 17 883 **H** 105 105 **J** 503 507

3. Round each number to the nearest thousand.

A 1295 **C** 5765 **E** 20 801 **G** 458 358

B 2603 **D** 15 976 **F** 135 053 **H** 755 555

4. Round **5 607 384** to the nearest

A ten. **C** thousand. **E** hundred thousand.

B hundred. **D** ten thousand. **F** million.

SET 3

1. Use multiples of 10, 100, or 1000 for each ▨.

A To estimate 526 + 385, find the sum 500 + ▨.

B To estimate 1285 − 938, find the difference ▨ − 900.

C To estimate 493 · 32, find the product 500 · ▨.

D To estimate 3207 + 376, find the quotient 3200 ÷ ▨.

E To estimate 543 · 208, find the product ▨ · 200.

F To estimate 6388 + 5892, find the sum 6000 + ▨.

G To estimate 41 850 ÷ 68, find the quotient ▨ ÷ 70.

2. Give an estimate for each of the following problems.

A 336 + 520	**D** 1538 ÷ 48	**G** 2436 − 498	**J** 7437 + 15 805
B 1640 − 1099	**E** 327 · 62	**H** 3208 ÷ 825	**K** 526 · 879
C 873 · 6	**F** 9051 + 832	**I** 3128 · 168	**L** 6172 · 983

SET 4

1. Find the sums.

A 56 + 33	**B** 35 + 57	**C** 158 + 49	**D** 351 + 493	**E** 1205 + 999	**F** 15 873 + 3 059
G 42 + 67 + 95	**H** 153 + 48 + 7	**I** 506 + 721 + 273	**J** 942 + 267 + 307	**K** 3841 + 1762 + 4211	**L** 76 437 + 8 649 + 3 001

2. Find the sums.

A 82 + 9 + 105	**G** 364 + 721 + 480	**M** 99 + 999 + 9999
B 86 + 157 + 19	**H** 6300 + 503 + 59	**N** 76 095 + 2649 + 850
C 135 + 49 + 63	**I** 78 + 151 + 35 + 103	**O** 45 083 + 30 906 + 1285
D 350 + 35 + 3500	**J** 756 + 8005 + 10 700	**P** 850 721 + 37 842 + 7655
E 943 + 17 + 306	**K** 3700 + 37 000 + 370	**Q** 937 210 + 653 213
F 77 + 103 + 520	**L** 538 + 5603 + 809	**R** 8 340 756 + 4 750 095

SET 5

1. Find the differences.

A 87 − 54	**B** 72 − 39	**C** 100 − 53	**D** 248 − 86	**E** 963 − 277	**F** 801 − 352
G 1576 − 347	**H** 2834 − 1843	**I** 6912 − 6853	**J** 5350 − 4795	**K** 9500 − 6306	**L** 10 000 − 591

2. Find the differences.

A 93 − 27	**F** 1586 − 958	**K** 51 753 − 1845	**P** 100 000 − 10 000
B 150 − 69	**G** 4081 − 3982	**L** 67 595 − 39 008	**Q** 100 000 − 1000
C 302 − 124	**H** 7800 − 839	**M** 30 000 − 3000	**R** 100 000 − 100
D 950 − 666	**I** 9040 − 150	**N** 67 805 − 1800	**S** 343 000 − 24 793
E 1500 − 875	**J** 3897 − 2999	**O** 97 031 − 950	**T** 956 000 − 955 311

SET 6

1. Find the products.

A 52 · 6	**D** 129 · 4	**G** 1576 · 2	**J** 12 511 · 3
B 73 · 8	**E** 456 · 7	**H** 4609 · 6	**K** 37 059 · 5
C 87 · 9	**F** 903 · 5	**I** 8080 · 9	**L** 90 903 · 7

2. Find the products.

A 54 · 18	**D** 253 · 35	**G** 1200 · 69	**J** 15 000 · 18
B 26 · 67	**E** 125 · 88	**H** 5306 · 81	**K** 37 112 · 23
C 46 · 22	**F** 605 · 76	**I** 7953 · 44	**L** 53 025 · 85

3. Find the products.

A 179 · 201	**D** 535 · 806	**G** 1734 · 213	**J** 6305 · 833
B 368 · 555	**E** 490 · 359	**H** 5877 · 303	**K** 5950 · 970
C 843 · 347	**F** 993 · 865	**I** 7951 · 537	**L** 8005 · 508

SET 7

1. Find the quotients and remainders.

A $5\overline{)74}$	**D** $6\overline{)384}$	**G** $3\overline{)501}$	**J** $6\overline{)1522}$	**M** $8\overline{)34\,300}$
B $7\overline{)105}$	**E** $4\overline{)149}$	**H** $5\overline{)419}$	**K** $4\overline{)1905}$	**N** $9\overline{)51\,355}$
C $8\overline{)125}$	**F** $9\overline{)570}$	**I** $7\overline{)959}$	**L** $2\overline{)9999}$	**O** $7\overline{)61\,499}$

2. Find the quotients and remainders.

A $42\overline{)183}$	**D** $54\overline{)1568}$	**G** $74\overline{)2075}$	**J** $19\overline{)51\,300}$
B $63\overline{)516}$	**E** $27\overline{)2245}$	**H** $68\overline{)8346}$	**K** $34\overline{)29\,806}$
C $36\overline{)172}$	**F** $49\overline{)1677}$	**I** $81\overline{)8000}$	**L** $99\overline{)81\,565}$

3. Find the quotients and remainders.

A $189\overline{)4160}$	**D** $351\overline{)4593}$	**G** $235\overline{)10\,815}$	**J** $262\overline{)37\,470}$
B $304\overline{)2463}$	**E** $705\overline{)6615}$	**H** $324\overline{)22\,043}$	**K** $936\overline{)105\,100}$
C $611\overline{)5500}$	**F** $931\overline{)9200}$	**I** $654\overline{)54\,321}$	**L** $677\overline{)200\,000}$

SET 8

1. Evaluate $2x - 1$ using the numbers in the set $\{1, 2, 3, 4\}$.

2. Find the missing numbers for each part.

A If $a = 6$, then $6 + a =$ ▒.
B If $b = 2$, then $17 - b =$ ▒.
C If $c = 3$, then $6c =$ ▒.
D If $d = 42$, then $d \div 6 =$ ▒.
E If $e = 17$, then $e - 5 =$ ▒.
F If $f = 4$, then $5f - 3 =$ ▒.
G If $g = 8$, then $(16 - g) + 2 =$ ▒.
H If $h = 3$, then $15 - 3h =$ ▒.
I If $k = 5$, then $4k \div 2 =$ ▒.
J If $j = 10$, then $j^2 =$ ▒.

3. Suppose $\{0, 1, 2, 3, 4, 5, 6, 7\}$ is the replacement set for $2x + 3$.
What is the replacement for x that will make the value of the expression

A 11? **B** 5? **C** 3? **D** 15? **E** 9?

SET 9

1. Solve the equations. Use {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} as the replacement set.

A $x - 3 = 4$

B $5a = 20$

C $81 \div r = 9$

D $6 + q = 15$

E $3y = 18$

F $19 - 2n = 11$

G $3t + 1 = 16$

H $36 \div 2m = 2$

I $4d - 6 = 22$

J $(13 - a) + 4 = 15$

K $16 + 3c = 16$

L $2h \div 5 = 2$

2. Find the solution set for each inequality. Use {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} as the replacement set.

A $x < 5$

B $y > 9$

C $a + 5 < 12$

D $d - 2 > 5$

E $2w > 4$

F $15 - t > 9$

G $3g + 1 < 10$

H $t^2 > 16$

I $5y - 2 < 19$

J $q^2 - 1 < 25$

K $x^2 + x < 50$

L $d^3 + 1 < 100$

SET 10

1. Solve the equations.

A $7 + c = 13$

B $21 - x = 17$

C $72 = r + 23$

D $8t = 40$

E $38 + y = 52$

F $t \div 9 = 4$

G $42 = r + 19$

H $21 - t = 16$

I $17x = 51$

J $7r = 42$

K $54 \div y = 54$

L $40w = 120$

M $t + 19 = 19$

N $28 \div x = 4$

O $15y - 3 = 27$

2. Give the number for the ▒. Then give the number for **x**.

A $3x + 4 = 10$
$3x = ▒$
$x = ?$

B $2x - 3 = 9$
$2x = ▒$
$x = ?$

C $4x + 2 = 14$
$4x = ▒$
$x = ?$

D $2x + 7 = 15$
$2x = ▒$
$x = ?$

E $5x - 4 = 6$
$5x = ▒$
$x = ?$

F $7x + 6 = 27$
$7x = ▒$
$x = ?$

G $4x - 6 = 14$
$4x = ▒$
$x = ?$

H $8x + 5 = 5$
$8x = ▒$
$x = ?$

SET 11

1. Give the number for the ▒. Then find the number for **x**.

A $15 + (x - 1) = 15$

$x - 1 =$ ▒

$x =$ **?**

B $3(x - 1) = 3$

$x - 1 =$ ▒

$x =$ **?**

C $6x + 23 = 23$

$6x =$ ▒

$x =$ **?**

D $3x + 2x = 45$

▒$x = 45$

$x =$ **?**

E $4x - 2x = 14$

▒$x = 14$

$x =$ **?**

F $5x + 3x - 4 = 20$

▒$x = 24$

$x =$ **?**

2. Solve the equations.

A $2x + 3x = 10$

B $6x - 2x = 12$

C $11x - 3x = 24$

D $(7x - 2x) + 4x = 36$

E $2x + (x + 3x) = 12$

F $4 + (3x - x) = 10$

3. Solve the equations.

A $(7x + 2x) - 5 = 40$

B $7 + (9x - 3x) = 37$

C $(5x + 2x) + x = 48$

D $(7x - 2x) - 3x = 6$

E $(8x - 4x) + 4 = 20$

F $(6x + 3x) - 2 = 7$

G $3 + (8x - 3x) = 13$

H $(10x - 7x) - 10 = 11$

I $(x + 2x) + 3x = 54$

SET 12

Write and solve an equation for each problem.

1. The product of a certain number and 8 is 72. What is the number?

2. Six times a certain number minus 2 is 34. What is the number?

3. When the sum of a certain number and 8 is multiplied by 6, the product is 66. What is the number?

4. When the sum of a certain number and 5 is divided by 7, the quotient is 13. What is the number?

5. Maria bought 3 cans of soup for 57 cents. How much did she pay for each can?

6. Ricardo had 93 cents. After he bought 3 pens, he had 6 cents left. How much for each pen?

7. Jan is 3 years older than Sam. In 5 years the sum of their ages will be 33.

A What is the sum of their ages now?

B How old is each person?

SET 13

1. Complete the table of numbers for the equation $x - y = 3$.

x	8	7	6	5	4	3
y	5					

2. Make a graph of the equation $x - y = 3$ using the pairs (x, y) from the table in Exercise 1.

3. Complete the table for the equation $2y - 4 = x$.

y	2	3	4	5	6	7
$2y$	4	6				
x	0					

4. Graph the equation $2y - 4 = x$ using the pairs (x, y) from the table in Exercise 3.

5. Make a table of pairs (x, y) for each equation. Then graph the equation.

   A $x + y = 8$ **B** $2x + y = 12$ **C** $y + 3 = x$

SET 14

1. Factor each number completely.

A 30	**D** 18	**G** 27	**J** 105	**M** 125	**P** 305
B 56	**E** 72	**H** 42	**K** 103	**N** 136	**Q** 489
C 24	**F** 44	**I** 51	**L** 110	**O** 150	**R** 512

2. Use exponents to give the prime factorization of each number.

A 15	**E** 54	**I** 336	**M** 308	**Q** 632	**U** 750
B 18	**F** 60	**J** 252	**N** 820	**R** 198	**V** 900
C 27	**G** 125	**K** 500	**O** 704	**S** 810	**W** 960
D 36	**H** 140	**L** 512	**P** 612	**T** 350	**X** 980

SET 15

1. Find the greatest common factor (GCF) of each pair.

A 6 and 18	**F** 24 and 36	**K** 28 and 42	**P** 90 and 120
B 11 and 12	**G** 14 and 35	**L** 30 and 70	**Q** 68 and 95
C 15 and 25	**H** 24 and 42	**M** 28 and 48	**R** 92 and 93
D 24 and 6	**I** 60 and 100	**N** 75 and 125	**S** 150 and 270
E 10 and 35	**J** 14 and 49	**O** 48 and 72	**T** 360 and 450

2. What is the GCF of the numerator and denominator of each fraction?

A $\frac{9}{12}$	**D** $\frac{20}{25}$	**G** $\frac{18}{27}$	**J** $\frac{60}{75}$	**M** $\frac{75}{300}$	**P** $\frac{240}{400}$
B $\frac{10}{40}$	**E** $\frac{20}{24}$	**H** $\frac{21}{31}$	**K** $\frac{25}{150}$	**N** $\frac{95}{85}$	**Q** $\frac{270}{280}$
C $\frac{44}{55}$	**F** $\frac{14}{44}$	**I** $\frac{27}{45}$	**L** $\frac{60}{150}$	**O** $\frac{33}{105}$	**R** $\frac{525}{2500}$

SET 16

1. Find the least common multiple (LCM) of each pair.

A 2 and 5	**F** 4 and 5	**K** 6 and 15	**P** 30 and 40
B 6 and 9	**G** 15 and 12	**L** 9 and 33	**Q** 10 and 13
C 10 and 4	**H** 20 and 25	**M** 10 and 12	**R** 18 and 30
D 2 and 7	**I** 15 and 20	**N** 12 and 16	**S** 16 and 24
E 8 and 3	**J** 2 and 17	**O** 21 and 15	**T** 90 and 60

2. Find the LCM of each group of numbers.

A 2, 6, and 10	**F** 4, 6, and 12	**K** 2, 7, and 5
B 3, 5, and 10	**G** 2, 4, and 7	**L** 5, 6, and 15
C 4, 6, and 15	**H** 2, 4, and 9	**M** 3, 8, and 12
D 10, 12, and 15	**I** 5, 10, and 15	**N** 10, 20, and 30
E 2, 3, and 5	**J** 8, 12, and 15	**O** 8, 12, and 16

SET 17

1. Find the sums.

A $^{-}3 + 2$	**F** $^{-}16 + 9$	**K** $^{-}50 + 70$	**P** $120 + {}^{-}119$
B $4 + {}^{-}6$	**G** $14 + {}^{-}12$	**L** $95 + {}^{-}30$	**Q** $^{-}98 + {}^{-}100$
C $^{-}8 + 5$	**H** $^{-}13 + 17$	**M** $72 + {}^{-}91$	**R** $175 + {}^{-}223$
D $^{-}9 + 9$	**I** $^{-}15 + {}^{-}8$	**N** $^{-}83 + {}^{-}17$	**S** $^{-}106 + {}^{-}106$
E $0 + {}^{-}7$	**J** $^{-}21 + {}^{-}21$	**O** $^{-}16 + 32$	**T** $^{-}935 + 0$

2. Find the sums.

A $6 + 4 + {}^{-}7$	**F** $15 + {}^{-}13 + 7$	**K** $(30 + 30) + (30 + {}^{-}30)$
B $6 + {}^{-}4 + {}^{-}7$	**G** $^{-}5 + 19 + {}^{-}3$	**L** $(30 + {}^{-}30) + (30 + {}^{-}30)$
C $6 + {}^{-}4 + 7$	**H** $26 + {}^{-}20 + {}^{-}8$	**M** $(30 + 30) + ({}^{-}30 + {}^{-}30)$
D $^{-}6 + {}^{-}4 + 7$	**I** $^{-}37 + {}^{-}1 + 35$	**N** $({}^{-}30 + 30) + ({}^{-}30 + {}^{-}30)$
E $^{-}6 + 4 + {}^{-}7$	**J** $^{-}86 + {}^{-}10 + {}^{-}1$	**O** $({}^{-}30 + {}^{-}30) + ({}^{-}30 + {}^{-}30)$

SET 18

1. Solve the equations. Think about missing addends.

S A A	S A A	S A A	S A A
A $0 - 3 = n$	**F** $2 - 6 = n$	**K** $5 - 9 = n$	**P** $7 - {}^{-}3 = n$
B $1 - 3 = n$	**G** $6 - {}^{-}2 = n$	**L** $^{-}5 - 3 = n$	**Q** $14 - {}^{-}6 = n$
C $0 - {}^{-}3 = n$	**H** $^{-}6 - 2 = n$	**M** $^{-}8 - {}^{-}2 = n$	**R** $^{-}4 - {}^{-}6 = n$
D $1 - {}^{-}3 = n$	**I** $^{-}6 - {}^{-}2 = n$	**N** $7 - {}^{-}6 = n$	**S** $^{-}12 - 3 = n$
E $^{-}1 - {}^{-}3 = n$	**J** $^{-}2 - {}^{-}6 = n$	**O** $^{-}6 - 9 = n$	**T** $^{-}12 - {}^{-}3 = n$

2. Find the differences.

A $17 - 8$	**G** $0 - {}^{-}5$	**M** $25 - 28$	**S** $38 - {}^{-}58$
B $5 - 13$	**H** $^{-}4 - {}^{-}14$	**N** $^{-}35 - {}^{-}3$	**T** $^{-}105 - 3$
C $0 - 19$	**I** $26 - {}^{-}19$	**O** $48 - {}^{-}5$	**U** $^{-}225 - {}^{-}1$
D $^{-}6 - 15$	**J** $^{-}22 - 13$	**P** $^{-}53 - 12$	**V** $^{-}65 - {}^{-}65$
E $20 - 30$	**K** $^{-}5 - {}^{-}8$	**Q** $10 - {}^{-}33$	**W** $543 - {}^{-}17$
F $^{-}6 - {}^{-}9$	**L** $12 - {}^{-}5$	**R** $75 - 47$	**X** $418 - 814$

SET 19

1. Find the products.

A $2 \cdot {}^{-}10$	**F** ${}^{-}12 \cdot {}^{-}6$	**K** $20 \cdot {}^{-}14$	**P** ${}^{-}1 \cdot 87$	**U** $200 \cdot {}^{-}8$
B ${}^{-}4 \cdot {}^{-}6$	**G** ${}^{-}100 \cdot 0$	**L** ${}^{-}15 \cdot {}^{-}8$	**Q** ${}^{-}1 \cdot {}^{-}87$	**V** ${}^{-}500 \cdot 0$
C $20 \cdot {}^{-}3$	**H** ${}^{-}1 \cdot {}^{-}53$	**M** ${}^{-}1 \cdot 43$	**R** $10 \cdot {}^{-}87$	**W** ${}^{-}35 \cdot {}^{-}4$
D ${}^{-}3 \cdot {}^{-}16$	**I** ${}^{-}30 \cdot {}^{-}3$	**N** ${}^{-}50 \cdot {}^{-}30$	**S** ${}^{-}10 \cdot {}^{-}87$	**X** ${}^{-}56 \cdot 22$
E ${}^{-}10 \cdot 15$	**J** $0 \cdot {}^{-}27$	**O** ${}^{-}17 \cdot {}^{-}2$	**T** ${}^{-}10 \cdot 87$	**Y** ${}^{-}30 \cdot {}^{-}45$

2. Find the products.

A $3 \cdot 3 \cdot 3$	**F** ${}^{-}5 \cdot {}^{-}4 \cdot {}^{-}1$	**K** $({}^{-}5 \cdot 4) \cdot ({}^{-}5 \cdot 4)$
B $3 \cdot 3 \cdot {}^{-}3$	**G** ${}^{-}6 \cdot 5 \cdot {}^{-}3$	**L** $({}^{-}5 \cdot {}^{-}4) \cdot (5 \cdot 4)$
C $3 \cdot {}^{-}3 \cdot {}^{-}3$	**H** ${}^{-}10 \cdot {}^{-}10 \cdot {}^{-}8$	**M** $({}^{-}5 \cdot {}^{-}5) \cdot (4 \cdot 4)$
D $3 \cdot {}^{-}3 \cdot 3$	**I** ${}^{-}3 \cdot 15 \cdot 4$	**N** $({}^{-}5 \cdot {}^{-}4) \cdot ({}^{-}5 \cdot 4)$
E ${}^{-}3 \cdot {}^{-}3 \cdot {}^{-}3$	**J** ${}^{-}12 \cdot 3 \cdot {}^{-}4$	**O** $({}^{-}5 \cdot {}^{-}4) \cdot ({}^{-}5 \cdot {}^{-}4)$

SET 20

1. Find the missing factor.

A $27 \div 3 = n$	**E** ${}^{-}22 \div {}^{-}2 = n$	**I** ${}^{-}72 \div {}^{-}8 = n$	**M** ${}^{-}100 \div 10 = n$
B $15 \div {}^{-}5 = n$	**F** $49 \div {}^{-}7 = n$	**J** ${}^{-}27 \div {}^{-}27 = n$	**N** ${}^{-}1000 \div {}^{-}10 = n$
C ${}^{-}18 \div 3 = n$	**G** ${}^{-}54 \div {}^{-}6 = n$	**K** $36 \div {}^{-}12 = n$	**O** ${}^{-}125 \div {}^{-}25 = n$
D $19 \div {}^{-}1 = n$	**H** ${}^{-}73 \div {}^{-}1 = n$	**L** ${}^{-}45 \div 45 = n$	**P** $0 \div {}^{-}128 = n$

2. Find the quotients.

A ${}^{-}15 \div 3$	**F** $55 \div {}^{-}11$	**K** ${}^{-}300 \div 60$	**P** $\frac{18}{{}^{-}9}$	**S** $\frac{({}^{-}15 + 3)}{{}^{-}6}$
B ${}^{-}18 \div {}^{-}2$	**G** ${}^{-}48 \div 16$	**L** ${}^{-}472 \div {}^{-}8$		
C $21 \div 3$	**H** ${}^{-}65 \div {}^{-}5$	**M** ${}^{-}1000 \div {}^{-}25$	**Q** $\frac{{}^{-}24}{{}^{-}8}$	**T** $\frac{({}^{-}10 + {}^{-}20)}{5 \cdot {}^{-}1}$
D $42 \div {}^{-}7$	**I** ${}^{-}33 \div {}^{-}1$	**N** ${}^{-}1000 \div {}^{-}10$		
E ${}^{-}13 \div 13$	**J** ${}^{-}120 \div {}^{-}40$	**O** ${}^{-}3500 \div 70$	**R** $\frac{(4 - 6)}{(16 - 15)}$	**U** $\frac{({}^{-}4 - {}^{-}6)}{(4 - 8)}$

SET 21

1. What is represented by each of the following?

A $|3|$ **C** $|5|$ **E** $|^-42|$ **G** $|4+2|$ **I** $|^-4+2|$ **K** $|^-4-^-2|$

B $|^-3|$ **D** $|^-8|$ **F** $|0|$ **H** $|4-2|$ **J** $|^-4+^-2|$ **L** $|4-^-2|$

2. Solve the equations.

A $x = |^-8|$ **D** $|^-3 \cdot 1| = x$ **G** $|^-14-^-7| = x$ **J** $|5-x| = 2$

B $|3| = x$ **E** $|^-5 \cdot 0| = x$ **H** $|21 \div ^-3| = x$ **K** $|x-^-6| = 30$

C $x = |4-2|$ **F** $|^-8+^-3| = x$ **I** $|35-50| = x$ **L** $|^-9+x| = 20$

3. Give the correct symbol ($<$, $=$, or $>$) for each ▨.

A $^-5$ ▨ $|^-5|$ **C** 5 ▨ $|^-5|$ **E** $|3|-|6|$ ▨ $3-6$

B $^-5$ ▨ 5 **D** $|^-5|$ ▨ $|0-5|$ **F** $|^-3|-|^-6|$ ▨ $|^-6|-|^-3|$

SET 22

1. Give the missing number for each ▨.

A If $4x+6=2$,
then $4x =$ ▨,
and $x =$ ▨.

B If $3x+16=13$,
then $3x =$ ▨,
and $x =$ ▨.

C If $2x-6=10$,
then $2x =$ ▨,
and $x =$ ▨.

2. Give the missing integers in each equation.

A $3x+4=^-5 \Rightarrow 3x =$ ▨ $\Rightarrow x =$ ▨.

B $2x-6=14 \Rightarrow 2x =$ ▨ $\Rightarrow x =$ ▨.

C $4x+5=^-3 \Rightarrow 4x =$ ▨ $\Rightarrow x =$ ▨.

D $3(x+2)=9 \Rightarrow x+2 =$ ▨ $\Rightarrow x =$ ▨.

E $2(x+14)=20 \Rightarrow x+14 =$ ▨ $\Rightarrow x =$ ▨.

3. Solve the equations.

A $3+x=^-2$ **D** $x \div 6 = ^-13$ **G** $^-4x+3=15$ **J** $^-4(x+1)=16$

B $^-5x=45$ **E** $(x+5)-3=^-8$ **H** $^-2x-8=10$ **K** $5(x+2)=^-35$

C $^-17+x=^-56$ **F** $3x+1=^-5$ **I** $5x \div 3 = ^-15$ **L** $3(x+6)=0$

SET 23

1. Write two fractions equivalent to each given fraction.

A $\frac{1}{4}$	**D** $\frac{2}{3}$	**G** $\frac{2}{5}$	**J** $\frac{5}{6}$	**M** $\frac{10}{9}$	**P** $\frac{5}{9}$	**S** $\frac{8}{5}$	**V** $\frac{1}{13}$
B $\frac{3}{4}$	**E** $\frac{3}{8}$	**H** $\frac{3}{2}$	**K** $\frac{7}{3}$	**N** $\frac{3}{20}$	**Q** $\frac{10}{11}$	**T** $\frac{1}{15}$	**W** $\frac{4}{21}$
C $\frac{1}{6}$	**F** $\frac{7}{10}$	**I** $\frac{5}{8}$	**L** $\frac{5}{12}$	**O** $\frac{4}{7}$	**R** $\frac{4}{4}$	**U** $\frac{3}{16}$	**X** $\frac{4}{30}$

2. Give the lowest-terms fraction for each given fraction.

A $\frac{14}{16}$	**D** $\frac{6}{21}$	**G** $\frac{36}{40}$	**J** $\frac{21}{27}$	**M** $\frac{18}{81}$	**P** $\frac{12}{45}$	**S** $\frac{48}{100}$	**V** $\frac{45}{55}$
B $\frac{15}{18}$	**E** $\frac{30}{40}$	**H** $\frac{15}{24}$	**K** $\frac{12}{25}$	**N** $\frac{30}{42}$	**Q** $\frac{16}{40}$	**T** $\frac{12}{90}$	**W** $\frac{24}{64}$
C $\frac{9}{9}$	**F** $\frac{5}{30}$	**I** $\frac{14}{40}$	**L** $\frac{15}{35}$	**O** $\frac{21}{7}$	**R** $\frac{60}{72}$	**U** $\frac{44}{55}$	**X** $\frac{95}{150}$

SET 24

1. Give the missing number for each ▒. Then tell whether the fractions are equivalent or not.

A $\frac{3}{5}$ and $\frac{5}{8}$: $3 \cdot 8 = ▒$, $5 \cdot 5 = ▒$

B $\frac{5}{6}$ and $\frac{20}{24}$: $5 \cdot 24 = ▒$, $6 \cdot 20 = ▒$

C $\frac{4}{9}$ and $\frac{20}{36}$: $4 \cdot 36 = ▒$, $9 \cdot 20 = ▒$

D $\frac{7}{5}$ and $\frac{42}{20}$: $7 \cdot 20 = ▒$, $5 \cdot 42 = ▒$

2. Give the correct symbol ($=$ or $\neq$) for each ▒.

A $\frac{2}{5}$ ▒ $\frac{5}{2}$	**D** $\frac{10}{15}$ ▒ $\frac{4}{6}$	**G** $\frac{2}{15}$ ▒ $\frac{10}{90}$	**J** $\frac{5}{9}$ ▒ $\frac{25}{45}$	**M** $\frac{54}{64}$ ▒ $\frac{7}{8}$
B $\frac{3}{4}$ ▒ $\frac{27}{32}$	**E** $\frac{7}{8}$ ▒ $\frac{28}{32}$	**H** $\frac{7}{6}$ ▒ $\frac{18}{21}$	**K** $\frac{21}{35}$ ▒ $\frac{10}{15}$	**N** $\frac{27}{63}$ ▒ $\frac{9}{27}$
C $\frac{11}{12}$ ▒ $\frac{55}{60}$	**F** $\frac{8}{56}$ ▒ $\frac{2}{14}$	**I** $\frac{20}{24}$ ▒ $\frac{5}{6}$	**L** $\frac{7}{5}$ ▒ $\frac{42}{25}$	**O** $\frac{25}{90}$ ▒ $\frac{5}{16}$

3. Find the integer for x in each equation.

A $\frac{2}{3} = \frac{x}{18}$	**C** $\frac{3}{5} = \frac{x}{35}$	**E** $\frac{x}{3} = \frac{20}{15}$	**G** $\frac{5}{27} = \frac{15}{x}$	**I** $\frac{3}{8} = \frac{x}{104}$
B $\frac{6}{5} = \frac{x}{20}$	**D** $\frac{x}{7} = \frac{24}{56}$	**F** $\frac{7}{8} = \frac{x}{48}$	**H** $\frac{9}{10} = \frac{72}{x}$	**J** $\frac{5}{16} = \frac{x}{400}$

SET 25

1. Find the sums.

A $\frac{5}{9} + \frac{1}{3}$ **D** $\frac{5}{5} + \frac{2}{3}$ **G** $\frac{-4}{9} + \frac{-5}{6}$ **J** $\frac{2}{3} + \frac{-3}{4}$ **M** $\frac{3}{20} + \frac{-1}{4}$

B $\frac{3}{8} + \frac{1}{4}$ **E** $\frac{1}{2} + \frac{5}{8}$ **H** $\frac{5}{12} + \frac{3}{4}$ **K** $\frac{-1}{6} + \frac{-3}{8}$ **N** $\frac{4}{7} + \frac{5}{8}$

C $\frac{3}{4} + \frac{1}{8}$ **F** $\frac{4}{5} + \frac{8}{15}$ **I** $\frac{9}{10} + \frac{-3}{4}$ **L** $\frac{1}{9} + \frac{5}{6}$ **O** $\frac{-3}{4} + \frac{3}{10}$

2. Find the sums.

A $22\frac{3}{8} + 17\frac{1}{16}$ **B** $36\frac{1}{2} + 9\frac{1}{4}$ **C** $53\frac{5}{6} + 26\frac{2}{3}$ **D** $8\frac{5}{6} + 14\frac{1}{2}$ **E** $76\frac{1}{2} + 24\frac{3}{5}$ **F** $106\frac{3}{10} + 321\frac{3}{5}$

G $33\frac{1}{4} + 44\frac{1}{2} + 9\frac{1}{3}$ **H** $8\frac{1}{9} + 10\frac{1}{5} + 6\frac{2}{3}$ **I** $31\frac{1}{10} + 15\frac{1}{100} + 8\frac{1}{50}$ **J** $31\frac{4}{15} + 19\frac{3}{10} + 15\frac{2}{5}$ **K** $18\frac{1}{6} + 14\frac{1}{9} + 38\frac{5}{12}$ **L** $155\frac{1}{20} + 73\frac{4}{5} + 128\frac{7}{10}$

SET 26

1. Find the differences.

A $\frac{5}{8} - \frac{1}{4}$ **D** $\frac{11}{10} - \frac{4}{5}$ **G** $\frac{1}{20} - \frac{-3}{10}$ **J** $\frac{-3}{5} - \frac{-3}{5}$ **M** $\frac{11}{12} - \frac{3}{4}$

B $\frac{7}{12} - \frac{1}{6}$ **E** $\frac{3}{8} - \frac{3}{8}$ **H** $\frac{5}{7} - \frac{1}{3}$ **K** $\frac{2}{3} - \frac{5}{9}$ **N** $\frac{1}{5} - \frac{-1}{9}$

C $\frac{2}{3} - \frac{1}{2}$ **F** $\frac{9}{10} - \frac{1}{4}$ **I** $\frac{9}{15} - \frac{-1}{5}$ **L** $\frac{3}{4} - \frac{-1}{9}$ **O** $\frac{3}{7} - \frac{-1}{3}$

2. Find the differences.

A $6\frac{1}{2} - 4\frac{1}{3}$ **B** $8\frac{1}{3} - 5\frac{1}{6}$ **C** $16\frac{7}{10} - 9\frac{9}{20}$ **D** $25\frac{3}{4} - 18\frac{1}{6}$ **E** $63\frac{5}{8} - 7\frac{1}{5}$ **F** $50\frac{1}{20} - 8\frac{1}{10}$

G $79\frac{5}{6} - 35\frac{9}{10}$ **H** $125\frac{3}{8} - 95\frac{3}{4}$ **I** $100\frac{1}{100} - 1\frac{1}{10}$ **J** $534\frac{7}{15} - 150\frac{4}{5}$ **K** $750\frac{7}{15} - 339\frac{3}{10}$ **L** $635\frac{19}{25} - 49\frac{17}{20}$

SET 27

1. Find the products.

A $\frac{1}{4} \cdot \frac{1}{5}$ **D** $\frac{3}{4} \cdot \frac{4}{3}$ **G** $2 \cdot \frac{3}{8}$ **J** $\frac{2}{3} \cdot 45$ **M** $\frac{11}{12} \cdot \frac{3}{3}$

B $\frac{3}{4} \cdot \frac{1}{2}$ **E** $\frac{1}{2} \cdot \frac{4}{5}$ **H** $\frac{3}{5} \cdot \frac{10}{6}$ **K** $\frac{7}{3} \cdot \frac{9}{8}$ **N** $\frac{7}{15} \cdot \frac{3}{8}$

C $\frac{2}{5} \cdot \frac{1}{4}$ **F** $\frac{3}{5} \cdot \frac{5}{9}$ **I** $\frac{1}{16} \cdot 8$ **L** $0 \cdot \frac{5}{6}$ **O** $\frac{8}{25} \cdot \frac{5}{6}$

2. Find the products.

A $\frac{-1}{3} \cdot \frac{1}{2}$ **D** $-1 \cdot \frac{1}{9}$ **G** $\frac{-1}{3} \cdot \frac{4}{5}$ **J** $-28 \cdot \frac{3}{4}$ **M** $\frac{15}{2} \cdot \frac{-9}{25}$

B $\frac{1}{5} \cdot \frac{-1}{3}$ **E** $5 \cdot \frac{-1}{10}$ **H** $\frac{-3}{4} \cdot \frac{-1}{5}$ **K** $\frac{-1}{3} \cdot \frac{-1}{3}$ **N** $\frac{-4}{7} \cdot 0$

C $\frac{-1}{7} \cdot \frac{-1}{3}$ **F** $-4 \cdot \frac{-1}{12}$ **I** $\frac{2}{9} \cdot \frac{-3}{8}$ **L** $\frac{-3}{5} \cdot \frac{15}{18}$ **O** $\frac{-7}{9} \cdot \frac{3}{21}$

3. Write each mixed numeral numeral as an improper fraction. Then find the product.

A $1\frac{1}{4} \cdot \frac{1}{3}$ **D** $\frac{2}{3} \cdot -2\frac{3}{4}$ **G** $\frac{-7}{8} \cdot -1\frac{5}{9}$ **J** $-10 \cdot 3\frac{5}{6}$ **M** $4\frac{7}{8} \cdot -5\frac{2}{3}$

B $2\frac{1}{2} \cdot 7$ **E** $\frac{-1}{2} \cdot -3\frac{1}{3}$ **H** $1\frac{1}{2} \cdot 1\frac{2}{3}$ **K** $8 \cdot -9\frac{3}{4}$ **N** $-15 \cdot 8\frac{1}{10}$

C $\frac{-4}{5} \cdot 1\frac{1}{2}$ **F** $3 \cdot -4\frac{1}{6}$ **I** $4\frac{1}{2} \cdot 3\frac{1}{3}$ **L** $\frac{-9}{10} \cdot 6\frac{2}{3}$ **O** $\frac{-9}{25} \cdot -2\frac{1}{4}$

SET 28

1. Find the quotients.

A $\frac{3}{4} \div \frac{1}{2}$ **D** $\frac{7}{6} \div \frac{7}{8}$ **G** $\frac{-5}{6} \div \frac{-1}{8}$ **J** $\frac{3}{4} \div -9$ **M** $\frac{-5}{8} \div -10$

B $\frac{7}{8} \div \frac{1}{4}$ **E** $\frac{1}{3} \div \frac{4}{5}$ **H** $1 \div \frac{7}{6}$ **K** $\frac{-8}{3} \div \frac{-5}{4}$ **N** $\frac{-9}{10} \div \frac{3}{8}$

C $\frac{4}{9} \div \frac{2}{3}$ **F** $\frac{-2}{3} \div \frac{2}{5}$ **I** $\frac{-3}{8} \div \frac{-4}{9}$ **L** $\frac{5}{8} \div \frac{-1}{3}$ **O** $\frac{-3}{4} \div \frac{-4}{15}$

2. Write each mixed numeral as an improper fraction. Then find the quotient.

A $2\frac{1}{3} \div \frac{1}{6}$ **D** $\frac{1}{4} \div 3\frac{1}{2}$ **G** $-2\frac{3}{4} \div \frac{1}{2}$ **J** $-25 \div 8\frac{1}{3}$ **M** $4\frac{1}{2} \div -18$

B $3\frac{1}{5} \div \frac{3}{10}$ **E** $2\frac{1}{3} \div 5\frac{1}{4}$ **H** $-14 \div 4\frac{2}{3}$ **K** $-4\frac{2}{3} \div -1\frac{3}{4}$ **N** $-3\frac{9}{10} \div 4\frac{1}{3}$

C $4\frac{1}{2} \div 15$ **F** $3\frac{1}{5} \div 1\frac{1}{7}$ **I** $-5 \div \frac{-5}{6}$ **L** $-3\frac{1}{3} \div 16\frac{2}{3}$ **O** $33 \div -5\frac{1}{2}$

SET 29

Write and solve an equation for each problem.

1. Carl is 12 years old. Sam is $\frac{3}{2}$ Carl's age. How old is Sam?

2. Marie bought a pair of shoes at a "$\frac{1}{2}$ off" sale. If the shoes were regularly priced at $15, what was the sale price of the shoes?

3. A furniture store is having a "$\frac{1}{3}$ off" sale. What is the sale price of a chair regularly priced at $105?

4. Jody played volleyball for $\frac{3}{4}$ hour yesterday and $\frac{2}{3}$ hour today. How much time did she play in two days?

5. If Mrs. Jackson drives 90 kilometres in $1\frac{1}{5}$ hours, what is her average speed in kilometres per hour?

★ 6. A football team won $\frac{3}{4}$ of its games. It won 4 more games than it lost. How many games did the team play?

SET 30

1. Find the sums.

A 6.5 + 1.43	E 0.366 + 6.2 + 0.18	I 4.7 + 32.643 + 825.6
B 12.32 + 0.471	F 243.2 + 3.61 + 4.438	J 55.89 + 5.589 + .5589
C 243.2 + 3.65	G 36 + 2.64 + 0.735	K 0.85 + 0.937 + 0.9539
D 0.538 + 8.469	H 0.7305 + 69.1 + 0.88	L 326.043 + 82.62 + 1.9

2. Find the differences.

A 6.28 − 3.7	E 35.8 − 6.45	I 30.2 − 4.86	M 84.27 − 62.99
B 15.75 − 9.39	F 4.85 − 0.769	J 7.06 − 2.49	N 500 − 65.37
C 38.6 − 27.7	G 10 − 1.58	K 23.56 − 17.742	O 830.7 − 179.03
D 0.937 − 0.66	H 18.6 − 5.935	L 326.1 − 44.78	P 5000 − 493.971

3. Find the sums.

A 3.75 + $^{-}$2.49	E $^{-}$0.97 + $^{-}$0.58	I 93 + $^{-}$76.7 + $^{-}$1.35
B 73.1 + $^{-}$69.5	F 0.935 + $^{-}$0.769	J $^{-}$9.5 + $^{-}$0.95 + 95
C $^{-}$5.34 + $^{-}$8.09	G $^{-}$28.9 + 33.73	K 83.7 + $^{-}$0.75 + $^{-}$78
D $^{-}$69.5 + 37.42	H $^{-}$0.0639 + 0.0115	L 60.8 + $^{-}$25 + 7.904

SET 31

1. Find the products.

A 4.72×6

B 32×0.4

C 43×0.07

D 22.1×7.1

E 52.6×4.2

F 3.86×0.4

G 76.4×0.34

H 8.26×0.65

I 436.2×0.03

J 2.51×0.361

K 64.3×2.66

L 1.43×0.953

M 23.7×0.564

N 8.63×22.6

O 92.7×0.365

P 9.731×42.56

Q 0.0631×5.2

R 504.6×7.31

S 32.2×0.3742

T 0.653×0.641

2. Find the products.

A $0.3 \times {}^{-}0.9$

B ${}^{-}2.1 \times 6$

C ${}^{-}3.5 \times 0.8$

D ${}^{-}1.7 \times {}^{-}2.5$

E ${}^{-}8.3 \times 0.43$

F ${}^{-}43.9 \times {}^{-}0.15$

G $5.96 \times {}^{-}1.7$

H ${}^{-}12.4 \times 68.4$

I ${}^{-}8.56 \times {}^{-}3.501$

J $143.6 \times {}^{-}340$

K $205.3 \times {}^{-}8.61$

L ${}^{-}95.1 \times {}^{-}37.05$

M $9.421 \times {}^{-}5.631$

N $0.075 \times {}^{-}0.99$

O ${}^{-}3.109 \times {}^{-}65.13$

SET 32

1. Represent each number in scientific notation.

A 400

B 3000

C 6300

D 20 000

E 54 000

F 350 000

G 9 000 000

H 45 600 000 000

I 8 030 000 000

J 7 905 000 000 000

K 0.5

L 8.49

M 0.14

N 0.07

O 0.003

P 0.0702

Q 0.00009

R 0.000453

S 0.0000008

T 0.0008002

2. Give the ordinary decimal for each number.

A 4.2×10^{4}

B 5.61×10^{2}

C 3.7×10^{6}

D 9.22×10^{1}

E 4.05×10^{5}

F 3.2×10^{9}

G 9.55×10^{0}

H 1.07×10^{8}

I 9.005×10^{4}

J 3.54×10^{7}

K 6×10^{-1}

L 8×10^{-3}

M 1.5×10^{-1}

N 3.4×10^{-2}

O 6.91×10^{-4}

P 7×10^{-8}

Q 4.4×10^{-10}

R 3.9×10^{-6}

S 8.01×10^{-7}

T 1.392×10^{-11}

SET 33

1. Find the quotients.

A $6\overline{)14.4}$ E $59\overline{)207.09}$ I $0.3\overline{)19.02}$ M $0.73\overline{)17.155}$

B $8\overline{)507.2}$ F $81\overline{)25.920}$ J $1.8\overline{)6.498}$ N $0.048\overline{)0.01728}$

C $15\overline{)139.5}$ G $1.3\overline{)0.3296}$ K $0.35\overline{)2.275}$ O $8.17\overline{)8.0883}$

D $26\overline{)98.80}$ H $542\overline{)504.06}$ L $4.05\overline{)1.4985}$ P $4.96\overline{)0.56544}$

2. Find the quotients. Annex zeros when necessary.

A $5\overline{)4.1}$ F $0.8\overline{)92.4}$ K $160\overline{)24}$ P $0.604\overline{)2.4764}$

B $12\overline{)61.8}$ G $0.55\overline{)22}$ L $8.8\overline{)1188}$ Q $5.56\overline{)4.0588}$

C $18\overline{)6.21}$ H $7.2\overline{)28.8}$ M $0.516\overline{)23.22}$ R $7.6\overline{)4.028}$

D $32\overline{)33.6}$ I $0.25\overline{)8.07}$ N $4.68\overline{)2.574}$ S $0.021\overline{)0.0714}$

E $95\overline{)5.89}$ J $1.35\overline{)621}$ O $0.38\overline{)2.47}$ T $9.012\overline{)67.69}$

SET 34

1. Round each number to one significant digit. Use scientific notation and estimate the products.

A 4264×7536 F 21.6×0.047 K 0.647×3846

B 567.3×384.5 G 8376×5.66 L $54\,230 \times 57.62$

C 62.7×218.8 H 325.7×478 M 7.06×0.407

D 0.736×0.38 I 0.731×537.9 N 6573×0.0045

E 74.2×6.78 J 865.7×0.338 O 0.00873×0.00531

2. Round each number to one significant digit. Use scientific notation and estimate the quotients.

A $19.76 \div 3.95$ E $88.6 \div 331.2$ I $0.00173 \div 0.0041$

B $382.5 \div 2.04$ F $75.9 \div 0.384$ J $0.083 \div 0.00437$

C $605.3 \div 28.4$ G $0.207 \div 0.0039$ K $0.44285 \div 0.21756$

D $86{,}730 \div 5.99$ H $6327 \div 4.578$ L $0.000872 \div 0.00285$

SET 35

1. Find the decimal approximation for $\frac{8}{15}$ to the nearest

A tenth. **B** hundredth. **C** thousandth.

2. Find the decimal approximation of each number to the nearest hundredth.

A $\frac{7}{8}$ **B** $\frac{3}{8}$ **C** $\frac{1}{9}$ **D** $\frac{4}{11}$ **E** $\frac{7}{18}$ **F** $\frac{20}{23}$

3. Find the decimal approximation of each number to the nearest thousandth.

A $\frac{11}{16}$ **B** $\frac{8}{9}$ **C** $\frac{15}{16}$ **D** $\frac{32}{45}$ **E** $\frac{41}{71}$ **F** $\frac{38}{39}$

SET 36

1. Solve the proportions.

A $\frac{3}{5} = \frac{x}{40}$ **C** $\frac{4}{9} = \frac{x}{45}$ **E** $3:4 = x:36$ **G** $x:9 = 16:24$

B $\frac{5}{6} = \frac{x}{54}$ **D** $\frac{x}{24} = \frac{15}{40}$ **F** $5:7 = 30:x$ **H** $9:x = 36:60$

2. Write and solve a proportion for each problem.

A Apples sell at 6 for 43 cents. How much for 18 apples?

B The ratio of boys to girls is 5:7. If there are 42 girls, how many boys are there?

C If a car averages 8 kilometres per litre of gasoline, how many kilometres can it go on 23 litres?

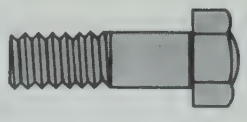
D A basketball player made 7 free throws out of every 10 attempts. How many free throws did he make in 130 attempts?

E On a map, 3 centimetres represents 75 kilometres. How many kilometres of actual distance is represented by 6.4 centimetres on the map?

SET 37

Solve the following problems involving scale drawings.

1. Measure the length of the bolt in centimetres. What is the actual length of the bolt, if the scale is



A 1:4? B 1:8? C 2:1? D 1:7.5?

2. If the scale on a map is 1 cm = 90 km, what is the actual distance between two cities whose distance apart on the map is 3.1 cm?

3. The actual distance between two cities is 720 km. What is the distance on the map if the scale for the map is 1 cm = 40 km?

4. A lot for Mr. Hoppe's new house measures 80 metres wide by 150 metres long. What would be the dimensions of a scale drawing of the lot using the scale 1 cm = 20 metres?

SET 38

1. Write a decimal for each percent.

A 18%	C 10%	E 5%	G 100%	I 2%	K 7.5%	M 1.5%
B 33%	D 9%	F 70%	H 1%	J 75%	L 4%	N 33.3%

2. Write a percent for each decimal.

A 0.38	C 0.2	E 1.5	G 1.30	I 0.093	K 16.3	M 0.017
B 0.65	D 0.03	F 0.15	H 2.05	J 5.5	L 0.178	N 0.309

3. Give the lowest-terms fraction for each percent.

A 37%	C 40%	E 75%	G 2%	I 15%	K 28%	M 520%
B 50%	D 30%	F 5%	H 1%	J 16%	L 8%	N 64%

4. Find a decimal for each fraction to the nearest hundredth, when necessary. Then express the decimal as a percent.

A $\frac{59}{100}$	C $\frac{1}{100}$	E $\frac{17}{50}$	G $\frac{9}{25}$	I $\frac{7}{18}$	K $\frac{5}{7}$	M $\frac{9}{15}$
B $\frac{3}{100}$	D $\frac{3}{4}$	F $\frac{11}{40}$	H $\frac{9}{16}$	J $\frac{5}{12}$	L $\frac{7}{23}$	N $\frac{27}{29}$

SET 39

Solve these percent problems.

1. Price of sweater: $15.75.
   Sale: 20% off.
   - A Find the amount of discount.
   - B What is the sale price?

2. Class: 35 students.
   Today: 28 present.
   What percent is present?

3. Earnings: $650 per month.
   Deductions: 26%.
   - A How much deducted?
   - B How much take-home pay?

4. Price of 10-speed bicycle: $155.
   Sales tax: 6%.
   - A What is the amount of sales tax?
   - B What is the total cost of the bicycle?

5. Purchase price of house: $38 000.
   Selling price: 120% of original purchase price.
   What is the selling price?

6. Which costs less? how much less?
   - [A] $25 coat at 20% off.
   - [B] $30 coat at 30% off.

SET 40

1. Write a decimal for each percent.

A 270%	C 1000%	E 1250%	G 0.3%	I $\frac{1}{2}$%	K 0.06%
B 850%	D 795%	F 3500%	H $\frac{7}{10}$%	J 0.35%	L $\frac{1}{8}$%

2. Find the amounts.

A 300% of 650	D 1200% of 525	G 0.6% of 750	J 0.04% of 3000
B 425% of 400	E 750% of 300	H $\frac{1}{2}$% of 988	K $\frac{3}{20}$% of 1500
C 170% of 1600	F 903% of 8000	I $\frac{1}{5}$% of 1280	L $\frac{1}{8}$% of 5000

3. Solve each percent problem.
   - A Population in 1940: 76 500.
     Population in 1970: 225% more.
     What is the population in 1970?
   - B Length of material: 3 metres.
     Percent of shrinkage: 0.25%.
     Find the greatest amount of shrinkage.

SET 41

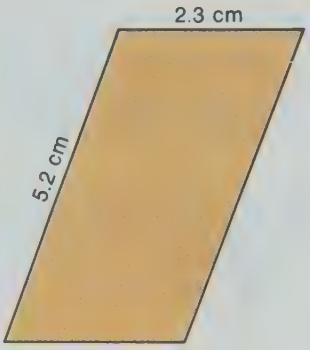
1. Express each measure in metres.

A 300 cm	**D** 15 km	**G** 4.6 km	**J** 0.1 km	**M** 0.1 cm
B 1300 mm	**E** 40 dm	**H** 40 dam	**K** 0.1 hm	**N** 0.1 mm
C 5 km	**F** 74 cm	**I** 530 mm	**L** 0.1 dam	**O** 0.003 km

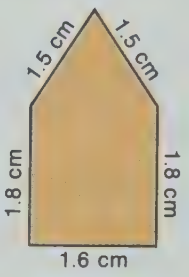
2. Find the perimeter of each region.

A

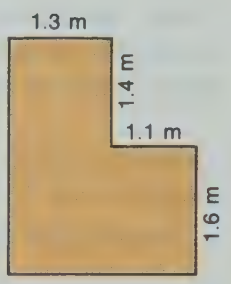
B



C



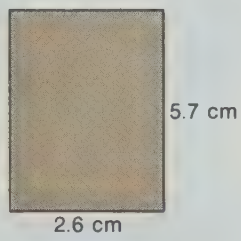
D



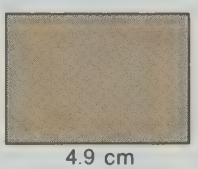
SET 42

1. Find the area of each rectangular region. Use the formula $A = lw$.

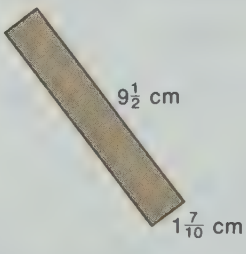
A



B



C

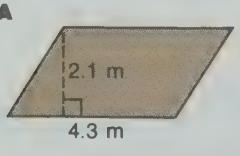


D $l = 65$ cm, $w = 32.1$ cm

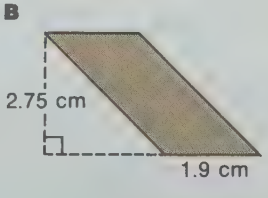
E $l = 8.75$ m, $w = 3.09$ m

2. Find the area of each parallelogram. Use the formula $A = bh$.

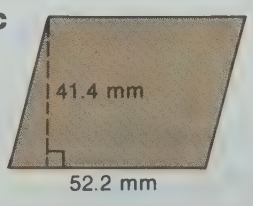
A



B



C



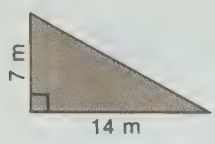
D $b = 48.3$ mm, $h = 56.1$ mm

E $b = 3.88$ dm, $h = 1.36$ dm

SET 43

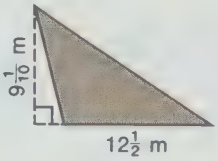
1. Find the area of each triangular region.

A

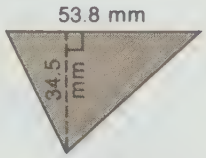


B

C



D



2. Find the area of each triangular region. Use the formula $A = \frac{1}{2}bh$.

A $b = 150$ cm
$h = 93$ cm

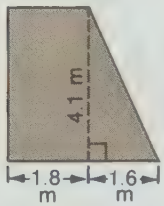
B $b = 8.5$ m
$h = 4.9$ m

C $b = 8\frac{1}{2}$ m
$h = 4\frac{3}{10}$ m

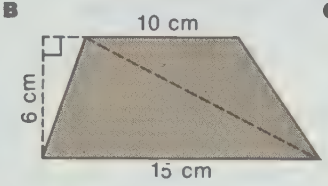
D $b = 9.54$ dm
$h = 9.07$ dm

★ **3.** Find the area of each region.

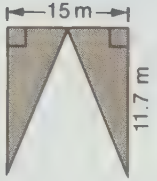
A



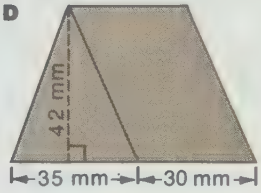
B



C



D



SET 44

1. Find the **sums** of the angle measures.

A $38°\,15'\,14''$
$27°\,33'\,42''$

B $22°\,34'\,47''$
$35°\,13'\,35''$

C $45°\,22'\,16''$
$53°\,49'\,33''$

D $62°\,15'\,28''$
$32°\,44'\,40''$

E $87°\,26'\,54''$
$43°\,15'\,\ 7''$

F $55°\,36'\,45''$
$57°\,48'\,48''$

G $73°\,48'\,16''$
$16°\,11'\,44''$

H $145°\,36'\,52''$
$38°\,14'\,18''$

2. Find the **differences** of the angle measures.

A $36°\,38'\,25''$
$14°\,12'\,19''$

B $48°\,37'\,15''$
$21°\,13'\,28''$

C $86°\,25'\,39''$
$35°\,41'\,28''$

D $123°\ 9'\,37''$
$95°\,48'\,50''$

E $136°\,38'\,14''$
$74°\,52'\,36''$

F $112°\,56'\,22''$
$88°\,55'\,28''$

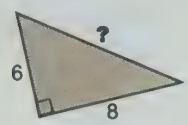
G $75°\,48'$
$49°\,59'\,11''$

H $90°$
$43°\,52'\,47''$

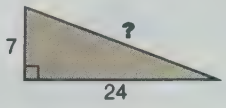
SET 45

1. Find the length of the hypotenuse of each right triangle.

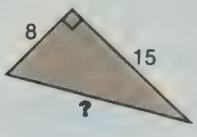
A



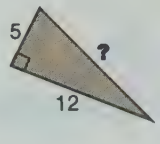
B



C

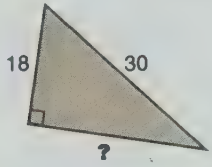


D

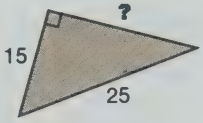


2. Find the length of the third side of each right triangle.

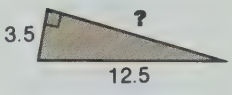
A



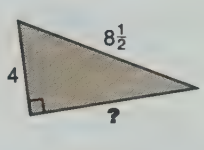
B



C



D



3. Find the length of the third side of each right triangle.

A $a = 15, b = 20$

B $a = 10, b = 24$

C $a = 18, b = 24$

D $a = 21, b = 28$

E $a = 8, \quad c = 15$

F $b = 48, c = 52$

G $a = 16, \quad c = 34$

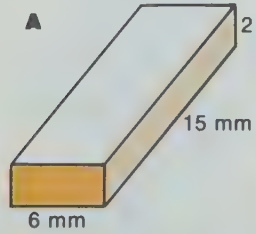
H $b = 4.4, c = 5.5$

I $a = 6, \quad c = 6.8$

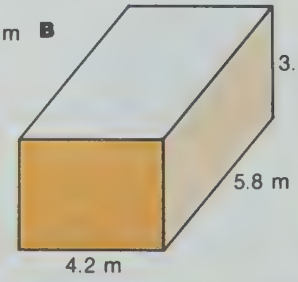
SET 46

1. Find the volume of each rectangular solid. Use the formula $V = lwh$.

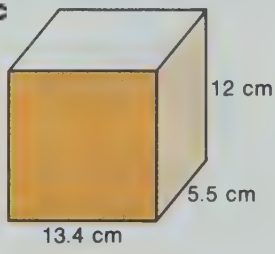
A



B



C

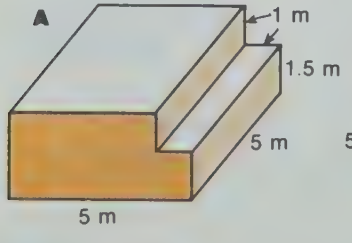


D $l = 220$ mm, $w = 175$ mm, $h = 30$ mm

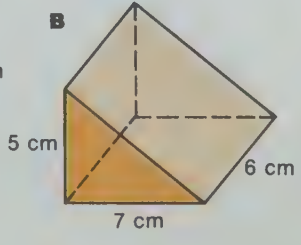
E $l = 8.6$ m, $w = 4.9$ m, $h = 7.05$ m

2. Find the volume of each space figure.

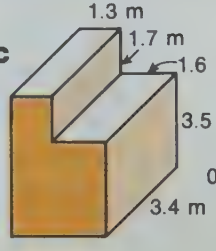
A



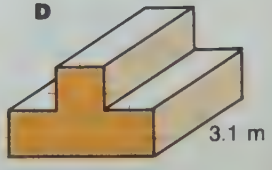
B



C



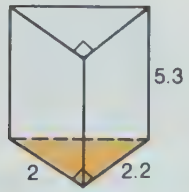
D



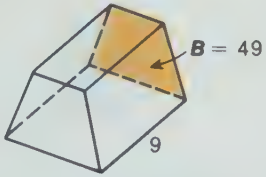
SET 47

1. Find the volume of each prism.

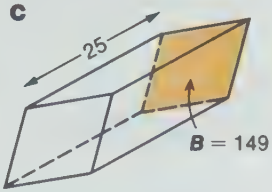
A



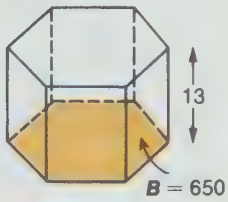
B



C

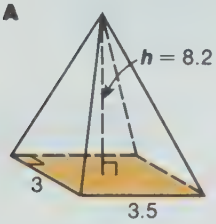


D

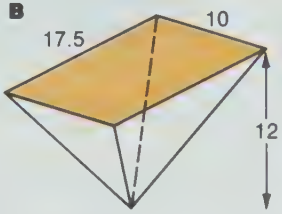


2. Find the volume of each pyramid.

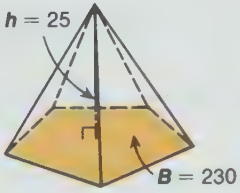
A



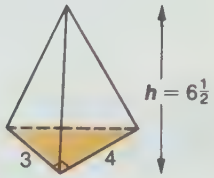
B



C

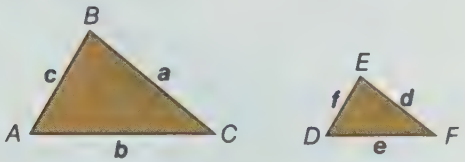


D



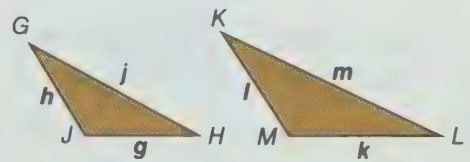
SET 48

1. $\triangle ABC \sim \triangle DEF$. The letters ***a***, ***b***, ***c***, ***d***, ***e***, and ***f*** represent the lengths of the sides.



A Find ***b***, if $a = 2$, $e = 9$, $d = 6$.
B Find ***f***, if $c = 16$, $e = 3$, $b = 6$.
C Find ***a***, if $d = 8$, $b = 24$, $e = 6$.
D Find ***d***, if $a = 21$, $c = 18$, $f = 6$.
E Find ***c***, if $a = 42$, $d = 14$, $f = 13$.
F Find ***e***, if $a = 6.4$, $b = 4.2$, $d = 3.2$.
G Find ***b***, if $e = 1\frac{1}{2}$, $f = 1\frac{3}{4}$, $c = 3\frac{1}{2}$.
H Find ***d***, if $a = 0.9$, $c = 0.8$, $f = 0.4$.

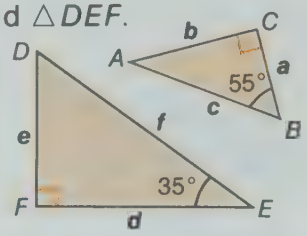
2. $\triangle GHJ \sim \triangle KLM$. The letters ***g***, ***h***, ***j***, ***k***, ***l***, and ***m*** represent the lengths of the sides.



A Find ***g***, if $j = 8$, $k = 7$, $m = 14$.
B Find ***m***, if $j = 8$, $l = 18$, $h = 12$.
C Find ***j***, if $m = 24$, $g = 12$, $k = 32$.
D Find ***h***, if $l = 12$, $j = 15$, $m = 36$.
E Find ***k***, if $g = 6$, $m = 72$, $j = 9$.
F Find ***l***, if $h = 10$, $k = 48$, $g = 16$.
G Find ***g***, if $h = 5.5$, $l = 11$, $k = 7.8$.
H Find ***j***, if $h = 3\frac{1}{3}$, $l = 10$, $m = 12$.

SET 49

1. Refer to $\triangle ABC$ and $\triangle DEF$.

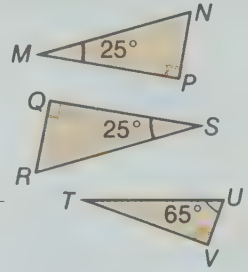


   A $\angle A \cong \angle ?$

   B $\frac{a}{b} =$ [illegible]

   C $\triangle ABC \sim \triangle ?$

2. Refer to $\triangle MNP$, $\triangle QRS$, and $\triangle TUV$.



   A $\frac{\overline{MN}}{\overline{NP}} = \frac{RS}{[illegible]} = \frac{[illegible]}{[illegible]}$

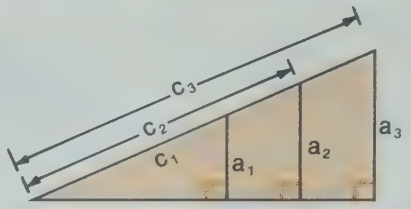
   B $\angle Q \cong \angle ? \cong \angle ?$

   C $\triangle MNP \sim \triangle ? \sim \triangle ?$

3. Suppose $\frac{a_1}{c_1} = \frac{2}{5}$. Give the ratios.

   A $\frac{a_2}{c_2}$

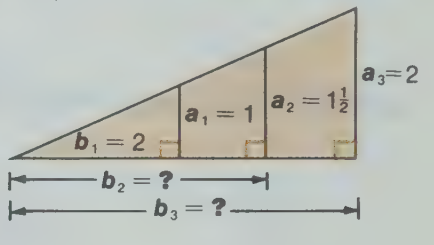
   B $\frac{a_3}{c_3}$



4. Suppose $\frac{a_1}{b_1} = \frac{1}{2}$. Find these lengths.

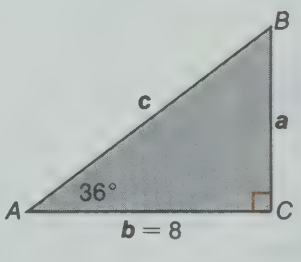
   A b_2

   B b_3



SET 50

1. In right $\triangle ABC$, $b = 8$, $\tan 36° \approx 0.73$, and $\cos 36° \approx 0.81$. Find a and c.



   A $\tan 36° = \frac{a}{8}$

   $\frac{a}{8} \approx 0.73$

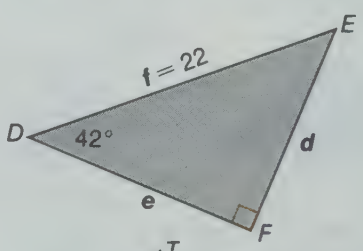
   $a \approx$ [illegible]

   B $\cos 36° = \frac{8}{c}$

   $\frac{8}{c} \approx 0.81$

   $c \approx$ [illegible]

2. In right $\triangle DEF$, $f = 22$, $\sin 42° \approx 0.67$, and $\cos 42° \approx 0.74$. Find d and e.



   A $\sin 42° = \frac{d}{22}$

   $\frac{d}{22} \approx 0.67$

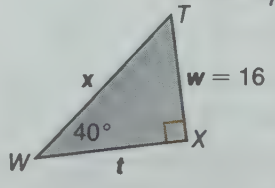
   $d \approx$ [illegible]

   B $\cos 42° = \frac{e}{22}$

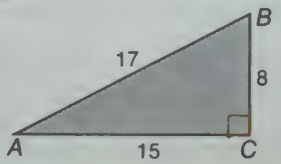
   $\frac{e}{22} \approx 0.74$

   $e \approx$ [illegible]

3. In right $\triangle TWX$, $w = 16$, $\sin 40° \approx 0.64$, $\tan 40° \approx 0.84$. Find t and x.



4. Use the figure to compute the trigonometric ratios for $\angle A$ to the nearest hundredth.



   A $\tan A$ B $\sin A$ C $\cos A$

SET 51

1. Find the circumference of each circle with the given diameter. Use the formula $C = \pi d$ and the fact $\pi \approx 3.14$.

A

B

C $d = 14$ cm

D $d = 23$ cm

E $d = 310$ mm

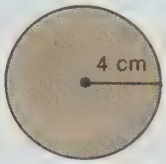
F $d = 1.6$ m

G $d = 17.3$ cm

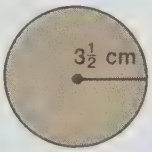
H $d = 0.9$ km

2. Find the circumference of each circle with the given radius. Use the formula $C = 2\pi r$ and the fact $\pi \approx 3.14$.

A



B



C $r = 22$ mm

D $r = 440$ m

E $r = 52\frac{1}{2}$ cm

F $r = 8.3$ m

G $r = 0.35$ mm

H $r = 6.15$ m

SET 52

1. Find the area of each circular region. Use the formula $A = \pi r^2$ and the fact $\pi \approx 3.14$.

A

B

C $r = 20$ m

D $r = 12$ cm

E $r = 1.5$ m

F $r = 3\frac{1}{10}$ m

G $r = 3.2$ dm

H $r = 0.4$ mm

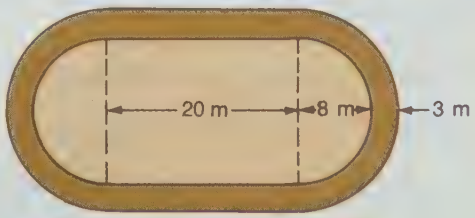
2. A circular hole is cut in the rectangular board at the right.

A If the diameter of the circle is the same as the width of the board, what is the area of the circle?

B What is the area of the rectangular region after the circular region is removed?

3. **A** Find the area of the inner region of the track.

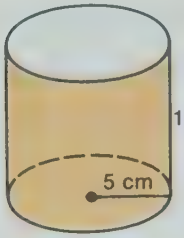
B Find the area of the track.



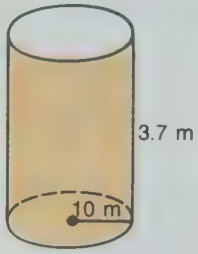
SET 53

1. Find the volume of each right circular cylinder. Use the formula $V = \pi r^2 h$ and the fact $\pi \approx 3.14$.

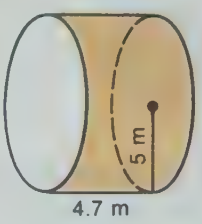
A



B



C



D $r = 10$ m, $h = 20$ m

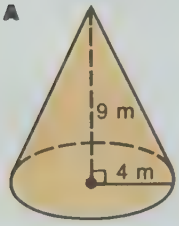
E $r = 12$ cm, $h = 15$ cm

F $r = 3.1$ m, $h = 5.2$ m

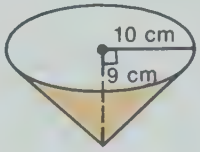
G $r = 2.2$ m, $h = 1.53$ m

2. Find the volume of each right circular cone. Use the formula $V = \frac{1}{3}\pi r^2 h$ and the fact $\pi \approx 3.14$.

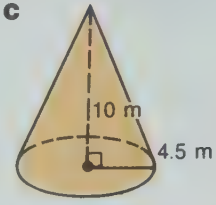
A



B



C



D $r = 5$ cm, $h = 7.5$ cm

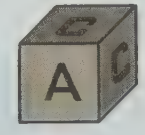
E $r = 40$ m, $h = 60$ m

F $r = 1.02$ m, $h = 4.2$ m

G $r = 0.8$ m, $h = 1.8$ m

SET 54

1. Suppose the cube has the letters **A**, **B**, **B**, **C**, **C**, and **C** on its faces. Give the probability that each event will appear.



A $P(\mathbf{A})$ **B** $P(\mathbf{B})$ **C** $P(\mathbf{C})$

2. Refer to the spinner. Give the probability that each event will occur.

A P(even) **C** P(multiple of 3)

B P(odd) **D** P(multiple of 4)

3. An experiment of tossing a die has a sample space $\{1, 2, 3, 4, 5, 6\}$. Give the probability that each event will occur.

A $P(x = 1)$ **D** $P(x > 5)$ **G** $P(1 < x < 4)$

B $P(x > 1)$ **E** $P(x < 5)$ **H** $P(x \text{ is even})$

C $P(x \leq 3)$ **F** $P(x \leq 4)$ **I** $P(x > 6)$

SET 55

1. Compute each of the following:

   A $4!$ **B** $6!$ **C** $7!$ **D** $2!\,4!$ **E** $3!\,5!$ **F** $6! \div 3!$

2. How many different arrangements of the letters in the word CART are possible? List them to verify your answer.

3. How many different arrangements of the letters in the word CANDY are possible?

4. How many different ways can 8 students be arranged in a row of 8 desks?

5. How many different ways can 9 pictures be arranged in a straight line on a wall?

6. Solve the equations. Give your answer as a factorial, when possible.

   A $7 \cdot 6! = n$
   B $9! \cdot n = 10!$
   C $5! \cdot n = 7!$
   D $6! \div 30 = n$
   E $5! \div 5 = n$
   F $8! \div n = 8$
   G $7! \cdot 8 \cdot 9 = n$
   H $6! \div 3! = n$
   I $6! \div 5! = n$

SET 56

1. From the set $\{3, 4, 5, 6, 7, 8\}$, a 2-digit numeral is to be selected.

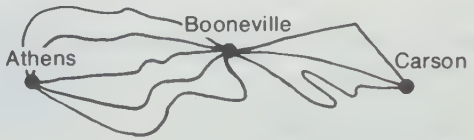
   A How many different digits can be used for the tens' digit?

   B How many different digits can be used for the ones' digit?

   C How many 2-digit numerals are possible?

2. Susan has 6 sweaters and 4 skirts. How many different outfits does she have if any sweater can be worn with any skirt?

3. There are 5 roads from Athens to Booneville and 3 roads from Booneville to Carson. How many different routes are there from Athens to Carson?



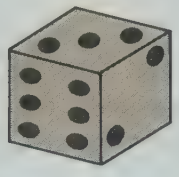
4. From a menu of 8 items, you can choose any 3 of them. How many choices do you have?

5. Compute the number of combinations indicated.

   A ${}_4C_2$ **B** ${}_6C_3$ **C** ${}_7C_5$ **D** ${}_8C_5$ **E** ${}_7C_3$ **F** ${}_9C_6$

SET 57

1. Suppose an experiment consists of tossing a die. The sample space is $\{1, 2, 3, 4, 5, 6\}$.



   A Let E be the event of getting a 1 or a 6. What is $P(E)$?

   B Let F be the event of getting a 2, 3, or 5. What is $P(F)$?

   C Find $E \cap F$. What is $P(E \cap F)$?

   D Find $E \cup F$. What is $P(E \cup F)$?

2. A spinner is divided into 8 parts of equal size and numbered 1 through 8.

   A Let E be the event of getting a 1, 2, or 3. What is $P(E)$?

   B Let F be the event of getting an odd number. What is $P(F)$?

   C What is $P(E \cap F)$? D What is $P(E \cup F)$?

3. An experiment consists of drawing two cards from those labeled E, F, G, H, I. Let E be the event of getting a vowel and let F be the event of getting a consonant.

   A What is $P(E)$? B What is $P(F)$? C What is $P(E \cup F)$?

SET 58

1. What is the average (arithmetic mean) of each set of numbers?

   A 25, 27, 29

   B 66, 65, 70, 67

   C 253, 250, 213, 248

   D 123, 101, 132, 130, 114

   E 208, 206, 199, 244, 235, 216

   F 4500, 4498, 4603, 4555

2. What is the average (arithmetic mean) to the nearest whole number?

   A 38, 50, 49, 61, 56

   B 305, 299, 316, 278, 366

   C 135, 164, 173, 122, 155, 90

   D 675, 701, 699, 693, 711, 700

   E 1266, 1313, 1288, 1295

   F 102, 99, 75, 87, 96, 92, 67

   G 58.2, 56.7, 52.5, 54.3

   H 48.73, 53.71, 59.77, 49.99

SET 59

1. Find the median of each set of numbers.

A 46, 48, 49, 51, 55

B 89, 84, 88, 80

C 120, 123, 128, 127

D 38, 35, 41, 42, 39, 44

E 1200, 1300, 1400, 1500

F 8, 8, 9, 10, 10, 11, 12, 12, 12

G 21, 19, 15, 17, 18, 19, 23, 20

H 303, 305, 333, 306, 325, 321

I 1560, 1545, 1598, 1534, 1555

J 72.6, 68.1, 93.1, 79.4

Solve the problems.

2. Test scores:
63, 52, 72, 75, 81, 76, 80, 77, 65
What is the median score?

3. Bowling scores:
156, 143, 152, 118, 121, 125, 104, 123, 118, 126, 142
What is the median score?

4. Masses of 6 teen-agers:
58.2, 56.7, 52.5, 54.3, 57.2, 55.4

A What is the median?

B What is the mean?

C Which shows the more "average" of the masses, the mean or the median?

SET 60

1. In 120 tosses of a die, a class found the frequency for each outcome as given in the table. What is the mode for the experiment?

Outcome	⚀	⚁	⚂	⚃	⚄	⚅
Frequency	14	16	11	35	22	21

2. A student survey of the number of books read in a month is shown in the table. How many books did the greatest number of students read?

Books read	1–2	2–3	3–4	4–5	5–6
Frequency	12	18	10	6	4

3. The table shows the temperatures of some cities on a certain day in March. What is the mode?

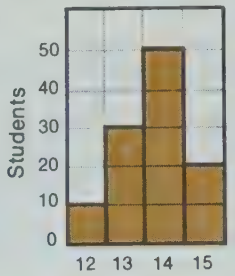
Temperature °C	1–5	6–10	11–15	16–20	21–25	26–30
Frequency	15	26	19	32	8	10

SET 61

1. **A** How many students are represented by this histogram?

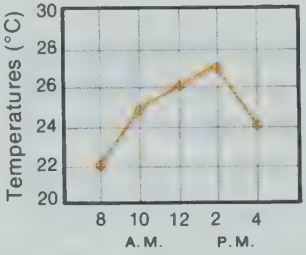
   B What percent are 13 years old?

   C What percent are over 13 years old?



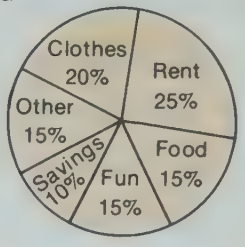
2. **A** What is the highest temperature?

   B What is the lowest temperature?



   C How many degrees did the temperature rise between 8 A.M. and noon?

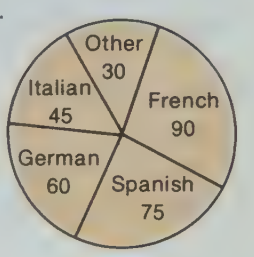
3. The circle graph shows how Marcia accounted for her monthly earnings of $600.



   A How much was spent for rent?

   B How much was spent for food?

4. The circle graph shows how 300 students are taking foreign language classes.



   A What percent take French?

   B What percent take Italian?

SET 62

1. Find the positive square root of each number.

A 169	**E** 1156	**I** 3136	**M** 98.01	**Q** 46.24
B 529	**F** 5041	**J** 7396	**N** 77.44	**R** 14.44
C 441	**G** 2916	**K** 6889	**O** 70.56	**S** 0.3259
D 256	**H** 2209	**L** 8836	**P** 37.21	**T** 0.1849

2. Find an approximation of each number to the nearest tenth.

A $\sqrt{14}$	**D** $\sqrt{150}$	**G** $\sqrt{500}$	**J** $\sqrt{12.5}$	**M** $\sqrt{51.9}$
B $\sqrt{23}$	**E** $\sqrt{425}$	**H** $\sqrt{9100}$	**K** $\sqrt{26.5}$	**N** $\sqrt{0.55}$
C $\sqrt{69}$	**F** $\sqrt{485}$	**I** $\sqrt{3000}$	**L** $\sqrt{41.3}$	**O** $\sqrt{0.11}$

Mathematical Symbols

$=$	Is equal to
$\neq$	Is not equal to
$>$	Is greater than
$<$	Is less than
$\leq$	Is less than or equal to
$\approx$	Is approximately
$\cong$	Is congruent to
$\sim$	Is similar to
$\{a, b, c\}$	Set with elements a, b, c
ϕ	Empty set
$\cup$	Union
$\cap$	Intersection

$\overleftrightarrow{AB}$	Line through points A and B
$\overrightarrow{AB}$	Ray AB with endpoint A
$\overline{AB}$	Segment with endpoints A and B
$\angle ABC$	Angle ABC with vertex B
$m\angle ABC$	Measure of angle ABC
$\triangle ABC$	Triangle ABC
$\overset{\frown}{RS}$	Arc with endpoints R and S
$35^\circ\ 14'\ 20''$	Thirty-five *degrees*, fourteen *minutes*, twenty *seconds*
%	Percent
π	Pi

Metric System Prefixes

kilo-	one thousand
hecto-	one hundred
deca-	ten
deci-	one tenth
centi-	one hundredth
milli-	one thousandth

Formulae

$P = a + b + c$	Perimeter of triangle
$P = 2 \cdot l + 2 \cdot w$	Perimeter of rectangle
$A = l \cdot w$	Area of rectangle
$A = b \cdot h$	Area of parallelogram
$A = \frac{1}{2} \cdot b \cdot h$	Area of triangle
$a^2 + b^2 = c^2$	Pythagorean Theorem
$V = l \cdot w \cdot h$	Volume of rectangular solid
$C = \pi \cdot d$	Circumference of circle
$A = \pi \cdot r^2$	Area of circle
$V = \frac{4}{3} \cdot \pi \cdot r^3$	Volume of sphere
$V = \pi \cdot r^2 \cdot h$	Volume of cylinder
$V = \frac{1}{3} \cdot \pi \cdot r^2 h$	Volume of cone
$V = B \cdot h$	Volume of prism (B = base area)
$V = \frac{1}{3} \cdot B \cdot h$	Volume of pyramid

Number n	Square n^2	Square root $\sqrt{n}$	Square root of 10 times n $\sqrt{10 \times n}$	Number n	Square n^2	Square root $\sqrt{n}$	Square root of 10 times n $\sqrt{10 \times n}$
1	1	1.000	3.162	51	2601	7.141	22.583
2	4	1.414	4.472	52	2704	7.211	22.804
3	9	1.732	5.477	53	2809	7.280	23.022
4	16	2.000	6.325	54	2916	7.348	23.238
5	25	2.236	7.071	55	3025	7.416	23.452
6	36	2.449	7.746	56	3136	7.483	23.664
7	49	2.646	8.367	57	3249	7.550	23.875
8	64	2.828	8.944	58	3364	7.616	24.083
9	81	3.000	9.487	59	3481	7.681	24.290
10	100	3.162	10.000	60	3600	7.746	24.495
11	121	3.317	10.488	61	3721	7.810	24.698
12	144	3.464	10.954	62	3844	7.874	24.900
13	169	3.606	11.402	63	3969	7.937	25.100
14	196	3.742	11.832	64	4096	8.000	25.298
15	225	3.873	12.247	65	4225	8.062	25.495
16	256	4.000	12.649	66	4356	8.124	25.690
17	289	4.123	13.038	67	4489	8.185	25.884
18	324	4.243	13.416	68	4624	8.246	26.077
19	361	4.359	13.784	69	4761	8.307	26.268
20	400	4.472	14.142	70	4900	8.367	26.458
21	441	4.583	14.491	71	5041	8.426	26.646
22	484	4.690	14.832	72	5184	8.485	26.833
23	529	4.796	15.166	73	5329	8.544	27.019
24	576	4.899	15.492	74	5476	8.602	27.203
25	625	5.000	15.811	75	5625	8.660	27.386
26	676	5.099	16.125	76	5776	8.718	27.568
27	729	5.196	16.432	77	5929	8.775	27.749
28	784	5.292	16.733	78	6084	8.832	27.928
29	841	5.385	17.029	79	6241	8.888	28.107
30	900	5.477	17.321	80	6400	8.944	28.284
31	961	5.568	17.607	81	6561	9.000	28.461
32	1024	5.657	17.889	82	6724	9.055	28.636
33	1089	5.745	18.166	83	6889	9.110	28.810
34	1156	5.831	18.439	84	7056	9.165	28.983
35	1225	5.916	18.708	85	7225	9.220	29.155
36	1296	6.000	18.974	86	7396	9.274	29.326
37	1369	6.083	19.235	87	7569	9.327	29.496
38	1444	6.164	19.494	88	7744	9.381	29.665
39	1521	6.245	19.748	89	7921	9.434	29.833
40	1600	6.325	20.000	90	8100	9.487	30.000
41	1681	6.403	20.248	91	8281	9.539	30.166
42	1764	6.481	20.494	92	8464	9.592	30.332
43	1849	6.557	20.736	93	8649	9.644	30.496
44	1936	6.633	20.976	94	8836	9.695	30.659
45	2025	6.708	21.213	95	9025	9.747	30.822
46	2116	6.782	21.448	96	9216	9.798	30.984
47	2209	6.856	21.679	97	9409	9.849	31.145
48	2304	6.928	21.909	98	9604	9.899	31.305
49	2401	7.000	22.136	99	9801	9.950	31.464
50	2500	7.071	22.361	100	10 000	10.000	31.62

abacus A device used for calculating, usually involving sliding beads or counters along a wire.

absolute value The undirected distance from a point on the number line to the origin. The absolute value of a number x is either x or ^{-}x, whichever is non-negative. We denote absolute value with vertical lines: $|^{-}5| = 5$; $|5| = 5$; $|0| = 0$.

acute angle An angle smaller than a right angle.

addend Any one member of a set of numbers to be added. In the equation $7 + 9 = 16$, 7 and 9 are addends.

addition An operation that combines a first number (*addend*) and a second number (*addend*) to give another number (this number is called the *sum* of the two addends).

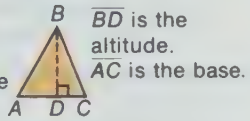
additive inverse Each of two numbers whose sum is zero is said to be the additive inverse of the other.

adjacent angles Two angles with a common vertex, a common side, and no common interior points. In the figure, $\angle 1$ and $\angle 2$ are adjacent angles.

adjacent sides Two sides of a polygon with a common vertex.

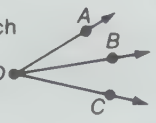
algorithm (algorism) A computation procedure. For example, the algorithm for finding the quotient of two numbers.

altitude (of a triangle) The segment from any vertex perpendicular to the opposite side. Also, see *height*.

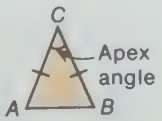


angle The union of two rays with a common endpoint (vertex).

angle bisector The ray which divides an angle into two congruent angles. In the figure $\overline{DB}$ bisects $\angle ADC$.



apex angle The angle of an isosceles triangle whose vertex is the intersection of the two congruent sides of the triangle.



approximation A number that is suitably "close" to another number. For example, a decimal approximation for $\frac{2}{9}$ is 0.22. The symbol "$\approx$" denotes "is approximately"; for example, $\pi \approx 3.14$.

arc If $\angle AOB$ is a central angle of a circle, arc ACB ($\overset{\frown}{ACB}$) is the set of points on the circle which are interior to $\angle AOB$ or on $\angle AOB$. $\overset{\frown}{ACB}$ denotes a *minor arc*. $\overset{\frown}{ADB}$ denotes a *major arc*.

area The measure of a plane region in terms of a chosen unit, usually a square.

arithmetic mean The quotient of the sum of numbers in a set divided by the number of numbers in that set (Often called the "average.")

associative (grouping) principle The principle which states that the sum (or product) of three or more numbers is the same regardless of grouping:
$(a + b) + c = a + (b + c)$
or $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

average See *arithmetic mean*.

axes (x-axis, y-axis) The coordinate axes in a plane are perpendicular number lines used to match each point in the plane with an ordered pair of numbers. Usually, the x-axis is the horizontal number line and is directed to the right while the y-axis is the vertical number line and is directed upward.

bar graph See *histogram*.

base (of numeration) The term "base" refers to the type of grouping involved in a system of numeration. For example, in base eight: $25_{(8)}$ means 2 eights and 5; $346_{(8)}$ means 3 sixty-fours, 4 eights, and 6.

base (of a polygon) Any side of a polygon may be referred to as a base. See figure for *altitude*.

base (of a space figure) See examples below.

bisect To divide into two congruent parts.

Celsius temperature Temperature based on the Celsius scale in which water freezes at 0° and boils at 100°.

centre A (fixed) point of symmetry for a geometric figure. See the figure for *circle*.

centi- A prefix meaning *one hundredth*.

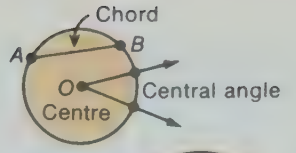
centimetre A unit of length that is $\frac{1}{100}$ of a metre.

central angle An angle with its vertex at the centre of the circle. See the figure for *circle*.

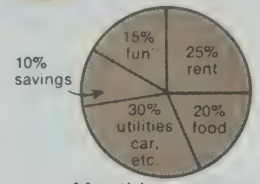
centroid (of a triangle) The point of intersection of the three medians of a triangle.

chord A segment with both endpoints on a circle. See the figure for *circle*.

circle The set of all points in a plane which are a specified distance from a fixed point called the *centre*.



circle graph A diagram used to show statistical information, particularly to show how a given quantity is divided into fractional parts.

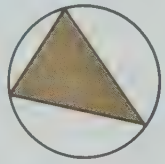


Monthly expenses

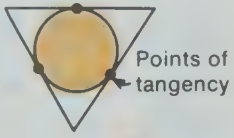
circumcentre The point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle circumscribing the triangle.

circumference The limit of the perimeters of regular inscribed polygons in a circle. The circumference of a circle is the product of its diameter and the number π. The formula is $c = \pi d$.

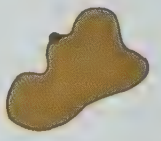
circumscribed circle A circle drawn through the vertices of a polygon. A circle can be circumscribed about any triangle but only about certain other polygons.



circumscribed polygon If the sides of a polygon are tangent to a circle, then the polygon is circumscribed about the circle. Also, see *inscribed circle*.



closed curve Intuitively, a curve is closed if you can begin at any point and trace the curve until you return to the starting point.



closure A property of a subset of a set *A* relative to an operation defined on the set *A*. For example, the operation of addition on the set of whole numbers is closed relative to the subset of even whole numbers, for the sum of any two evens is even. It is not closed upon the subset of odd whole numbers, for the sum of two odds is not odd. In the set of rational numbers the subset of integers is closed under subtraction, but the subset of whole numbers is not.

collinear (points) Two or more points lying on the same line.

combination Essentially, the word combination means the same as "subset." We say that from a set of 4 elements, 6 combinations of 2 elements can be formed, meaning that the set of 4 has 6 subsets of 2. We denote this as ${}_4C_2$.

common factor A number that is a factor of each of two or more given numbers. For example, 3 is a common factor of 6 and 9.

common multiple A number that is a multiple of each of two or more given numbers. For example, 18 is a common multiple of 2, 3, and 9.

commutative (order) principle The principle which states that the sum (or product) of any two numbers is the same regardless of the order in which they are added (or multiplied):
$a + b = b + a$ or $a \cdot b = b \cdot a$

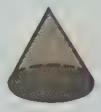
complementary angles Two angles whose measures have a sum of 90°.

complex fraction A fraction that has a fraction for the numerator or the denominator, or both.

composite number Any whole number greater than 1 which is not prime. Composite numbers have more than two factors.

concave figure See *convex figure*.

cone A space figure formed by a closed plane curve and all line segments from a point not in the plane of the curve to all the points of the curve.



congruent angles Angles which have the same measure.

congruent figures Two figures are congruent if one of the figures is the image of the other by one of the *rigid motions* (reflection, rotation, translation) or by any combination of motions.

congruent integers Two integers a and b are congruent, modulo m if the difference $a - b$ is divisible by m. We write $a \equiv b \pmod{m}$.
$23 \equiv 7 \pmod 4$ because $23 - 7$ is divisible by 4.
$5 \equiv {}^-13 \pmod 3$ because $5 - {}^-13$ is divisible by 3.

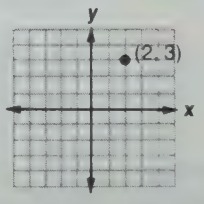
congruent segments Segments whose endpoints are equally far apart; segments which have the same length.

convex figure A figure is convex if every segment joining any two points on the boundary of the figure contains only points on the figure or in its interior.

convex figure

non-convex or **concave** figure

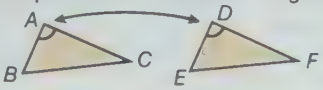
coordinate axes Two intersecting perpendicular number lines used for graphing ordered number pairs (*coordinates*).



coordinates An ordered number pair matched with a point in the coordinate plane. See the figure for *coordinate axes*.

correspondence A matching of elements in one set with those of another set.

corresponding angles Two angles matched by any one-to-one correspondence between angles of two triangles. For example, $\angle A$ and $\angle D$.



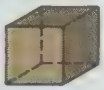
corresponding points Two points matched by any one-to-one correspondence between points of two triangles. For example, point C corresponds to point F above.

corresponding segments Two segments matched by any one-to-one correspondence between segments of two figures. For example, $\overline{AB}$ and $\overline{DE}$ are corresponding segments above.

cosine See *trigonometric ratios*.

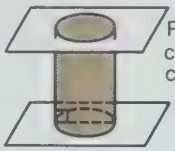
cross section The intersection of a space figure and a plane.

cube (geometry) A regular polyhedron each of whose six faces is a square. It is also a regular hexahedron.



cube (numeration) A number raised to the third power. 8 is the cube of 2 because $2^3 = 8$. Also, to raise a number to the third power.

cylinder The space figure formed by two congruent curves in parallel planes and the parallel segments connecting corresponding points of the curves.



deca- A prefix meaning *ten*.

decametre A unit of length that is 10 metres.

deci- A prefix meaning *one tenth*.

decimal (numeral) Any base-ten numeral written using a decimal point.

decimal system of numeration A system of numeration in which powers of ten and place value are used.

decimetre A unit of length that is $\frac{1}{10}$ of a metre, or 10 centimetres.

degree A unit angle that is $\frac{1}{90}$ of a right angle.

denominator For each fraction $\frac{a}{b}$, $b \neq 0$, b is the denominator.

diagonal A segment connecting two nonconsecutive vertices of a polygon.

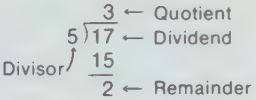
diameter Any chord that contains the centre of a circle.

difference The number resulting from subtraction.

digits The basic symbols used in a place-value system of numeration. In base ten, the symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

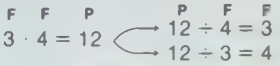
distributive principle The fundamental principle connecting addition and multiplication. $a \cdot (b + c) = a \cdot b + a \cdot c$.

dividend The number to be divided in a division problem.



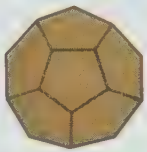
divisible A given number is divisible by a second number if the remainder is zero. If $a = b \cdot c$ with $b \neq 0$ and $c \neq 0$, then a is divisible by either b or c.

division The inverse of the operation of multiplication.



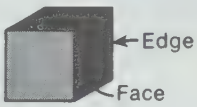
divisor See *dividend*.

dodecahedron A polyhedron which has 12 faces. A regular dodecahedron has 12 congruent pentagonal faces.



domain The set of all input numbers for a function.

edge A segment that is a side of any face of a space figure. A cube has 12 edges.



element A member of a set.

empty set The set containing no elements. It is denoted by "{ }" or the symbol ϕ.

equal sets Sets containing the same elements. They may be listed in different order.

equality The relation of identity. A statement of equality asserts that two symbols name the same object or number.

equally likely outcomes Outcomes or results that have the same chance of occurring.

equation A mathematical sentence using the equality symbol (=). $7 + n = 9$ is an equation.

equiangular triangle A triangle all of whose angles have the same measure.

equilateral triangle A triangle having all sides the same length.

equivalent decimals Decimals which represent the same number. For example, 0.5, 0.50, and 0.500 are equivalent decimals.

equivalent fractions Fractions which represent the same number, such as $\frac{1}{2}$, $\frac{5}{10}$ and $\frac{50}{100}$.

equivalent ratios Two ratios which represent the same rational number. For example, the ratio 5:10 is the same as 1:2.

equivalent sets Sets having the same number of elements.

estimate An approximation for a given number. Often used in the sense of a rough calculation.

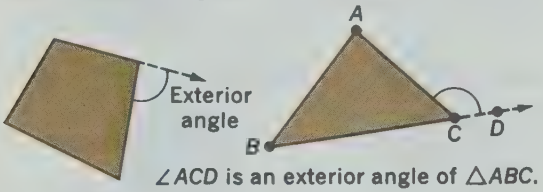
even number A whole-number multiple of 2. The set of even numbers is $\{0, 2, 4, 6 \ldots\}$

event Any subset of a sample space for a probability experiment. If a die is tossed, the event of getting a number greater than 3 has occurred if any number in the subset $\{4, 5, 6\}$ results from the toss.

expanded notation A representation of a number as a sum of powers of ten such as:
$3424 = 3 \cdot 10^3 + 4 \cdot 10^2 + 2 \cdot 10^1 + 5 \cdot 10^0$.

exponent In a symbol such as 5^4, the numeral 4 is an *exponent* and 5 is the *base*. Positive exponents indicate how many times the base is to be used as a factor; $5^4 = 5 \cdot 5 \cdot 5 \cdot 5$. Negative exponents are used to indicate reciprocals of powers of numbers; $6^{-3} = \frac{1}{6^3}$, $10^{-4} = \frac{1}{10^4}$.

exterior angle of a polygon An angle formed by extending any one side of a polygon. A triangle has six exterior angles.



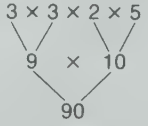
$\angle ACD$ is an exterior angle of $\triangle ABC$.

extremes The first and last terms in a proportion written in the form $a:b = c:d$. The numbers a and d are extremes and b and c are called the *means*.

face Any one of the bounding polygonal regions of a space figure. See the figure for *edge*.

factor Any one member of a set of numbers to be multiplied. In the equation $5 \cdot 7 = 35$, 5 and 7 are factors.

factor tree A diagram suggestive of a tree showing the prime factorization of a number.



factorial If n is a positive whole number, factorial n denoted by $n!$, is defined to be
$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-2) \cdot (n-1) \cdot n$.
$0!$ is defined to be 1.

flow chart A diagram which gives instructions in a logical order.

formula A general fact or rule expressed by using symbols. For example, the area any parallelogram with base b and height h is given by $b \cdot h$. The formula is usually presented with the equation $A = b \cdot h$.

fraction A symbol for a fractional number of the form $\frac{a}{b}$ such that $b \neq 0$.

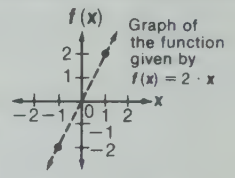
fractional number The set of non-negative rational numbers.

frequency The number of times an outcome occurs in a probability experiment or the number of times a bit of data occurs in a collection of data.

function A correspondence that associates each element in a first set with a unique element of a second set. Different elements in a first set may possibly be matched with the same element of the second set.

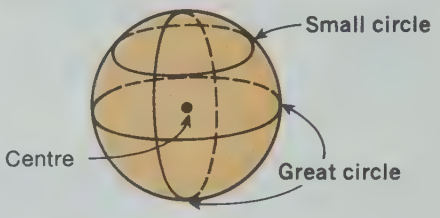
gram A unit of mass. The mass of one cubic centimetre of water at 4°C.

graph A set of points associated with a given set of numbers or number pairs showing a relation or function.



graph (statistical) A picture used to illustrate a given collection of data. It may be in the form of a bar graph, a circle graph, a line-segment graph, or a pictograph.

great circle A circle formed by the intersection of a sphere and a plane containing the centre of the sphere.



greater than A number x is greater than a number y (written $x > y$) if $x = y + p$ and p is positive. $^-3$ is greater than $^-5$, for $^-3 = {}^-5 + 2$. We write, $^-3 > {}^-5$.

greatest common factor (GCF) The largest, or greatest, number that is a factor of each of a set of numbers.

greatest possible error Half of the basic unit in which a measurement is given. For example, if a length is given as 18 centimetres to the nearest centimetre, then the basic unit is the centimetre and the greatest possible error is 0.5 centimetre.

hecto- A prefix meaning *one hundred*.

height (of a prism) The perpendicular distance between the planes of the parallel faces.

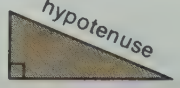
heptagon A 7-sided polygon.

hexagon A 6-sided polygon.

Hindu-Arabic numeral A numeral for a whole number formed from the basic Hindu-Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

histogram A bar graph showing the frequency distribution of a set of data.

hypotenuse The side of a right triangle opposite the right angle.



icosahedron A polyhedron of 20 faces. A regular icosahedron has equilateral triangles for its faces.

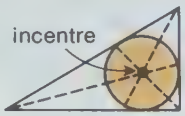
identity element For addition, zero is the identity element since $a + 0 = a$ for any number a. Similarly, for multiplication, 1 is the identity element since $a \cdot 1 = a$.

identity principles For any number a, $a + 0 = a$ and $a \cdot 1 = a$.

image The figure obtained after a rigid motion of a given figure. From a functional viewpoint, y is the image of x if there is a function f such that y is the output when x is the input; that is, $f(x) = y$.

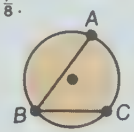
improper fraction A fraction whose numerator is greater than or equal to its denominator.

incentre The point of intersection of the angle bisectors of each angle of a triangle. The centre of the circle inscribed in a triangle.



inequality A mathematical statement using either of the symbols $<$ or $>$ such as $\frac{3}{4} < \frac{7}{8}$.

inscribed angle An angle that contains three points of the circle and has its vertex on the circle.



inscribed polygon See *circumscribed circle*.

integers The set consisting of the natural numbers, the negatives of the natural numbers, and zero; $\{\ldots {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}$.

interpolation When we wish to find a number between two entries in a table, we interpolate. For example, if a table of square roots shows

$$\sqrt{7} = 2.646 \quad \text{and} \quad \sqrt{8} = 2.828$$

then a good approximation for $\sqrt{7.5}$ can be interpolated between these numbers by choosing the number midway between them:

$$\sqrt{7.5} \text{ is approximately } \frac{2.646 + 2.828}{2} = 2.737.$$

intersection The intersection of two sets is the set of all elements common to both sets.

inverse (operation) Operations which are opposite in effect such as addition and subtraction or multiplication and division.

irrational number A real number that cannot be expressed as the quotient of two integers; a nonrational number. Irrational numbers have infinite nonrepeating decimal representations.

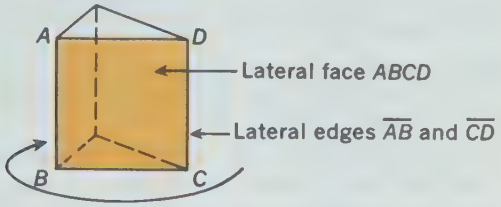
isosceles triangle A triangle with at least two congruent sides.

kilo- A prefix meaning *one thousand*.

kilogram A unit of mass that is 1000 grams.

kilometre A unit of length that is 1000 metres.

lateral edges An edge of a prism connecting the parallel faces.



lateral face One of the faces of a prism containing two lateral edges. (See figure for lateral edges.)

least common denominator The least common multiple of the denominators of two or more fractions. for example, the LCD of $\frac{5}{6}$ and $\frac{3}{4}$ is 12.

least common multiple (LCM) The smallest nonzero number that is a multiple of each of two or more given numbers. The LCM of 4 and 6 is 12.

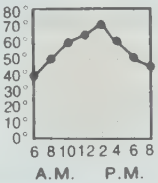
legs (of a right triangle) The perpendicular sides of a right triangle.

length The measure of a segment in terms of a chosen unit.

less than a is less than b (written $a < b$) if b is greater than a. (*See* greater than.)

line of symmetry Line t is a line of symmetry of a set S of points if for each point A of S there is a corresponding point B such that t is the perpendicular bisector of $\overline{AB}$.

line-segment graph A diagram used to show statistical information, particularly that of a continuous nature.



litre The basic unit of capacity in the metric system. 1000 cubic centimetres.

lowest-terms (simplest) fraction A fraction is in lowest terms if the numerator and denominator of the fraction have no common factor other than 1.

mass The amount of matter in a physical object. Except in scientific work, the word "weight" is used synonymously with mass. A rock whose *mass* is 1 kilogram on earth will have a mass of 1 kilogram on the moon, but its weight on the moon will be only about $\frac{1}{6}$ its earth weight.

mean See *arithmetic mean*.

means The second and third terms in a proportion written in the form $a : b = c : d$. See *extremes*.

median (of a set) The middle number of a set containing an odd number of elements that are arranged in order. If a set contains an even number of elements, the average of the two middle numbers is the median.

median (of a triangle) The segment from the vertex to the midpoint of the opposite side of a triangle.

metre The basic unit of length in the metric system. Defined as 1 650 763.73 times the wave length of the orange-red spectral emission line of krypton-86.

metric system A system of measurement in which the metre, the litre, and the kilogram are the basic units of length, capacity, and mass.

midpoint (middle point) A point which divides a segment into two congruent segments.

milli- A prefix meaning *one thousandth*.

minute A unit of angular measure that is $\frac{1}{60}$ of a degree.

mixed decimal numeral A combination of a decimal and a fraction such as $.4\frac{1}{3}$ and read as "four and one-third tenths."

mixed numeral A numeral such as $4\frac{2}{3}$ indicating the sum of a whole number and a fractional number; $4\frac{2}{3} = 4 + \frac{2}{3}$.

mode The most frequently occurring element of a set. There may be more than one mode of a set.

multiple A first number is a multiple of a second number if there is a whole number that multiplies by the second number to give the first number. For example, 24 is a multiple of 4 and 6 since $4 \times 6 = 24$.

multiplication An operation that combines a first number (*factor*) and a second number (*factor*) to give another number (this result is called the *product* of the two factors).

multiplicative inverse See *reciprocal*.

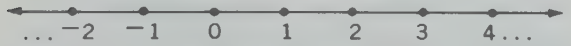
negative integers The set of numbers $\{^{-}1, ^{-}2, ^{-}3, \ldots\}$.

nonagon A 9-sided polygon.

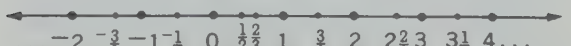
nonrepeating decimal A decimal not formed from any point on by repeating blocks of digits. Each nonrepeating decimal represents an irrational number, while each repeating decimal represents a rational number.

number line A line with a subset of its points matched with a subset of the real numbers. We say that the rational number line has "holes" in it because some points are not matched with rational numbers. The real number line is said to be "complete" because each point is matched with some real number. (*See* real number line.)

Whole number line



The integer number line



The rational number line

number pair An ordered pair of numbers such as (3, 4); often, co-ordinates of a point in a plane.

numeral A symbol for a number.

numerator For each fraction $\frac{a}{b}$, a is the numerator.

oblique prism A prism in which the base is not perpendicular to its lateral edges.

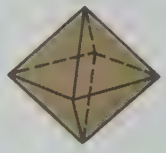
Oblique prism

Right prism

obtuse angle An angle greater than a right angle and smaller than a straight angle.

octagon An 8-sided polygon.

octahedron A polyhedron having eight faces. Each face of a regular octahedron is an equilateral triangle.



odd number Any number in the set $\{1, 3, 5, 7, \ldots\}$.

one principle For any number a, $a \cdot 1 = a$. See *identity principle* for multiplication.

one-to-one correspondence Two sets S and T are in one-to-one correspondence if there is a matching so that each element in S is matched with exactly one element in T, and vice versa.

operation A binary operation like "multiplication" associates each order paired of numbers with one number. Usually we speak of the four fundamental operations: addition, subtraction, multiplication, and division.

opposites principle For any integer a, $a + {}^{-}a = 0$.

origin The intersection of the co-ordinate axes; the point associated with the number pair (0, 0).

outcome A possible result in a probability experiment.

parallel lines Lines in the same plane which do not intersect.

parallelepiped A space figure all of whose faces are parallelograms.

parallelogram A quadrilateral whose opposite sides are parallel.

pentagon A 5-sided polygon.

percent Literally means "per hundred". A symbol such as 3% is an abbreviation for "3 per 100" or "$\frac{3}{100}$."

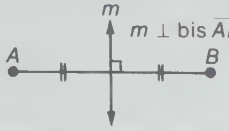
perimeter The distance around a polygon.

period Each set of three digits in a numeral. In the numeral 421 365 761 409, there are 421 billions, 365 millions, 761 thousands, and 409 units.

permutation Any ordered subset of a set is referred to as a permutation. For example, six permutations of subsets of 2 can be formed from any set of three elements.

$\{a, b, c\} \longrightarrow (a, b), (b, a), (a, c), (c, a), (b, c), (c, b)$

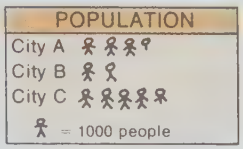
perpendicular bisector A line which bisects a segment as well as being perpendicular to it.



perpendicular lines Two intersecting lines that form right angles.

pi (π) The ratio of the circumference of a circle to its diameter. $\pi \approx 3.14159$.

pictograph A diagram used to show statistical information with the use of pictures.



plane figure A set of points in one plane.

point of tangency If a line and circle intersect in just one point, the intersection is the point of tangency.

polygon A closed plane figure whose boundary is made up of segments.

polygonal path A path consisting of a union of segments which are connected end to end.

polyhedron A space figure each of whose faces is a polygonal region.

positive integer Any number in the set $\{1, 2, 3, \ldots\}$.

power In the statement $a = b^n$, a is the n^{th} power of b. For example, for $1000 = 10^3$, 1000 is the third power of 10.

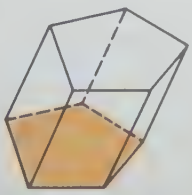
precision One measurement is *more precise* than another if its *relative* error is smaller than that of the other.

prime factorization An expression of a composite number as a product of prime factors.

prime number A whole number greater than 1 whose only factors are itself and 1.

prism A 3-dimensional (space) figure whose bases are congruent polygonal regions in parallel planes and whose faces are parallelograms.

Triangular prism



Pentagonal prism

probability The ratio of the number of times a certain outcome can occur to the number of total possible outcomes.

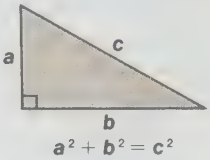
product The number $a \cdot b$ which results from applying the multiplication operation to the numbers a and b.

proportion An equation stating that two ratios are equal: $a:b = c:d$ or $\frac{a}{b} = \frac{c}{d}$.

protractor An instrument for measuring angles.

pyramid A 3-dimensional (space) figure with a polygonal base and triangular lateral faces.

Pythagorean Theorem In any right triangle, the sum of the areas of the squares on the legs is equal to the area of the square on the hypotenuse.



$a^2 + b^2 = c^2$

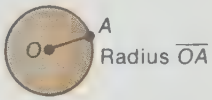
quadrilateral A 4-sided polygon.

quotient The number $a \div b$ (or $b \div a$) which results from applying the division operation to the numbers a and b.

radian A unit of angular measure. A central angle of a circle subtended by an arc whose length is equal to the radius; approximately 57.3°; exactly $(180 \div \pi)°$.

radical sign The sign $\sqrt{\ }$ which indicates that a root is the be extracted. For example, $\sqrt[3]{7}$ is the cube root of 7 and $\sqrt{8}$ is the positive square root of 8.

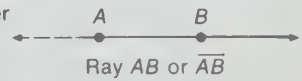
radius Any segment from the centre of a circle to a point on the circle.



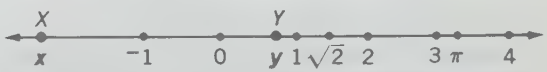
ratio The ratio of two numbers a and b is their quotient, $\frac{a}{b}$.

rational number The quotient of two integers, the divisor not being zero.

ray A half-line together with the point determining it (endpoint).



real number line Any line with its points matched one to one with the real numbers is called a real number line.

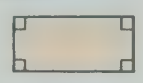


The real number line. A point for each real number and a number for each point.

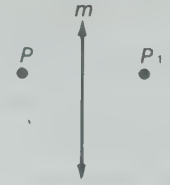
real numbers All numbers, either positive, negative, or zero, having decimal representations. Those real numbers with nonrepeating decimals are the irrational numbers.

reciprocal If $a \cdot b = 1$, each of the numbers a and b is the reciprocal (multiplicative inverse) of the other. Each nonzero real number has a unique reciprocal. Zero has no reciprocal.

rectangle A parallelogram with four right angles.

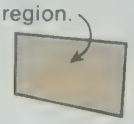


reflection A rigid motion that maps the points of a plane onto itself. If point P is reflected in a line with a reflection image P_1, then the line is the perpendicular *bisector of* $\overline{PP_1}$.



region All the points in the part of a plane bounded by a simple closed curve.

The interior of a quadrilateral is a region.



This curve forms two regions.

regular polygon A polygon that is equiangular and equilateral; for example, a square.

relative error The ratio of the greatest possible error of a measurement to the measurement itself. For example, if a measurement is given as 10 centimetres to the nearest centimetre and the greatest possible error is 0.5 centimetre, then the relative error is the ratio of 0.5 to 10 or 5%.

remainder In whole-number division, if dividend D is divided by divisor d and the relation $D = (q \cdot d) + r$ is obtained where $0 < r < d$, then r is called the remainder. Note that $r = 0$ if d is a factor of D (see *dividend*). Also, the difference in a subtraction problem.

repeating decimal A decimal whose digits can be grouped from some point on in repeating periods. Examples: 0.333 . . .; 0.2727 . . .; 4.273149149 . . .. In these examples, the repeating decimals have (respectively) periods of 1, 2, and 3 digits and can be written as $0.\overline{3}$, $0.\overline{27}$, and $4.273\overline{149}$.

rhombus An equilateral parallelogram which is not a square.

right angle An angle whose measure is 90°.

rigid motion A transformation of the points of the plane that preserves all distances.

Roman numerals The numerals I, V, X, L, C, D, M, and combinations of the numerals used in the Roman numeration system.

rotation A rigid motion in which the points of the plane are turned about a fixed point.

rounding A process of replacing a number by an approximation (another number) with fewer significant digits. For example, rounding 456 789 to the nearest thousand would be 457 000.

sample space The set of all possible outcomes of a probability experiment.

scale drawing A drawing of an object made so that distances in the drawing are proportional to actual distances. A scale of 1 : 10 indicates that distances in the drawing are $\frac{1}{10}$ of the actual distances.

scalene triangle A triangle with no pair of congruent sides.

scientific notation A notation used for writing any number as the product of a power of ten and a number between 1 and 10 (including 1).

second A unit of angular measure that is $\frac{1}{60}$ of a minute.

segment A set of points consisting of two points A and B and all points between them.

Segment AB or $\overline{AB}$

semicircle An arc that is exactly half a circle. One of two arcs cut off by a diameter.

set A group or collection of objects.

Sieve of Eratosthenes A process of determining the set of all primes that are less than a finite number n.

significant digits All nonzero digits reported in a measurement are significant, but if terminal zeros occur, then it must be specified whether or not they are significant. For example, suppose a population is reported as 43 000.

43 000: If only the 4, 3 are significant, then the population is expected to fall between 42 500 and 43 500.

43 000: If the first zero is also significant, then the population is expected to fall between 42 950 and 43 050.

In scientific notation, all digits in the factor times the power of ten are significant.

2.3×10^4 has 2 significant digits.

2.370×10^4 has 4 significant digits.

similar polygons Two polygons are similar if there is a one-to-one correspondence between the vertices such that corresponding angles are congruent and corresponding sides are proportional. Congruence is a special case of similarity.

sine See *trigonometric ratios.*

skew lines Lines that are not in the same plane. They do not intersect, yet they are not parallel.

space The set of all points. This term usually denotes a 3-dimensional quality. We also refer to a plane as 2-dimensional space and a line as a 1-dimensional space.

space figure A geometric figure whose points are not all in the same plane.

sphere The set of all points in space at a fixed distance from a given point.

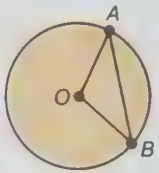
square root If $x^2 = y$, then x is a square root of y. We use the sign $\sqrt{\ }$ to indicate the operation of extracting the positive square root of a number. For example, in the set of integers, 9 has two square roots, 3 and $^-3$. The symbol $\sqrt{9}$ by agreement denotes only the positive square root, 3.

statistics The science of analyzing numerical information.

straightedge An unmarked ruler.

subset Set A is a subset of set B if every element of A belongs to B.

subtend Literally means "to be opposite to." For example, central angles and inscribed angles in a circle are said to subtend the arcs they cut off. Also, chords and arcs in a circle subtend the angles of a circle.



$\overset{\frown}{AB}$ subtends $\angle AOB$ or
$\overset{\frown}{AB}$ is subtended by $\angle AOB$

subtraction The operation that is the inverse of the addition operation.

sum The number $a + b$ which results when the operation of addition is applied to the numbers a and b.

supplementary angles Two angles whose measures have a sum of 180°.

surface area The total area of the polygonal regions (faces) of a polyhedron.

symmetric figure A plane figure which can be divided into two congruent parts by a line (of symmetry).

tangent A line which intersects a circle in just one point. See also *trigonometric ratios*.

$\overleftrightarrow{AB}$ is tangent to circle O at point A.

terminating decimal A decimal such as 0.5, 1., 1.24, 0.0307, and so on, that represents the quotient of a whole number and a power of 10:

$$0.5 = \frac{5}{10},\ 1.24 = \frac{124}{10^2},\ 0.0307 = \frac{307}{10^4}$$

tessellation A repeated pattern of geometric figures which will completely "cover a plane" without any gaps or overlapping.

tetrahedron A polyhedron having 4 faces.

theorem A statement to be proved true by means of basic assumptions and mathematical reasoning.

translation A rigid motion in which each point of the plane is moved the same distance and in the same direction.

trapezoid A quadrilateral with a pair of parallel sides.

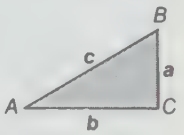
triangle A 3-sided polygon.

trigonometric ratios If one of the acute angles of a right triangle is chosen, then each of the six possible ratios that can be formed between two of the sides of the triangle is called a trigonometric ratio of that angle.

$\frac{a}{c}$ is the sine of $\angle A$ (sin A).

$\frac{b}{c}$ is the cosine of $\angle A$ (cos A).

$\frac{a}{b}$ is the trangent of $\angle A$ (tan A).

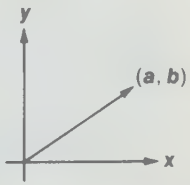


union If A and B are sets, then the union of A and B ($A \cup B$) is the set consisting of all elements that belong to at least one of the two sets.

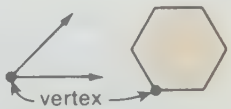
unit The object adopted as a standard of measurement. For length, we choose a *unit* segment; for area, we choose a *unit* square; and for volume, a *cube*.

variable A symbol, usually a letter, used to represent any number in a given set. For example, we may say: "If x is a whole number, then $2 \cdot x$ is an even number."

vector A directed line segment. In the co-ordinate plane a vector is often denoted by an arrow emanating from the origin and is denoted by the ordered pair of the point at the tip of the arrow for the vector.



vertex A point that two rays of an angle have in common. Also, the common point of any two sides of a polygon.



vertical angles Two pairs of angles formed by two intersecting lines.

volume The measure of a space figure in terms of a chosen unit.

whole numbers Any number in the set $\{0, 1, 2, 3, \ldots\}$.

zero principle The principle which states that for any number a, $a + 0 = a$. Also, see *identity element* for addition.

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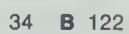
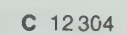
UNIT E: MODULE 1

E-3 USING THE IDEAS

1. A 34 **B** 122 **C** 12 304 **D** 230 123 **E** 1 020 262
F 2 200 306 **2. A** 7 **B** 35 **C** 43 **D** 620 **3. A** 27
B 1616 **C** 2877 **4. A** 5 **B** 6 **C** 4 **D** 10 **E** 11
F 9 **G** 60 **H** 40 **I** 600 **J** 400 **K** 1100 **L** 900
M 1159 **N** 178 **O** 1970
5. A  **B** 
C CCXLVII **6.** MMMCLXXIX
Think 1776

E-5 USING THE IDEAS

1. B 900 **C** 4000 **D** 600 000 **E** 5 000 000
2. A 874 **B** 3691 **C** 28 437 **D** 100 101 **E** 67 080
F 3 700 000 **3. A** 6 000 000 000 000 000
B 73 000 000 000 **C** 276 000 000 000 000 000 000 000 000
D 1 000 000 000 000 000 000 000 000 000 000 000
E 439 000 000 000 000 000 000 000 000 000 **F** 89 000 000
4. 1 000 000 000 000
Think 61

E-7 USING THE IDEAS

1. A 14 **B** 13 **C** 25 **D** 80 **E** 200 **F** 1300 **G** 8
H 5 **I** 7 **J** 0 **K** 8 **L** 138 **M** 35 **N** 72 **O** 0 **P** 1
Q 3 **R** 8 **2. A** 9 **B** 7 **C** 3 **D** 9 **E** 7 **F** 90
G 502 **H** 10 **I** 63 **3. A** 729 **B** 779 **C** 786
D 1265 **E** 1295 **F** 1303 **G** 563 **H** 513 **I** 507
J 317 **K** 277 **L** 269
Think *A, C, D, F*

E-9 SOME SPECIAL FUNCTION MACHINE RULES

1. A 24 **B** 120 **C** 200 **2. A** 18 **B** 34 **C** 90
3. A 6 **B** 20 **C** 28 **4. A** 110 **B** 82 **C** 200
5. A 11 **B** 30 **C** 800 **6. A** 21 **B** 45 **C** 39
7. cubeachadd
Think **1.** 4, 15 **2.** 5, 24 **3.** 6, 35 **4.** 10, 99
5. 20, 399 **6.** 100, 9999

E-11 USING THE IDEAS

1. A 7^2 **B** 3^3 **C** 2^5 **D** 10^4 **E** 8^6 **F** 10^5 **G** 5^2
H 9^7 **I** 4^3 **2. A** 3 **B** 2 **C** 6 **D** 7 **E** 19 **F** 50
G 1000 **3. A** 1000 **B** 100 000 **C** 1 000 000
D 100 000 000 **E** 1 000 000 000 **F** 1 000 000 000 000
G 10 **4. A** 10^4 **B** 10^5 **C** 10^5 **D** 10^4 **E** 10^7 **F** 10^6
G 10^8 **H** 10^9 **I** 10^{13} **5. A** 10^2 **B** 10^3 **C** 10^1
D 10^3 **6. A** 2 **B** 3 **C** 4 **7.** D **8. A** $(2 \cdot 10) + (7 \cdot 10^0)$
B $(4 \cdot 10^2) + (7 \cdot 10^1) + (3 \cdot 10^0)$ **C** $(5 \cdot 10^2) + (7 \cdot 10^0)$
D $(9 \cdot 10^3) + (6 \cdot 10^2) + (2 \cdot 10^1) + (5 \cdot 10^0)$ **E** $(8 \cdot 10^4) + (7 \cdot 10^3) + (3 \cdot 10^1) + (4 \cdot 10^0)$ **F** $(4 \cdot 10^6) + (6 \cdot 10^5) + (3 \cdot 10^4) + (2 \cdot 10^3) + (4 \cdot 10^2) + (7 \cdot 10^1)$
Think 2^{1000}

E-13 USING THE IDEAS

1. A 6000 **B** 6000 **2. A** 2 857 460 **B** 2 857 500
C 2 857 000 **D** 2 860 000 **E** 2 900 000 **3. A** 437 000
B 2 084 000 **C** 1 639 000 **D** 31 000 **4. A** 9 976 000
B 10 000 000 **5. A** 6100 **B** 6000 **6. A** 8 500 000
B 9 000 000
Think 639

E-15 USING THE IDEAS

1. A 600 **B** 700 **C** 80 **D** 70 **E** 700 **F** 4000
G 500 **H** 300 **I** 90 **J** 8000 **K** 400 **2. A** 900
B 6300 **C** 4800 **D** 30 **E** 500 **F** 24 000 **G** 900
H 18 000 **I** 230 **J** 10 000 **K** 280 000 **3. A** 1400
B 1100 **C** 400 **D** 2400 **E** 200 **F** 49 000 **G** 9000
H 200 **I** 100 000 **J** 1000 **K** 11 000 **L** 18 000 **4.** 5
5. $20 000 000 000 **6. A** 10 **B** 2 **C** USSR

E-17 USING THE IDEAS

1. A 525 **B** 2442 **C** 881 188 **D** 69 696 **E** 15 851
F 43 911 934 **2. A** 1089 **B** 1089 **C** 1089 **D** 1089
3. A 4416 **B** 4416 **C** 215 418 **D** 215 418 **4. A** 52
B 3713, R90 **C** 839

5. A
$$\begin{array}{r} 5837 \\ +\ 9678 \\ \hline 15515 \end{array}$$

B
$$\begin{array}{r} 4007 \\ -\ 2989 \\ \hline 1018 \end{array}$$

C
$$\begin{array}{r} 476 \\ 999 \\ \hline 4284 \\ 4284 \\ 4284 \\ \hline 475524 \end{array}$$

D
$$\begin{array}{r} 11 \\ 76\overline{)836} \\ 76 \\ \hline 76 \\ 76 \\ \hline 0 \end{array}$$

E-19 USING THE IDEAS

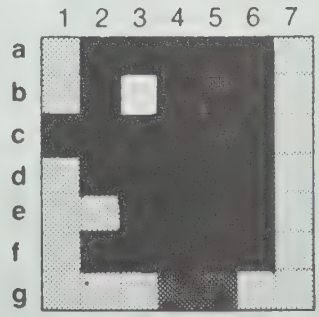
1. A 1000110 **B** 1101111 **C** 0110001 **D** 1010011
E 1111010 **F** 1100010 **2. A** 102 **B** 39 **C** 85 **D** 59
E 44 **F** 126 **3. A** 65 **B** 53 **C** 95 **D** 77 **E** 56
F 117 **4. A** 111 **B** 1101 **C** 11010 **D** 11110
E 101010 **F** 110010 **G** 0111101 **H** 1000011
I 1001001 **J** 1010000 **K** 1100100 **L** 1111101

E-21 USING THE IDEAS

1. A twos, 1×2^1 **B** sixteens, 1×2^4
C sixty-fours, 1×2^6 **2. A** 1 **B** 2 **C** 7 **D** 4 **E** 5
F 6 **G** 3 **H** 12 **I** 9 **J** 10 **K** 11 **L** 15 **3.** 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192 **4. A** 14
B 8 **C** 28 **D** 16 **E** 31 **F** 32 **G** 119 **H** 170
5. A $11_{(2)}$ **B** $101_{(2)}$ **C** $11000_{(2)}$ **D** $101110_{(2)}$
6. A $10011100_{(2)}$ **B** $101110100_{(2)}$ **C** $1000011011_{(2)}$
D $1010101100_{(2)}$ **E** $1110000100_{(2)}$ **F** $1111111111_{(2)}$
G $10011111111_{(2)}$ **H** $11000110000_{(2)}$
Think **A** 23 **B** 133 **C** 225 **D** 511 **E** 433

E-23 USING THE IDEAS

1. A 41 **B** black **2.** 111110 **3.** 000001
4. 011111, 100000
5.



E-25 USING THE IDEAS

1. A 11 **B** 101 **C** 110 **D** 10010 **E** 10110
F 1011 **2. A** 10 **B** 1 **C** 1 **D** 10 **E** 11011 **F** 111

3. A 11 **B** 0 **C** 100 **D** 1000 **E** 10000 **F** 1001
G 1111 **H** 110001 **I** 11000 **J** 10101 **K** 100011
L 11110 **4. A** 101 **B** 11 **C** 1011 **D** 1000 **5. A** 10
B 1 **C** 10 **D** 11 **E** 11 **F** 1110 **G** 100 **H** 100 **I** 1

Think

+	0	1	2	3	4	5	6	7	8	9	D	E
0	0	1	2	3	4	5	6	7	8	9	D	E
1	1	2	3	4	5	6	7	8	9	D	E	10
2	2	3	4	5	6	7	8	9	D	E	10	11
3	3	4	5	6	7	8	9	D	E	10	11	12
4	4	5	6	7	8	9	D	E	10	11	12	13
5	5	6	7	8	9	D	E	10	11	12	13	14
6	6	7	8	9	D	E	10	11	12	13	14	15
7	7	8	9	D	E	10	11	12	13	14	15	16
8	8	9	D	E	10	11	12	13	14	15	16	17
9	9	D	E	10	11	12	13	14	15	16	17	18
D	D	E	10	11	12	13	14	15	16	17	18	19
E	E	10	11	12	13	14	15	16	17	18	19	1D

×	0	1	2	3	4	5	6	7	8	9	D	E
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	D	E
2	0	2	4	6	8	D	10	12	14	16	18	1D
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	D	13	18	21	26	2E	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2E	36	41	48	53	5D	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
D	0	D	18	26	34	42	50	5D	68	76	84	92
E	0	E	1D	29	38	47	56	65	74	83	92	D1

E-26 REVIEWING THE IDEAS

1. A 110213 **B** 1668 **C** 1464 **2. A** 3 000 000 000
B 748 000 000 **3.** first 3 on left. **4.** $53 - 27 = n$
5. $234 \div 9 = y$ **6. A** commutative (+) **B** one
C commutative (×) **D** associative (+) **E** zero
7. A 6^3 **B** 4^5 **C** 10^2 **D** 2^7 **8.** 0 **9. A** 10^5 **B** 2^7
C 10^4 **D** 8^3 **E** 7^5 **F** 10^1 **10. A** 7 463 500
B 7 463 000 **C** 7 000 000 **11.** *B* **12.** 500
13. A 18 481 **B** 5638 **C** 399 280 **D** 247, R14
14. 10100 **15. A** 101 **B** 1100 **C** 10011 **D** 11101
E 100000 **F** 1000010 **16. A** 6 **B** 11 **C** 8 **D** 27
E 15 **F** 43 **17 A** 1010 **B** 1001 **C** 100011 **D** 1001

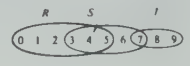
UNIT E: MODULE 2

E-31 USING THE IDEAS

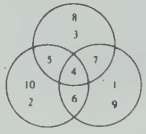
1. A infinite **B** infinite **C** finite **D** finite
2. A true **B** false **C** false **D** false **E** true **F** false
G true **H** false **I** true **J** false **3.** $\varnothing$, {2, 4, 6}, {2}, {4}, {6}, {2, 4}, {2, 6}, {4, 6} **4.** 16 **5.** $B \subset A$; $C \subset A$; $C \subset B$ **6.** $Q \subset M$; $Q \subset N$; $N \subset M$ **7.** $J \subset L$ **8.** $A \subset C$
Think 18

E-33 USING THE IDEAS

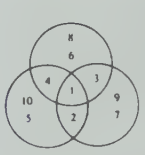
1. A {1, 3, 5, 6, 7, 9, 12} **B** {3, 9}
2. A {0, 1, 2, 3, 4, 5, 6, 7} **B** {0, 1, 2, 3, 4, 5, 7, 8, 9}
C {3, 4, 5, 6, 7, 8, 9} **D** {3, 4, 5} **E** $\varnothing$ **F** {7}
3. A {7} **B** {3, 4, 5, 7, 8, 9} **C** $\varnothing$
D {0, 1, 2, 3, 4, 5, 7} **E** {3, 4, 5} **F** {0, 1, 2, 3, 4, 5, 7}
G {7} **H** {3, 4, 5, 7, 8, 9} **I** {3, 4, 5, 6, 7} **J** {3, 4, 5}
K {3, 4, 5, 7} **L** {3, 4, 5, 7} **4.**



5. A {4, 8, 12, 16, . . .} **B** {6, 12, 18, 24, . . .}
C {12, 24, 36, . . .} **D** common multiples of 4 and 6
Think
1.



2.



E-35 USING THE IDEAS

1. 10, 11, 12 **2.** 7, 6, 5, 4 **3.** −2, 4, 10, 16 **4. A** 10
B 54 **C** 3 **D** 49 **E** 29 **F** 2 **G** 8 **H** 6 **5.** 8
6. 1, 3, 7, 13, 21 **7. A** 4, 9, 16 **B** 4, 9, 16
Think 20^{20}, then pattern repeats

E-37 USING THE IDEAS

1. A 8 **B** 5 **C** 3 **D** 1
E {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} **F** 10 **G** 7 **H** 0
I {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} **2. A** {0, 1, 2, 3}
B {6, 7, 8, 9} **C** {0, 1, 2, 3, 4, 5} **D** {5, 6, 7, 8, 9}
E $\varnothing$ **F** {6, 7, 8, 9} **G** {0, 1, 2, 3, 4, 5, 6}
H {0, 1, 2, 3, 4} **I** {6, 7, 8, 9} **3.** *A, C, D, F, G, H, K* are generalizations **4.** *A, C, D, F, G, H, K*
Think 6

E-39 USING THE IDEAS

1. A 5, 7 **B** 21, 7 **C** 8, 11 **D** 6, 18 **E** 9, 5
F 13, 7 **2. A** 4 **B** 6 **C** 19 **D** 1 **E** 8 **F** 26 **G** 38
H 7 **I** 90 **J** 10 **K** 12 **L** 42 **M** 24 **N** 19 **O** 3
P 12 **3. A** 8, 4 **B** 20, 5 **C** 20, 10 **D** 18, 6 **E** 14, 7
F 20, 4 **4. A** 5 **B** 10 **C** 3 **D** 0 **E** 1 **F** 6

E-41 USING THE IDEAS

1. A 0, 1 **B** 1, 2 **C** 0, 2 **D** 1, 4 **E** 0, 11 **F** 7, 4
G 8, 6 **H** 0, 0 **I** 3, 11 **J** 10, 7 **K** 15, 12 **L** 2
2. A 8 **B** 5 **C** 5 **D** 2 **E** 7 **F** 6 **G** 5 **H** 6 **I** 4

Think

4728
3524
6953 — 9999
9999 — 6475
3046
24726

22367

E-43 USING THE IDEAS

1. A $12x$ **B** $11x$ **C** $6x$ **D** $12x$ **E** $7x$ **F** $8x$ **G** x
H $9x$ **I** $18x$ **J** $21x$ **K** $23x$ **L** $8x$ **2. A** $3x$ **B** $3x$
C $2x$ **D** $6x$ **E** $9x$ **F** $4x$ **G** $8x$ **H** $8x$ **I** $3x$ **J** $4x$
K $10x$ **L** 0 **3. B** $4x$ **C** $8x$ **D** $7x$ **E** $2x$ **F** $8x$ **G** x
H x **I** $6x$ **J** x **K** x **4. A** 45 **B** 90 **C** 63 **D** 48
E 120 **F** 140 **G** 14 **H** 60 **I** 0 **J** 0 **K** 5 **L** 2
Think SPOMVEEMLATETHTFEURNS

E-45 USING THE IDEAS

1. A 3 **B** 6 **C** 7 **D** 7, 1 **E** 3, 4 **F** 27, 3 **G** 20, 4
H 35, 7 **I** 14, 2 **2. A** 2 **B** 4 **C** 4 **D** 5 **E** 5 **F** 2
G 3 **H** 5 **I** 2 **3. A** 3 **B** 3 **C** 4 **D** 6 **E** 5 **F** 3
G 5 **H** 5 **I** 6
Think 11, 15

E-47 USING THE IDEAS

1. A $(x + 10)4 = 56$ **B** $6y - 8 = 10$ **C** $5z + z = 18$
2. A 4 **B** 3 **C** 3 **3.** 8 **4.** 13 **5.** 13 **6.** 15 **7.** 7
8. A 33 **B** Beth 19; Tina 14 **9.** Dave 18; Chuck 15

Think

E-49 USING THE IDEAS

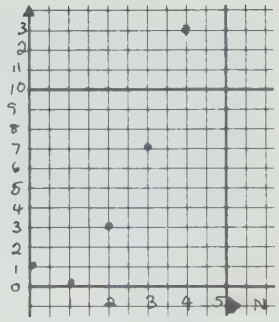
1. A $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$; $B = \{6, 7, 8, 9, 10\}$;
$A \cap B = \{6, 7\}$ **B** $C = \{3, 4, 5, 6, 7, 8, 9, 10\}$;
$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$; $C \cap D = \{3, 4, 5, 6, 7, 8\}$
C $E = \{0, 1, 2\}$; $F = \{8, 9, 10\}$; $E \cup F = \{0, 1, 2, 8, 9, 10\}$
D $G = \{6, 7, 8, 9, 10\}$; $H = \{2\}$;
$G \cup H = \{2, 6, 7, 8, 9, 10\}$ **2. A** $2 < x < 10$
B $12 < x < 19$ **C** $0 < x < 11$ **D** $a < x < b$ **E** $x \leq 9$
F $x \geq 3$ **3. A** $\{6, 7, 8, 9\}$ **B** $\{3, 4, 5, 6\}$
C $\{0, 1, 2, 8, 9, 10\}$ **D** $\{7, 8, 9, 10\}$
E $\{0, 1, 2, 3, 4\}$ **F** $\{6, 7, 8, 9, 10\}$
4. A $\{24, 25, 26, 27, 28, 29, 30\}$ **B** $\{4, 5, 6, \ldots, 18, 19\}$
C $\{4998, 4999, 5000, 5001\}$ **D** $\{1, 2, 3\}$
E $\{101, 102, \ldots, 998, 999\}$ **F** $\varnothing$
G $\{244, 245, \ldots, 341, 342\}$ **H** $\{577, 578, \ldots, 674, 675\}$
I $\{27, 28, \ldots, 80, 81\}$ **5. A** $\{9, 10, 11\}$
B $\{0, 1, 2, 3, 4, 5, 6, 7\}$ **C** $\{1, 2, 3, 4\}$ **D** $\{13\}$
E $\{3, 4, 5, 6, 7, 8, 9\}$ **F** $\{14\}$ **G** $\{8, 9, 10, 11, 12\}$
H $\varnothing$ **I** set of whole numbers

E-51 USING THE IDEAS

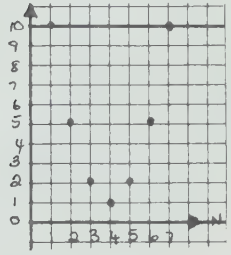
1. A 81 **B** 9 **C** 0 **2. A** 42 **B** 72 **C** 216 **3. A** 39
B 54 **C** 69 **4. A** 18 **B** 100 **C** 448 **D** 900 **5. A** 3
B 4 **C** 9 **D** 99 **6. A** 9 **B** 17 **C** 33 **D** 65
7. $10n - 1$
Think 55

E-53 USING THE IDEAS

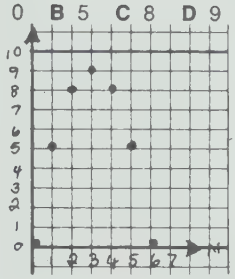
1. A 3 **B** 7 **C** 13



2. A 5 **B** 2 **C** 1 **E** 5



3. A 0 **B** 5 **C** 8 **D** 9 **E** 8 **F** 5 **G** 0



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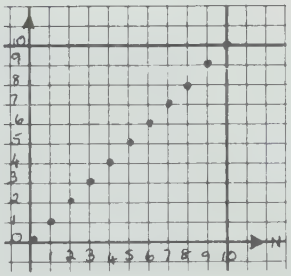

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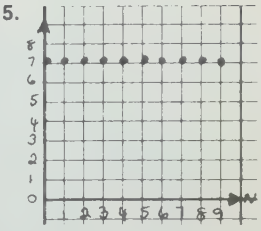

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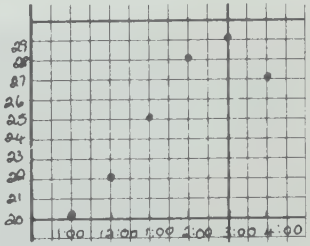
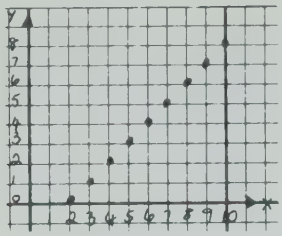


Think 4

E-55 USING THE IDEAS

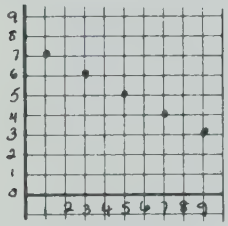
1. 6, 5, 4, 3, 2, 1, 0

2.



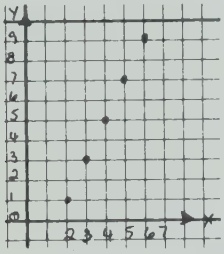
3. 5, 10; 4, 8; 3, 6

4.

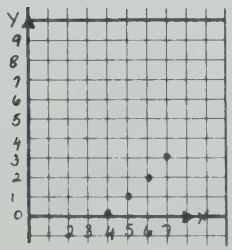


5. 5; 10, 7; 12, 9

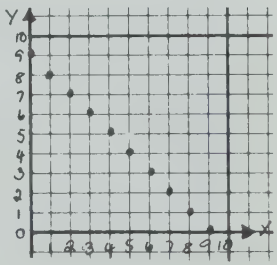
6.



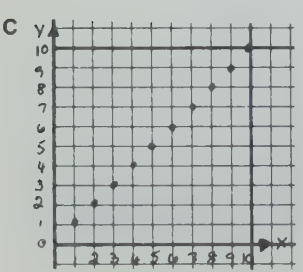
7. A



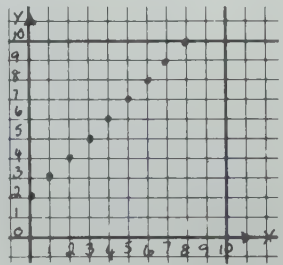
B



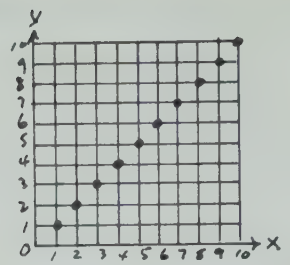
C



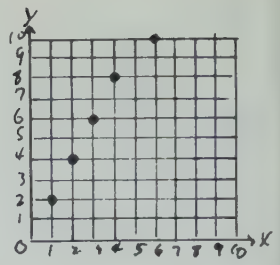
D



E



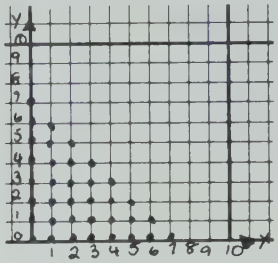
F



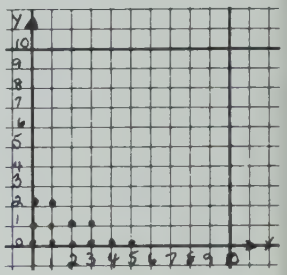
Think Chris: 3: Juanita: 6

E-57 USING THE IDEAS

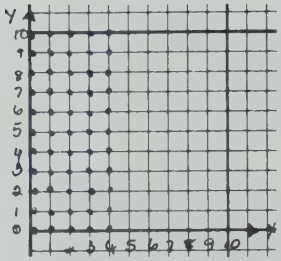
1. A



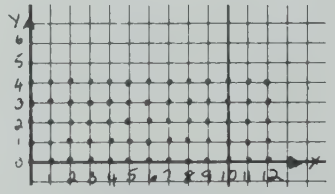
B



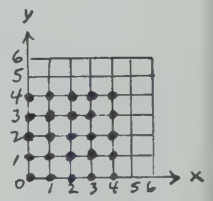
C



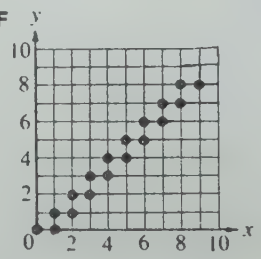
D



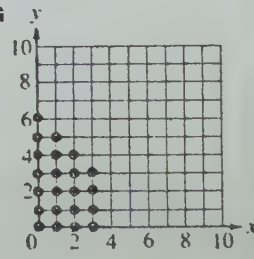
E



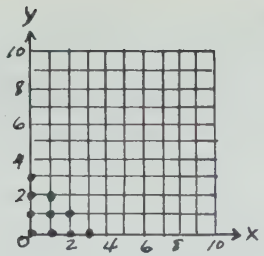
F



G

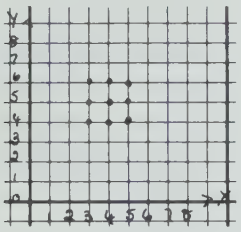


H

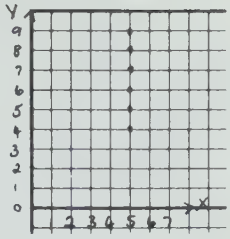


2. A $x + y \leq 1$ **B** $x < 3$

3.



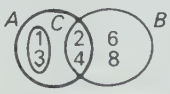
4.



E-58 REVIEWING THE IDEAS

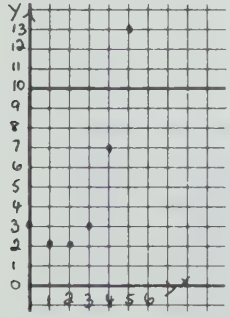
1. A yes **B** yes **C** no **D** yes **2. A** true **B** false **C** true **D** true **E** false **3. A** $\{1, 2, 3, 4, 6, 8\}$ **B** $\{2, 4\}$ **C** $\{1, 2, 3, 4\}$ **D** $\{1, 3\}$ **E** $\{1, 2, 3, 4, 6, 8\}$ **F** $\varnothing$

4.

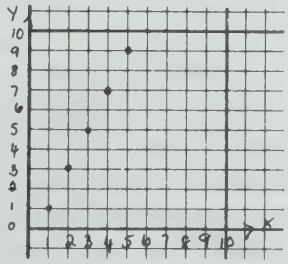


5. 6 **6.** 50 **7. A** $\{9\}$ **B** $\{8\}$ **C** $\{0, 1, 2, 3\}$ **D** $\{8, 9, 10\}$ **8.** yes **9. A** 7 **B** 3 **10. A** 11 **B** 9 **11. A** 11, 4 **B** 4, 5 **12.** $8x + 9 = 33$ **13.** 3 **14.** 7 **15. A** $\{6, 7\}$ **B** $\{0, 1, 10\}$ **C** $\{0, 1, 2, 3, 4, 6, 7, 8, 9, 10\}$ **D** $\{3, 4, 5, 6, 7, 8\}$ **16. A** 3 **B** 4 **C** 7 **D** 12

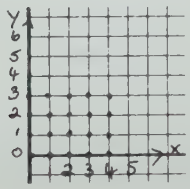
16.



17.



18.



UNIT E: MODULE 3

E-63 USING THE IDEAS

1. A 84 **B** 42 **C** 28 **D** 21 **E** 14 **F** 12 **2.** 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84 **3.** 4 **4.** 6 **5. A** 1, 2, 5, 7, 10, 14, 35, 70 **B** 1, 2, 3, 6, 11, 22, 33, 66 **C** 1, 2, 3, 4, 6, 8, 12, 16, 24, 48 **D** 1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, 150 **C** 1, 2, 3, 5, 6, 7, 10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 60, 70, 84, 140, 210, 420 **6. A** yes **B** no **C** yes **D** yes **E** no **F** yes **7. A** 28 **B** 496 **8.** 6, 10, 15, 30, 1 **9. A** 1, 3, 5, 7, 13, 15, 21, 35, 39, 65, 91, 105, 195, 273, 455, 1365 **B** 1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 21, 22, 30, 33, 35, 42, 55, 66, 70, 77, 105, 110, 154, 165, 210, 231, 330, 385, 462, 770, 1155, 2310

Think no

E-65 USING THE IDEAS

2. 4 **3. A** 3 **B** 6 – 3, 2 **C** 9 **D** 5 **4. B** 25

E-67 USING THE IDEAS

1. A 79 **B** 179 **C** 379 **D** 479 **2.** no; 1, 3, 9, 31, 93, 279 **3. A** 64 **B** 89 **C** 5 **D** 120 **E** 198 **F** 167 **4.** 13, 113 **5.** 101, 401, 601, 701, 1201 **6. A** 3 and 5; 5 and 7; 11 and 13; 17 and 19; 29 and 31; 41 and 43; 59 and 61; 71 and 73 **B** 1019 and 1021; 1031 and 1033; 1049 and 1051; 1061 and 1063; 1091 and 1093 **C** $1217 - 2 = 1215$; $5.243 = 1215$ and $1217 + 2 = 1219$; $25.53 = 1219$ **7.** 3, 5, 7 **8.** 5, 7, 11 **9. A** 17, 23, 31, 41, 53, 67, 83, 101, 121, 143 **B** No, 121 and 143 are not prime **10.** 17, 19, 23, 29, 37, 47, 59, 73, 89, 107, 127, 149, 173, 199, 227, 257, 289, 323, 359, 397; No, 289 and 323 are not prime numbers.

Think 1, 4, 9, 16, 25, 36, 49; perfect squares

E-69 USING THE IDEAS

1. A $5 \times 5 \times 3$ **B** $2 \times 5 \times 2 \times 2$ **C** $5 \times 5 \times 3 \times 3$ **2. A** 3×5 **B** $2 \times 2 \times 3$ **C** 2×11 **D** 3×13 **E** $2 \times 5 \times 5$ **F** $2 \times 5 \times 7$ **G** $3 \times 3 \times 5$ **H** $2 \times 2 \times 2 \times 2$ **I** $3 \times 3 \times 3 \times 3 \times 3$ **J** $2 \times 2 \times 3 \times 5$ **K** $2 \times 3 \times 11$ **L** $2 \times 2 \times 5 \times 5$ **3. A** $2 \times 7 \times 7$ **B** $3 \times 3 \times 7$ **C** $2 \times 3 \times 2 \times 5$ **D** $2 \times 2 \times 3 \times 13$ **E** $2 \times 2 \times 7 \times 7$ **F** $2 \times 3 \times 5 \times 5$ **G** $11 \times 2 \times 2 \times 3$ **H** $3 \times 3 \times 2 \times 2 \times 2$ **I** $3 \times 5 \times 2 \times 2 \times 2 \times 2$ **J** $3 \times 3 \times 3 \times 7$ **K** $2 \times 2 \times 2 \times 2 \times 3 \times 3$ **L** $2 \times 5 \times 7 \times 7$ **4. A** $2^3 \cdot 3$ **B** $2^3 \cdot 3^2$ **C** $2 \cdot 5^3$ **D** 3^4 **E** $2^3 \cdot 3^3$ **F** 10^3 **G** $2^3 \cdot 3 \cdot 5$ **H** $2^4 \cdot 3$ **I** $2^2 \cdot 3^3$ **J** $2^2 \cdot 11^2$ **K** $2^5 \cdot 3^2$ **L** $3^2 \cdot 5^2$ **M** $3^2 \cdot 7^2$ **N** $2^2 \cdot 3^2 \cdot 17$ **O** $3^2 \cdot 11^2$ **5. A** 8 **B** 8 **C** 9 **D** 12 **E** 16 **F** 8 **6. A** $\{1, 2, 3, 4, 6, 8, 12, 24\}$ **B** $\{1, 2, 3, 5, 6, 10, 15, 30\}$ **C** $\{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ **D** $\{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$ **E** $\{1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 120, 125, 250, 500, 1000\}$ **F** $\{1, 7, 11, 13, 77, 91, 143, 1001\}$

Think

1.

2.

E-71 USING THE IDEAS

1. A {1, 2, 3, 4, 6, 12} **B** {1, 2, 3, 6, 9, 18}
C {1, 2, 3, 6} **D** 6 **2. A** {1, 2, 3, 4, 6, 9, 12, 18, 36}
B {1, 2, 3, 6, 9, 18, 27, 54} **C** {1, 2, 3, 6, 9, 18} **D** 18
3. A 4 **B** 2 **C** 7 **D** 15 **E** 4 **F** 10 **G** 7 **H** 24
I 17 **J** 10 **4. A** 5 **B** 8 **C** 3 **D** 24 **E** 14 **F** 24
G 22 **5. A** relatively prime **B** 3 **C** relatively prime
D relatively prime **E** relatively prime **F** 3 **G** relatively prime **H** relatively prime **I** 3 **J** 13 **6. A** 15 **B** 2
C 14 **D** 66 **E** 90 **F** 315
Think 3

E-73 USING THE IDEAS

1. A 14 **B** 23 **C** 23 **D** 27 **E** 35 **F** 64 **G** 16
H 37 **I** 113 **J** 19 **K** 56 **L** 213 **3. A** 1 **B** 0 **4.** no, 11 **5. A** 32 **B** 15
Think yes

20

E-75 USING THE IDEAS

1 A 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66 **B** 20, 40, 60
C 60 **D** 60 **2. A** 12 **B** 30 **C** 36 **D** 30 **E** 56
F 40 **G** 40 **H** 80 **I** 42 **J** 120 **3. A** 60 **B** 12
C 36 **D** 60 **E** 24 **F** 12 **G** 120 **H** 180 **I** 12
J 60 **K** 36 **L** 120 **4. A** 28 **B** 100 **C** 210 **D** 72
E 165 **F** 180 **G** 108 **H** 432
Think **1.** 20 hrs **2.** 1200 km

E-77 USING THE IDEAS

1. A 12, 20, 24, 30, 48, 70 **B** 15, 20, 30, 35, 70
C 20, 30, 70 **2. A** true **B** false **C** true **D** true
3. A 6, yes **B** 5, no **C** 12, yes **D** 18, yes **E** 3, yes
F 16, no **G** 27, yes **4.** *A*, *B*, *C*, *E*, *F*, *G*, *I* **5.** *A*, *D*, *F*, *G*, *H*, *I*, *J*, *K*, *L* **6. A** 3×37 **B** $2 \times 3 \times 19$ **C** 3×89
D $3 \times 5 \times 13$ **E** $3 \times 2 \times 5 \times 367$

E-79 USING THE IDEAS

1. *A*, *B*, *E*, *F*, *G*, *H*, *I*, *J* **2.** *A*, *B*, *D*, *E*, *F*, *G*, *I*, *J*, *K*, *L*
3. A yes **B** yes **C** yes **4.** *A*, *B*, *D*, *E* **5.** *A*, *D*, *E*, *G*, *I*
6. *C*, *E*, *F*, *G*, *H*
Think 2520

E-81 USING THE IDEAS

1. A 11, 13, 15 **B** 10 000; 100 000; 1 000 000
C 32, 64, 128 **D** 23, 27, 31 **2. A** 15, 19, 23
B 10, 15, 21 **C** 16, 25, 36 **D** 11, 18, 27
3. 2, 5, 8, 11, 14 **4.** 1, 0, 1, 0, 1 **5. A** $3n$ **B** $n + 1$
C $n(n + 1)$ **D** n^2 **6. A** 34, 55, 89, 144
B $(n - 2)^{th}$ term + $(n - 1)^{th}$ term
Think 2, 4, 8, 15, 26, 64

E-82 SQUARE NUMBERS

1. 16, 25, 36, 49 **2.** $1 + 3 + 5 + 7 = 16$;
$1 + 3 + 5 + 7 + 9 = 25$; $1 + 3 + 5 + 7 + 9 + 11 = 36$
3. A 9 **B** 16 **C** 25 **D** 36 **E** 100 **F** 2500 **4. A** *D*: $1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$; *E*: $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 5^2$; **B** $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 8^2$ **C** $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 10^2$ **5. A** 6^2
B 7^2 **C** 9^2

E-83 TRIANGULAR NUMBERS

1. 10, 15, 21, 28 **2.** 15, 21, 28, 36, 45, 55, 66, 78, 91, 105 **3.** $15 = 1 + 2 + 3 + 4 + 5$; $21 = 1 + 2 + 3 + 4 + 5 + 6$; $28 = 1 + 2 + 3 + 4 + 5 + 6 + 7$; $36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$ **4. A** $10 + 15 = 25$; $15 + 21 = 36$; $21 + 28 = 49$
B $45 + 55 = 100$ **5.** 28; 55; 210; 5050; 500 500

E-85 USING THE IDEAS

1. A 22 **B** 76 **C** 148 **D** 244 **E** 313 **F** 2998
2. A yes **B** no **C** yes **D** yes **E** no **F** no **G** no
H no **I** no **J** yes **3. A** {1, 4, 7, 28} **B** {1, 4, 16}
C {1, 10} **D** {1, 34} **E** {1, 7, 10, 70}
F {1, 4, 19, 76} **G** {1, 25} **H** {1, 4, 10, 25, 100}
I {1, 4, 13, 52} **J** {1, 7, 13, 91} **4. A** yes **B** yes
C yes **D** no **E** no **F** yes **5.** 4 **6. A** $4 \cdot 4$ **B** $4 \cdot 22$
C $7 \cdot 10$ **D** $13 \cdot 4$ **E** $7 \cdot 7$ **7.** $10 \cdot 10$; $4 \cdot 25$

E-86 REVIEWING THE IDEAS

1. A {1, 2, 3, 6, 9, 18} **B** {1, 2, 3, 5, 6, 10, 15, 30}
C {1, 2, 11, 22} **D** {1, 2, 5, 10, 25, 50}
E {1, 2, 4, 8, 16} **F** {1, 2, 4, 7, 14, 28}
G {1, 3, 5, 15, 25, 75} **H** {1, 2, 4, 8, 16, 32, 64, 128}
2. 4 **3.** 97 **4.** 69 **5. A** 2×3^3 **B** $2^3 \times 5$ **C** $2^2 \times 5$
D 3^4 **E** $2 \times 3 \times 5 \times 7$ **F** $2^2 \times 5 \times 11$ **G** $2^2 \times 71$
H $2^2 \times 5^3$ **I** $2^2 \times 3^2 \times 5^2$ **6. A** 19 **C** 41 **E** 29 **F** 61
H 73 **7. A** 8 **B** 5 **C** 3 **D** 7 **E** 22 **F** 15 **G** 1
H 13 **I** 24 **8. A** 21 **B** 37 **C** 43 **D** 47 **9. A** 6, 12, 18, 24, 30, 36, 42, 48, 54, 60 **B** 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 **C** 30, 60, 90, 120 **D** 30 **10. A** 24 **B** 40
C 36 **D** 42 **E** 60 **F** 20 **G** 77 **H** 36 **I** $3^2 \cdot 5^2$
J $5^2 \cdot 7^2$ **K** $2^2 \cdot 3 \cdot 5$ **L** $7^2 \cdot 11^2 \cdot 17$ **11. A** 12 **B** 24
C 120 **D** 105 **E** 36 **F** 60 **G** 60 **H** 54 **I** 180
12. A 54, 66, 70, 222 **B** 54, 66, 111, 222 **C** 54, 66, 222
D 65, 70 **13. A** yes **B** no **C** yes **D** no **14.** if sum of digits is divisible by 9 **15.** *B*, *C*, *D* **16. A** 26, 31, 36, 41
B 81, 243, 729, 2187 **17.** 5, 8, 11, 14 **18.** triangular; prime; none; square; triangular

UNIT E: MODULE 4

E-91 USING THE IDEAS

1. A −3 **B** −7 **C** −10 **D** −100 **E** 4 **F** 8 **G** 11
H 99 **I** 89, 376 **D** −10 000 **K** 1 000 000
L −10 000 001 **2.** −87 **3.** −86 **4. A** positive
B negative **C** negative **D** negative **E** positive
F negative **G** negative **H** positive **I** negative
J zero **5. A** negative **B** positive **C** negative
D positive **E** zero **6.** −392

E-93 USING THE IDEAS

1. A 9 **B** 0 **C** 3 **D** −30 **E** −25 **2. A** 3 **B** −2
C −70 **D** 20 **E** −600 **3. A** −3 **B** −1 **C** −3
D −3 **E** −2 **F** −7 **G** −1 **H** −6 **I** −8 **J** −3
4. A 18 **B** −7 **C** −33 **D** 37 **E** 4 **F** 2 **G** 0 **H** 0
I 2 **5. A** −10 **B** -10^0 **C** $5.00 **D** 22 km south

E-95 USING THE IDEAS

1. A 0, 0 **B** 9, 9 **C** 0, 0 **D** 0, 0 **2. A** 4, 4 **B** 5, 5
C −4, −4 **D** −5, −5 **E** −2, −2 **F** −7, −7 **G** 7, 7
H 9, 9 **I** −10, −10 **J** −5, −5 **K** −8, −8 **L** −8, −8

3. A 1 **B** 2 **C** −12 **D** −3 **E** −1 **F** 5 **G** 3 **H** 4
I −17 **J** −3 **K** −3 **L** −2 **M** −7 **N** −1 **O** 7
P 8 **Q** 11 **R** −8 **S** −13 **T** −36 **U** −26 **V** −200
W 161 **X** −1103 **4. A** 37 **B** −14 **C** 2 **D** 0 **E** 3
F −1 **5. A** −1 **B** −11 **C** −9 **D** −9 **E** −4 **F** 20
G 31 **H** −20 **6. A** 7 **B** −8 **C** 0 **D** −8 **E** 9
F −2 **G** −8 **H** −7 **I** 6

Think

E-97 USING THE IDEAS

1. A −4 **B** 5 **C** −9 **D** 7 **E** −9 **F** 4 **G** −6
H −10 **I** 6 **J** −10 **K** −9 **L** −5 **M** −11 **N** 5
O −8 **P** 12 **Q** −4 **R** −12 **S** 4 **T** 0 **2. A** 5, 6
B 5, 7 **C** 10, 11 **D** −6, −4 **E** −6, −3 **F** 8, 11
3. A 11 **B** 9 **C** 12 **D** 10 **E** −5 **F** −10 **G** −7
H −11 **I** −4 **J** −1 **K** −1 **L** −6 **4. A** 15 **B** −15
C 21 **D** −21 **E** 14 **F** −14 **G** 15 **H** −24 **I** −18
J −13 **K** −63 **L** 39 **5. A** 10 **B** 16 **C** 10

Think *D*: 30; *E*: 55 **2.** 204

E-99 USING THE IDEAS

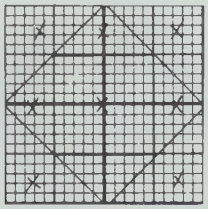
1. A 3 **B** 2 **C** 5 **D** −3 **E** −7 **F** −4 **G** −1
H 6 **I** 4 **J** −15 **K** 6 **L** 4 **2. A** 3 **B** −10 **C** 5
D −11 **E** 20 **F** −16 **G** −10 **H** −5 **I** 6 **J** 0
K −143 **L** −22 **3. A** −5 **B** 5 **C** −6 **D** 2 **E** −4
F −4 **G** 8 **H** 1 **I** −16 **J** 3 **K** 2 **L** 12 **4. A** 4
B 13 **C** 3 **D** −6 **E** −4 **F** −8 **G** −1 **H** 14 **I** −17

Think Check *AO* bag. If *A* then *AA*, and *OO*, *AO*, *AA*. If *O* then *AO*, *AA*, *OO*.

E-101 USING THE IDEAS

1. B 5 · −3 = −15 **C** −5 · −3 = 15 **D** −10 · 4 = −40
E 10 · −4 = −40 **F** −10 · −4 = 40 **G** 15 · 2 = 30
H −15 · −4 = 60 **I** 12 · −5 = −60 **2. A** −12 **B** −16
C −8 **D** 10 **E** −24 **F** 16 **G** −70 **H** 18 **I** −49
J −90 **K** −40 **L** −10 **3. A** 12 **B** 8 **C** 4 **D** 0
E −4 **F** −8 **G** −12 **H** −12 **I** −8 **J** −4 **K** 0
L 4 **M** 8 **N** 12

Think



E-103 USING THE IDEAS

1. A 19, 19 **B** −8, −8 **C** −24, −24 **D** 86, 86
2. A −12, −12 **B** −18, −18 **C** −36, −36 **D** −2, −2
E −126, −126 **F** −12, −12 **3. A** −16, −16
B −3, −3 **C** −84, −84 **D** −27, −27 **E** −384, −384
F −7, −7 **G** −48, −48 **H** −360, −360
4. A 0, 0, 0, 0, −12 **B** 0, 0, 0, 0, −10 **C** 0, 0, 0, 0, 0
5. A −72 **B** −42 **C** 45 **D** 0 **E** −161 **F** −448
G 686 **H** 1200 **I** −972 **J** −2436 **K** 2813 **L** 0
M −920 **N** −4800 **O** 62000 **P** 10000 **Q** 0
R −125 **S** 256 **T** −360 **6. A** negative **B** positive
C zero

E-105 USING THE IDEAS

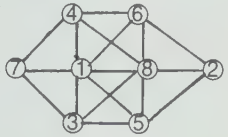
1. A −3 **B** 3 **C** −1 **D** −15 **E** 0 **F** −5 **G** 13
H −8 **I** 0 **J** 86 **K** −7 **L** −3 **M** −8 **N** −2
O −2 **2. B** 4 **C** −6 **D** −1 **E** 3 **F** −5 **G** −9
H −1 **I** 0 **J** −8 **K** −1 **L** −6 **M** −24 **3. A** −3
B −6 **C** 2 **D** −4 **E** −7 **F** −7 **G** −4 **H** −4
I −3 **J** −1 **4. A** positive **B** negative **C** positive

Think b = 9, a = 10

E-107 USING THE IDEAS

1. A −11 **B** 9 **C** −3 **D** 7 **E** 15 **2. A** −10 **B** 6
C −28 **D** 8 **E** 0 **3. A** −17 **B** 5 **C** −13 **D** 15
E 95 **4. A** 5 **B** 5 **C** 26 **D** 26 **E** 2 **5. A** 8 **B** −1
C 64 **D** −27 **E** −2 **6. A** −2 **B** 4 **C** 64 **D** 3
E 5 **7. A** 5 **B** −60 **C** 72 **D** −16 **E** 27 **F** −9
G 25 **H** −46 **I** −3 **8. A** −5 **B** −92 **C** 6 **D** −4
E 14 **F** −1 **G** −4

Think



E-109 USING THE IDEAS

1. A 2 **B** 4 **C** 7 **D** 98 **2. A** > **B** > **C** >
D < **E** > **F** < **G** > **H** > **I** < **J** < **K** > **L** >
3. A −10 **B** 10 **C** −1, −2, . . . , −10
D 2, 1, 0, −1, −2 **E** −9, −10 **F** −3, −2, −1 **G** −3
H −8, −9, −10, 8, 9, 10 **4. A** less **B** greater **C** less
D greater **E** less

Think

6	−8	−7	3
−5	1	0	−2
−1	−3	−4	2
−6	4	5	−9

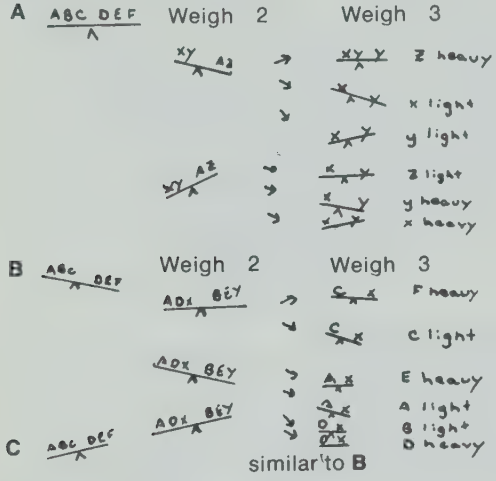
E-111 USING THE IDEAS

1. A {−2, −1, 0, 1, 2, 3, 4} **B** {−4, −3, −2, −1, 0, 1}
C {−2, −1, 0, 1} **D** {−4, −3, 4}
E {−2, −1, 0, 1, 2, 3, 4} **F** {−4, −3, −2, −1, 0, 1}
2. A {−3, −2, −1, 0, 1} **B** {−3, −2}
C {−2, −1, 0, 1, 2, 3, 4, 5} **D** {−3, −2, −1}
E {−3, −2, −1, 0} **F** {−3, −2, −1} **G** {1, 2, 3, 4, 5}
H {−3, −2, −1} **I** {−1, 0, 1, 2, 3, 4, 5} **J** {−2}
K {0, 1, 2} **L** {0, 1} **M** {−3, 3, 4, 5}
N {−3, −2, 2, 3, 4, 5} **O** {−3, 3, 4, 5}
P {0} **Q** {−2, −3} **R** ∅ **3. A** {8, 9, 10}
B {−8, −9, −10} **C** {−10}
D {−4, −3, −2, −1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
E {−10, −9, −8, −7, −6, −5, −4}
F {−10, −9, −8, −7, −6, −5, −4, −3}
G {−4, −5, −6, −7, −8, −9, −10}
H {−10, −9, −8, −7, −6} **I** {5, 6, 7, 8, 9, 10}

J $\{-2, -1, 0, 1, 2\}$
K $\{-2, -3, -4, -5, -6, -7, -8, -9, -10\}$
L $\{-3, 4, -4, 5, -5, 6, -6, 7, -7, 8, -8, 9, -9, 10, -10\}$
Think $19 \times 3 = 55 + 2$

E-113 USING THE IDEAS

1. A 8, 2, 20, 53, 3, 16 **B** 4, 2, 2, 3, 4, 6
C 1, 1, 6, 2, 2, 6 **2. A** 5 **B** 5 **C** 18 **D** 2 **E** 47 **F** 4
G 4 **H** 3 **I** 6 **J** 20 **3. A** 5 **B** 6 **C** 5 **D** 5 **E** 8
F 19 **G** 9 **H** 4 **4. A** 3, −3 **B** 8, −8 **C** 4, −4
D 10, −10 **E** 5, −5 **F** 3, −5 **G** 4, −2 **H** 3, −3
5. A −7 **B** 5 **C** −3 **D** 2 **E** −2 **F** −13
Think label coins *A*, *B*, *C*, *D*, *E*, *F*, *x*, *y*, *z*
Weigh 1



E-115 USING THE IDEAS

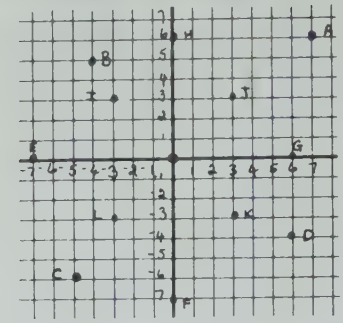
1. A 8 **B** 12 **C** 8 **D** 12 **E** 15 **F** 2 **G** 49
H 100 **I** 14 **J** 14 **K** 15 **L** 11 **2. A** 0, 12 **B** 0, 8
C 0, −4 **D** 0, −18 **E** 1, 11 **F** 1, 7 **G** −1, −3
H −1, −19 **3. A** = **B** > **C** = **D** > **E** = **F** >
4. A $\{-10, -9, 9, 10\}$ **B** $\{-3, -2, -1, 0, 1, 2, 3\}$
C $\{0, 1, 2, 3, 4\}$
D $\{-10, -9, -8, -7, -6, -5, -4, -3, 9, 10\}$
Think **1.** *B* **2.** *A*: \$8800; *B*: \$8900 **3.** *A*: \$16 400; *B*: \$16 600 **4.** *A*

E-117 USING THE IDEAS

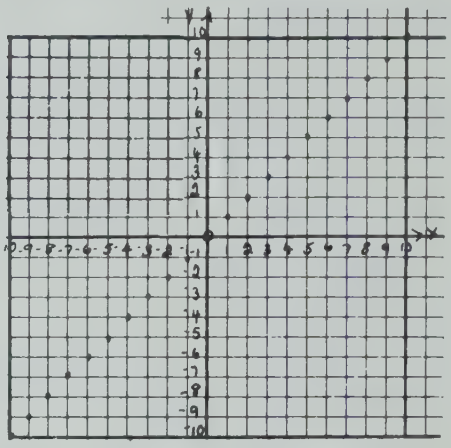
1. A −3 **B** 8 **C** −5 **D** −8 **E** −10 **F** 4
2. A −9, −3 **B** −6, −3 **C** −4, −1 **3. A** 10, 5
B −6, −2 **C** 8, −4 **D** −3, −6 **E** 3, −1 **4. A** 2
B −4 **C** −9 **D** 10 **E** −3 **F** 5 **G** −8 **H** 5 **I** −2
5. B 12, −12, 4, −4 **C** −5, 5, −8, 2
Think INTEGERS; ADDITION

E-119 USING THE IDEAS

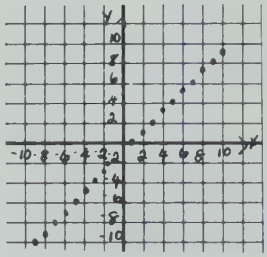
1.



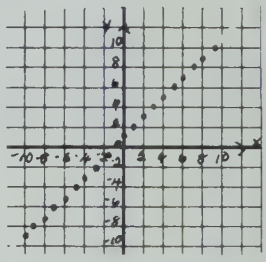
2.



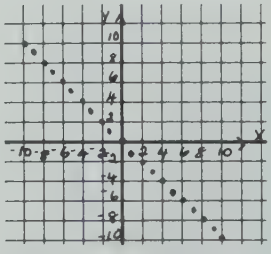
3.



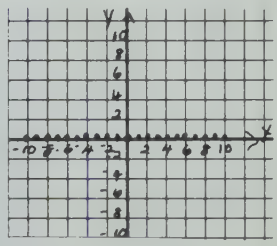
4.



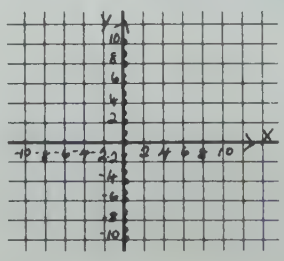
5.



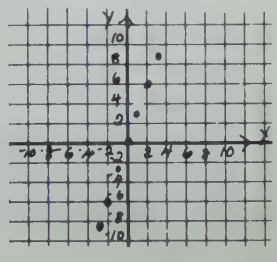
6.



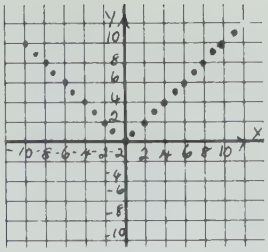
7.



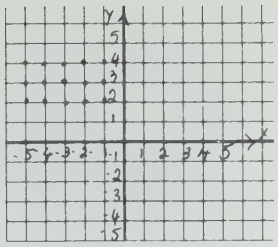
8.



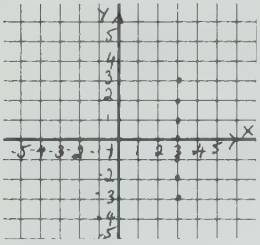
9.



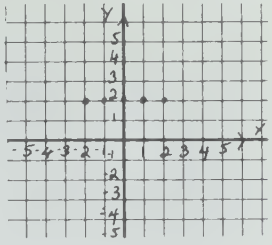
10. A



B

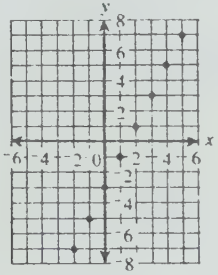


C

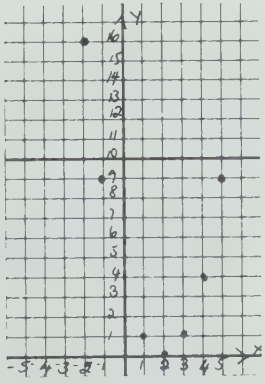


E-121 USING THE IDEAS

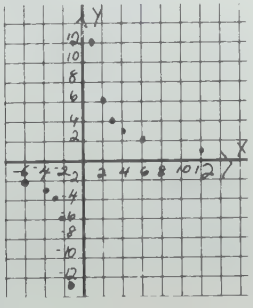
1. A −3, −1, 1, 3, 5, 7



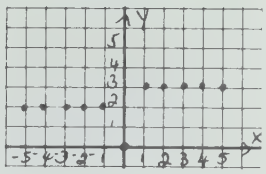
B 4, 1, 0, 1, 4, 9, 16



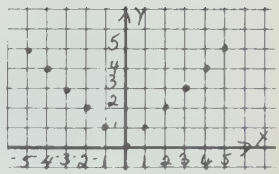
C 2, 3, 4, 6, 12, −12, −6, −4, −3, −2



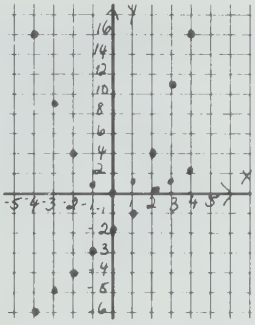
2.



3.



4. A



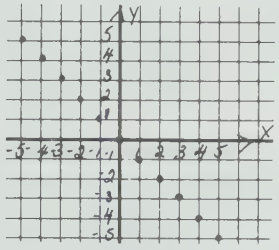
Think

256	2	64
8	32	128
16	512	4

E-123 USING THE IDEAS

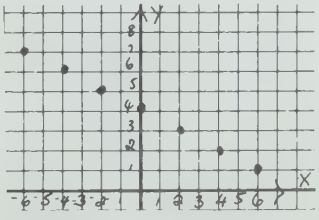
1. −4, 3, −3, 2, 1, −1, 0

2.



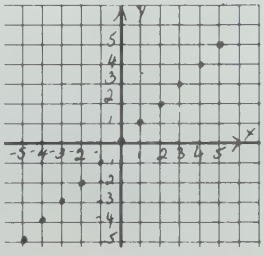
3. 2, 1, 6, 7

4.



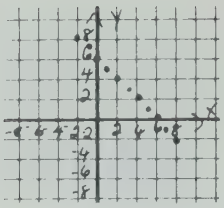
5. Answers will vary.

6.

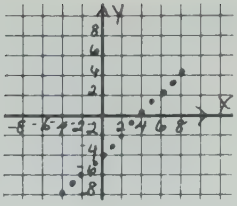


7. $y = x + 2$

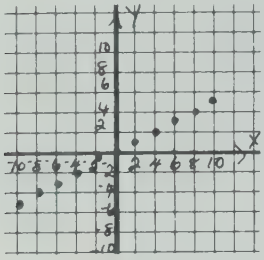
8. **A**



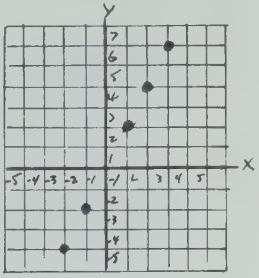
B



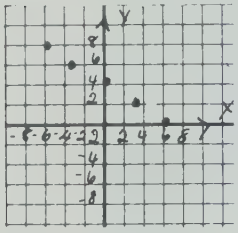
C



D



E



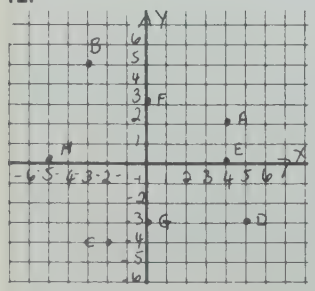
Think

L	I	K	E
L	A	K	E
M	A	K	E
M	A	T	E
M	A	T	H

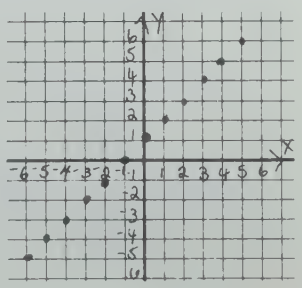
E-124 REVIEWING THE IDEAS

1. A 5 **B** −8 **C** 12 **2.** zero **3. A** 9 **B** −7 **C** −31
4. A 2 **B** −2 **C** 10 **D** 0 **5. A** 16 (zero)
B −9 (one) **C** −8 (opposite) **D** 17 (commutative +)
E 8 (associative +) **F** 2 (distributive)
G −20 (associative ×) **6. A** 42 **B** −42 **C** 42
7. A −9 **B** −6 **C** 8 **D** 0 **8. A** $\{-5, -4, -3, -2\}$
B $\{0, 1, 2, 3, 4, 5\}$ **C** $\{-1, 0, 1, 2\}$ **D** $\{4, 5, -5\}$
9. A 9 **B** 9 **10. A** 13 **B** 27 **C** 6 **D** 3 **E** 3, 1
F 2, 6 **11. A** 7, 2 **B** −6, 2 **C** 3, 5 **D** 1, −4
E −8, −5

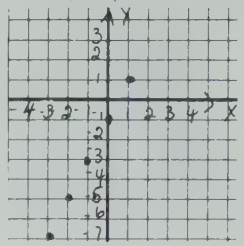
12.



13.



14. A −5, −1, 1

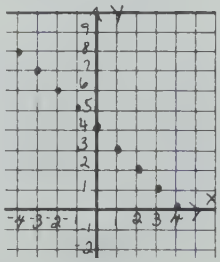


B 0, −3, −4, −3

15. A *b* **B** *a* **C** *b* **D** *b*

E-126 CUMULATIVE REVIEW

1. A MDCCCXCIII **B** 1617 **2.** 7^4 **3. A** 10^6 **B** 2^7
C 10^2 **D** 1 **4. A** 5 004 039 **B** $(7 \cdot 10^7) + (8 \cdot 10^6) + (4 \cdot 10^5) + (6 \cdot 10^4) + (3 \cdot 10^3) + (0 \cdot 10^2) + (9 \cdot 10^1) + 7$
5. A 747 000 **B** 750 000 **C** 700 000 **6.** 32 000
7. 700 **8. A** 1176 **B** 8424 **C** 631 896 **D** 1795, R4
9. A 101 **B** 11001 **C** 1001101 **10. A** 1111 **B** 110
C 1111 **11. A** $\{1, 3, 5, 7, 9\}$ **B** $\{0, 2, 4, 6, 8, 10\}$ **C** *X*
12. *B* **13.** 22 **14. A** $\{4\}$ **B** $\{0, 1, 2, 3\}$ **C** $\{3, 4, 5\}$
D $\{3, 4, 5\}$ **15. A** $7x$ **B** $15x$ **C** $12x$ **D** $10x$
16. $6x - 4 = 26$ **17.** 5 **18. A** $\{15, 16, 17, 18\}$
B $\{3, 5\}$ **C** $\{3, 6, 9\}$ **D** $\{1, 3\}$ **19. A** 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 **B** 4, 6, 8, 9, 10, 12, 14, 15, 16, 18
20. A $2 \times 7 \times 7$ **B** $2 \times 3 \times 17$ **C** $5 \times 5 \times 5 \times 5$
21. A 2 **B** 10 **22. A** 40 **B** 105 **23. A** 3, 5
B 2, 3, 5, 10 **C** 2, 3 **D** 3, 5, 9 **24.** 4, 7, 12, 19, 28
25. A 46 **B** −97 **C** 1000 **26. A** 18 **B** −8
27. A −6 **B** −13 **C** 12 **D** 25 **28. A** −32 **B** 45
C 0 **29. A** −5 **B** −9 **C** 50 **D** −3 **30. A** 3
B −6 **C** −225 **D** −4
31. A −1, −8, 10, 0 **B** −2, 10, 0 or 3, 40 **32. A** >
B < **C** < **D** > **33. A** $\{4, 6\}$
B $\{-4, -2, 0, 2, 4, 6\}$ **C** $\{-4, -2\}$ **D** $\{-4, -2\}$
34. A 4 **B** 12 **C** 68 **D** 68 **E** 0 **F** 1 **35. A** 0, 7
B 1, 5 **36. A** 2, 1, 5, 6, 7, 8
B



UNIT F: MODULE 1

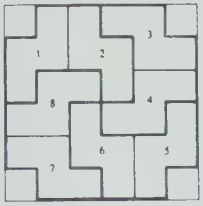
F-3 USING THE IDEAS

1. $\frac{2}{5}, \frac{4}{10}, \frac{6}{15}$, or $\frac{8}{20}$ **2. A** $\frac{4}{16}, \frac{5}{20}, \frac{6}{24}$ **B** $\frac{8}{12}, \frac{10}{15}, \frac{12}{18}$ **C** $\frac{12}{8}, \frac{15}{10}, \frac{18}{12}$
D $\frac{16}{20}, \frac{20}{25}, \frac{24}{30}$ **E** $\frac{20}{28}, \frac{25}{35}, \frac{30}{42}$ **F** $\frac{36}{32}, \frac{45}{40}, \frac{54}{48}$ **3. A** $\frac{2}{10}, \frac{3}{15}$ **B** $\frac{8}{6}, \frac{12}{9}$
C. $\frac{10}{16}, \frac{15}{24}$ **D** $\frac{6}{20}, \frac{9}{30}$ **E** $\frac{10}{6}, \frac{15}{9}$ **F** $\frac{2}{16}, \frac{3}{24}$ **G** $\frac{14}{16}, \frac{21}{24}$ **H** $\frac{10}{24}, \frac{15}{36}$
I $\frac{10}{12}, \frac{15}{18}$ **J** $\frac{22}{20}, \frac{33}{30}$ **K** $\frac{14}{32}, \frac{21}{48}$ **L** $\frac{14}{40}, \frac{21}{60}$ **4. A** 6, $\frac{2}{3}$ **B** 4, $\frac{2}{3}$
C 24, 1 **D** 7, $\frac{6}{7}$ **E** 2, $\frac{5}{2}$ **F** 3, $\frac{2}{3}$ **G** 4, $\frac{5}{3}$ **H** 5, $\frac{2}{3}$ **I** 5, $\frac{5}{8}$
J 10, 0 **K** 7, $\frac{9}{10}$ **L** 15, $\frac{3}{4}$ **M** 16, $\frac{3}{4}$ **N** 13, $\frac{3}{4}$ **O** 12, 1

5. **A** $\frac{16}{24}$ **B** $\frac{12}{24}$ **C** $\frac{10}{24}$ **D** $\frac{9}{24}$ **E** $\frac{6}{24}$ **F** $\frac{20}{24}$ **G** $\frac{21}{24}$
6. **A** improper **B** proper **C** proper **D** improper **E** proper **F** improper
Think 622

F-5 USING THE IDEAS

1. **A** $\frac{3}{10}$ **B** $\frac{1}{2}$ **C** $\frac{9}{10}$ **D** $\frac{2}{5}$ **E** $\frac{4}{5}$ **2.** **A** 45, 45, yes **B** 24, 24, yes **C** 120, 150, no **3.** **A** yes **B** no **C** yes **D** yes **E** yes **F** no **G** yes **H** yes **4.** **A** 3 **B** 15 **C** 45 **D** 8 **E** 12 **F** 1 **G** 25 **H** 60 **I** 2 **J** 66 **K** 135 **L** 125
Think



F-7 USING THE IDEAS

1. **A** $-\frac{3}{8}$ **B** $\frac{5}{6}$ **C** $\frac{1}{4}$ **D** $-\frac{11}{5}$ **C** $\frac{37}{45}$ **2.** 0 **3.** **A** $\frac{6}{5}$ **B** $-\frac{7}{5}$ **C** $\frac{4}{5}$ **D** $-\frac{4}{5}$ **E** $-\frac{2}{5}$ **F** $\frac{1}{5}$ **G** $-\frac{9}{5}$ **4.** **A** $-\frac{3}{4}$ **B** $\frac{5}{8}$ **C** $-\frac{11}{4}$ **D** $\frac{7}{8}$ **E** $\frac{11}{6}$ **F** $-\frac{8}{3}$ **G** -2 **H** $-1\frac{3}{4}$ **I** $\frac{2}{3}$ **J** 0 **K** $\frac{5}{4}$ **L** 0 **M** 4 **N** 0 **O** $-\frac{5}{2}$ **5.** $-\frac{4}{8}, -\frac{5}{10}, -\frac{6}{12} \ldots$
6. **A** negative **B** positive **C** zero
Think 19 units minimum

F-9 USING THE IDEAS

1. **A** $-\frac{4}{3}$ **B** $\frac{5}{6}$ **C** $-\frac{3}{5}$ **D** $\frac{6}{10}$ **E** $\frac{9}{9}$ **2.** **A** T **B** F **C** F **D** T **E** T **3.** **A** *P* **B** *Q* **C** *P* **D** *Q* **E** *Q* **F** *Q* **4.** **A** *S* **B** *R* **C** *U* **D** *S* **E** *S* **F** *R* **G** *T* **H** *T* **I** *R* **J** *S* **K** *S* **L** *U*
Think 3

F-11 USING THE IDEAS

1. **A** $\frac{5}{6}$ **B** $\frac{1}{10}$ **C** $-\frac{1}{8}$ **D** $-\frac{5}{4}$ **E** 1 **F** $\frac{2}{10}$ **G** $-\frac{2}{100}$ **H** $-\frac{12}{16}$ **I** $-\frac{4}{8}$ **J** $\frac{4}{5}$ **K** $\frac{10}{5}$ **L** $-\frac{10}{9}$ **2.** **A** $-\frac{1}{7}$ **B** $\frac{2}{4}$ **C** $-\frac{7}{10}$ **D** $-\frac{5}{8}$ **E** $\frac{4}{6}$ **F** $\frac{6}{5}$ **G** $-\frac{2}{12}$ **H** $-\frac{6}{4}$ **I** $-\frac{9}{10}$ **J** 0 **K** $\frac{19}{100}$ **L** $-\frac{6}{9}$ **3.** **A** $\frac{7}{4}$ **B** $\frac{3}{8}$ **C** $\frac{1}{10}$ **D** 0 **E** $-\frac{1}{2}$ **F** $-\frac{7}{6}$ **G** $-\frac{2}{10}$ **H** $\frac{1}{3}$ **I** 1 **4.** **A** 1 **B** -1 **C** $-\frac{1}{3}$ **D** $-\frac{6}{8}$ **E** 1 **F** $-\frac{9}{10}$ **G** $-\frac{19}{20}$ **H** -1 **I** $\frac{4}{15}$ **J** 1 **K** $-\frac{1}{6}$ **L** $\frac{6}{4}$ **M** $-\frac{4}{6}$ **N** $\frac{3}{2}$ **O** -1 **P** $\frac{13}{100}$ **Q** $-\frac{3}{10}$ **R** $\frac{34}{100}$
Think **A** 4 **B** 5 **C** 5 **D** 5

F-13 USING THE IDEAS

1. **A** 3, 4, 7 **B** 4, 9, 13 **C** $-30, 7, -23$ **D** $-9, -20, -29$ **2.** **A** $\frac{4}{7}$ **B** $-\frac{1}{2}$ **C** $-\frac{2}{5}$ **D** $-\frac{3}{4}$ **E** $\frac{1}{6}$ **F** $-\frac{3}{8}$ **G** $-\frac{1}{2}$ **H** $-\frac{4}{3}$ **I** $\frac{9}{8}$ **J** $\frac{9}{20}$ **K** $-\frac{1}{6}$ **L** $\frac{1}{12}$ **M** $-\frac{19}{24}$ **N** $\frac{17}{12}$ **O** $\frac{1}{3}$ **P** $-\frac{19}{40}$ **3.** **A** $101\frac{5}{6}$ **B** $21\frac{13}{16}$ **C** 46 **D** $53\frac{1}{12}$ **E** $68\frac{3}{8}$ **F** 733 **G** $184\frac{57}{100}$ **H** $1197\frac{23}{100}$ **4.** **A** $27\frac{7}{8}$ **B** $18\frac{4}{5}$ **C** $12\frac{1}{2}$ **D** $15\frac{3}{10}$
Think

7	−7	−6	4
−4	2	1	−1
0	−2	−3	3
−5	5	6	−8

F-15 USING THE IDEAS

1. **A** 1 **B** $-\frac{4}{5}$ **C** $2\frac{2}{5}$ **D** -1 **E** 1 **F** $\frac{1}{2}$ **G** 1 **H** -1 **I** $-\frac{2}{5}$ **J** $1\frac{1}{4}$ **K** $-1\frac{1}{2}$ **L** $\frac{1}{2}$ **M** $\frac{4}{25}$ **N** $-1\frac{133}{1000}$ **O** $-\frac{2}{5}$ **2.** **A** $\frac{3}{8}$ **B** $\frac{5}{12}$ **C** $\frac{7}{12}$ **D** $\frac{1}{5}$ **E** $\frac{1}{5}$ **F** $\frac{1}{12}$ **G** $\frac{5}{20}$ **H** $\frac{1}{24}$ **I** $\frac{13}{100}$ **J** $\frac{357}{1000}$ **3.** **A** $1\frac{1}{8}$ **B** -1 **C** $-\frac{1}{3}$ **D** $\frac{5}{6}$ **E** $1\frac{3}{8}$ **F** $1\frac{13}{30}$ **G** $-1\frac{3}{10}$ **H** $\frac{7}{20}$ **I** $-\frac{1}{2}$ **J** $-\frac{7}{8}$ **K** $\frac{13}{20}$ **L** $\frac{1}{10}$ **M** $1\frac{1}{6}$ **N** $\frac{1}{6}$ **O** $-\frac{1}{6}$ **4.** **A** $5\frac{7}{12}$ **B** $24\frac{3}{16}$ **C** $22\frac{3}{10}$ **D** $12\frac{5}{6}$ **E** $46\frac{7}{8}$
5. **A** $5\frac{2}{5}$ **B** $3\frac{3}{20}$ **C** $2\frac{4}{5}$

Think

1.

−1	1	−1
1	1	1
−1	1	−1

2.

1	−1	1
−1	−1	−1
1	−1	1

F-17 USING THE IDEAS

1. **A** $\frac{1}{12}$ **B** $\frac{1}{16}$ **C** $\frac{1}{5}$ **D** $\frac{1}{3}$ **E** $\frac{1}{42}$ **F** $\frac{1}{10}$ **G** $\frac{1}{10}$ **H** $\frac{1}{2}$
2. **A** $\frac{4}{3}$ **B** $\frac{3}{4}$ **C** 0 **D** 1 **E** 4 **3.** 1 **4.** **A** one **B** one **C** commutative × **D** associative × **E** commutative × **F** distributive **5.** **A** $-\frac{1}{4}$ **B** $-\frac{3}{10}$ **C** $-\frac{2}{3}$ **D** $-\frac{3}{8}$ **E** $-\frac{5}{6}$ **F** $-\frac{17}{100}$ **6.** additive; inverse
7. **A** $\frac{1}{3}$ **B** $\frac{3}{8}$ **C** $\frac{9}{10}$ **D** $\frac{3}{4}$ **8.** **A** $-\frac{1}{4}$ **B** $-\frac{1}{10}$ **C** $\frac{7}{8}$ **D** -1
9. **A** $-\frac{1}{6}$ **B** $-\frac{1}{24}$ **C** $\frac{1}{18}$ **D** $\frac{1}{10}$ **E** $-\frac{1}{1000}$ **F** $\frac{1}{100}$

F-19 USING THE IDEAS

1. **A** $\frac{8}{15}$ **B** $\frac{5}{21}$ **C** $\frac{3}{2}$ **D** $\frac{9}{32}$ **E** $\frac{1}{2}$ **F** $\frac{5}{3}$ **G** 1 **H** $1\frac{1}{2}$ **I** $\frac{2}{3}$ **J** $\frac{1}{4}$ **K** 0 **I** $2\frac{2}{9}$ **J** $3\frac{1}{3}$ **K** $\frac{9}{10}$ **L** $\frac{27}{100}$ **2.** **A** $-\frac{9}{16}$ **B** $-\frac{5}{21}$ **C** $\frac{1}{4}$ **D** $-\frac{4}{15}$ **E** $\frac{1}{12}$ **F** $\frac{1}{2}$ **G** -1 **H** -2 **I** $\frac{1}{4}$ **J** $\frac{4}{9}$ **K** $-\frac{5}{2}$ **L** $\frac{20}{9}$ **M** -6 **N** $-\frac{21}{10}$ **O** $-\frac{15}{2}$ **P** $-\frac{21}{5}$ **Q** 45 **R** 1 **S** 1 **T** 63 **3.** **A** positive **B** positive **C** negative **4.** **A** 14 **B** -14 **C** 12 **D** -25 **E** 9 **F** 16 **5.** **A** $\frac{32}{15}$ **B** $\frac{21}{16}$ **C** -2 **D** $-\frac{35}{9}$ **E** $\frac{15}{4}$ **F** 40 **G** -25 **H** 25 **I** 20
Think Start at 5 (49 lots)

F-21 USING THE IDEAS

1. **A** $\frac{5}{2}$ **B** -3 **C** $\frac{8}{7}$ **D** $\frac{15}{16}$ **E** $-\frac{1}{4}$ **F** $\frac{7}{3}$ **G** 1 **H** $-\frac{6}{11}$ **I** $\frac{3}{10}$ **J** $\frac{5}{37}$ **K** $\frac{1}{10}$ **L** -100 **2.** **A** $\frac{8}{3}$ **B** $\frac{4}{5}$ **C** $\frac{7}{8}$ **D** $\frac{5}{6}$ **E** $\frac{7}{12}$ **F** $\frac{9}{4}$ **G** $\frac{1}{10}$ **H** 6 **I** $\frac{2}{5}$ **J** $\frac{6}{41}$ **K** $\frac{3}{50}$ **L** 1 **3.** **A** $\frac{1}{15}$ **B** $-\frac{1}{28}$ **C** 8 **D** $-2\frac{1}{2}$ **E** $-2\frac{2}{3}$ **F** $4\frac{2}{7}$ **4.** **A** $\frac{3}{2}$ **B** $\frac{9}{10}$ **C** 2 **D** $-\frac{3}{8}$ **E** $\frac{7}{5}$ **F** $\frac{10}{7}$
Think **1.** yes **2.** yes for A and B but not for C

F-22 ELECTRICAL ENERGY

1. 50 **2.** $1\frac{7}{20}$ **3.** $19\frac{1}{6}$ **4.** $4\frac{1}{2}$ **5.** $\frac{1}{3}$ **6.** 21 **7.** 10 000 **8.** $1\frac{28}{75}$ **9.** **A** \$11.70 **B** $1\frac{1}{2}$¢ **C** \$65.70 **D** 50.4¢

F-23 IMPROVING COMPUTATIONAL SKILLS

1. **A** $45\frac{4}{5}$ **B** $20\frac{1}{2}$ **C** $220\frac{3}{5}$ **D** 100 **E** $199\frac{1}{30}$ **F** $1\frac{1}{30}$ **G** $1\frac{47}{1000}$ **H** $1\frac{13}{60}$ **I** $3\frac{73}{100}$ **J** $\frac{111}{200}$ **2.** **A** $\frac{14}{25}$ **B** $\frac{12}{25}$ **C** $\frac{1}{15}$ **D** $-6\frac{4}{25}$ **E** $\frac{59}{90}$ **F** $8\frac{1}{5}$ **G** $8\frac{3}{4}$ **H** $7\frac{1}{10}$ **I** $45\frac{3}{5}$ **J** $91\frac{1}{8}$ **3.** **A** $\frac{7}{25}$ **B** $\frac{1}{6}$ **C** $\frac{5}{16}$ **D** $17\frac{1}{3}$ **E** 20 **F** $15\frac{1}{4}$ **G** 13 **H** $183\frac{3}{4}$ **I** 100 **J** $88\frac{2}{3}$
Think **A** $\frac{1}{9}$ **B** 0 **C** $\frac{1}{2}$ **D** 4

F-25 USING THE IDEAS

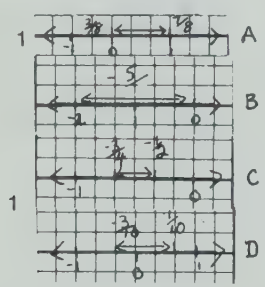
1. **A** $\frac{3}{2}$ **B** $\frac{3}{5}$ **C** $\frac{2}{5}$ **D** $\frac{3}{2}$ **E** -2 **F** $-\frac{2}{3}$ **G** $\frac{15}{8}$ **H** $\frac{9}{2}$ **I** $-\frac{12}{5}$ **2.** **A** $\frac{2}{3}$ **B** $\frac{5}{2}$ **C** $\frac{3}{2}$ **D** $\frac{7}{9}$ **E** $-\frac{3}{2}$ **F** $-\frac{3}{4}$ **G** -1 **H** 4 **I** $\frac{3}{2}$ **J** $\frac{2}{3}$ **K** $\frac{2}{3}$ **L** $-\frac{3}{10}$ **3.** **A** $3\frac{1}{3}$ **B** $1\frac{4}{5}$ **C** $1\frac{11}{14}$ **D** $2\frac{1}{4}$ **E** $-\frac{1}{4}$ **F** -3 **G** 2 **H** $2\frac{2}{3}$ **4.** **A** $\frac{5}{6}$ **B** 25 **C** 3

D $-\frac{3}{4}$ **E** $1\frac{1}{4}$ **F** $-\frac{2}{3}$ **G** $-\frac{4}{7}$ **H** $-2\frac{1}{2}$ **5. A** $1\frac{7}{8}$ **B** $1\frac{7}{15}$
C $1\frac{7}{15}$ **D** $1\frac{7}{8}$ **E** $1\frac{1}{7}$ **F** $4\frac{2}{3}$
Think 17

F-27 USING THE IDEAS

1. A 2 **B** 12 **C** −8 **D** 5 **E** −15 **F** −25 **G** 7
H 77 **I** 2 **J** 7 **2. A** 4, 5; < **B** 10, 9; >
C −24, −25; > **D** 28, 33; < **E** 140, 120; >
F 15, −20; > **3. A** < **B** < **C** > **D** > **E** > **F** <
G > **H** = **I** < **J** > **K** < **L** = **M** < **N** <
O > **P** > **4. A** $-\frac{3}{2} < x < \frac{3}{4}$ **B** $-\frac{15}{4} < x < 0$
C $-\frac{7}{3} < x < \frac{9}{10}$

5.

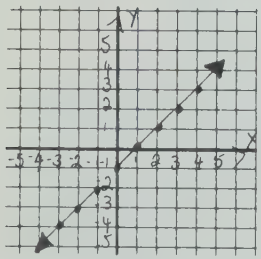


F-29 USING THE IDEAS

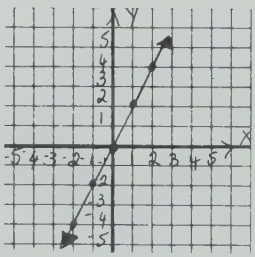
1. $72.67 **2.** $135 **3.** $9 **4. A** $4.80 **B** $4.72
5. 16 **6.** 60 km/h **7.** 1000 **8.** 9 **9.** $\frac{1}{4}$
Think A $16 **B** *A*: $2; *B*: $2; *C*: $2

F-31 USING THE IDEAS

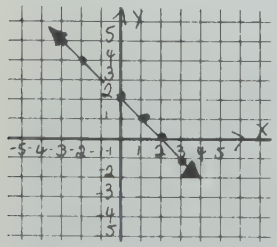
1. $-2, -2\frac{1}{2}$



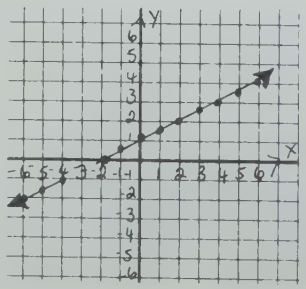
2. $-1, 0, 3$



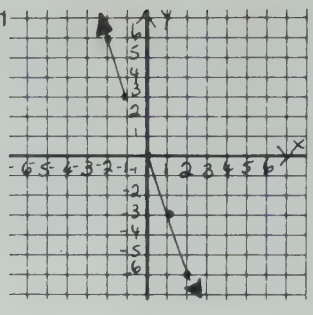
3. $3, 4, 2\frac{1}{2}$



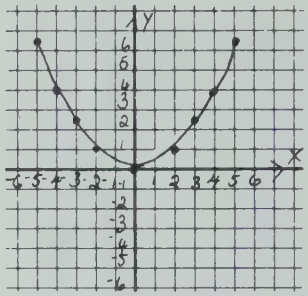
4. $2, \frac{5}{2}, \frac{1}{2}$



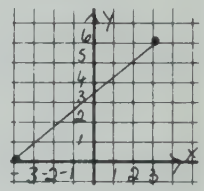
5. $0, -1, -\frac{3}{2}, 3, 1$



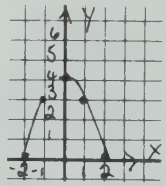
6. $0, \frac{1}{4}, 1, \frac{9}{4}, \frac{1}{4}$



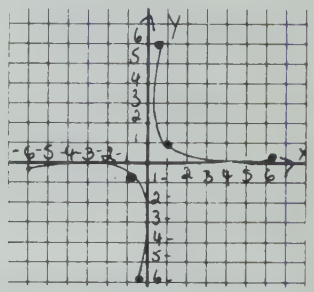
7. $1, 2, \frac{5}{2}, 3, 4$



8. $3, 0, 1\frac{3}{4}, 1\frac{3}{4}$

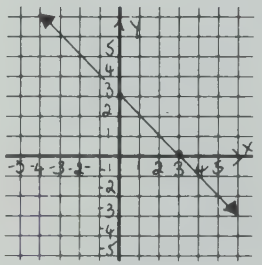


9. $4, \frac{1}{2}, -4, -\frac{1}{2}$

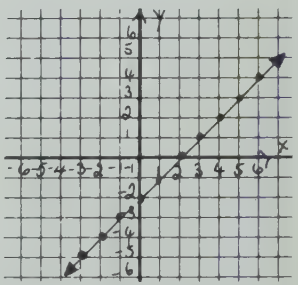


F-33 USING THE IDEAS

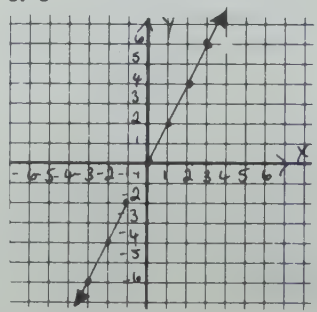
1. 1, 5



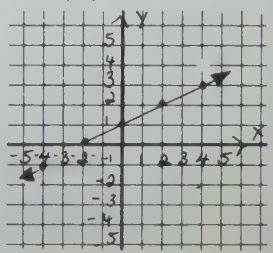
2. −4



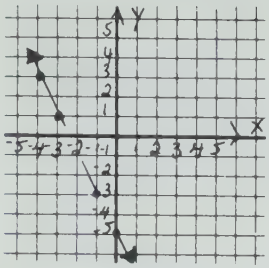
3. 8



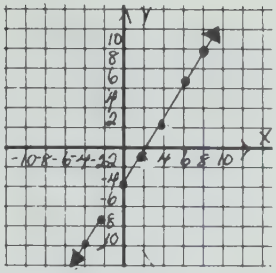
4. 1, 3, −1



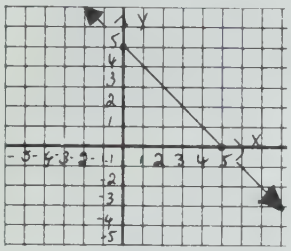
5. −5, 5, −7



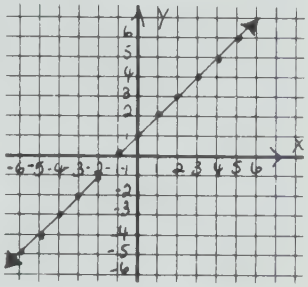
6. −1, −4, −10



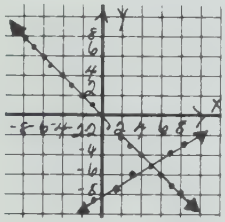
7. A



B

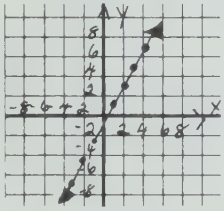


C

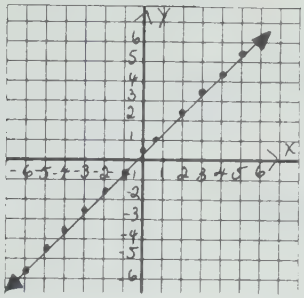


8. (3, 1)

9. (5, −5)

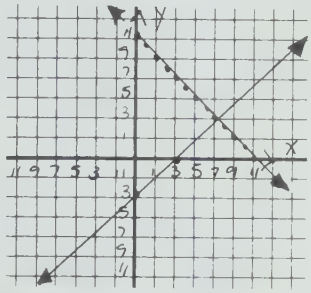


12.



13. Answers may vary. Some pairs are: (−1, 13), (2, 11), (1, 9), (2, 7), . . .

14.



F-34 REVIEWING THE IDEAS

1. A $-\frac{3}{7}$ **B** $\frac{2}{3}$ **C** $\frac{5}{7}$ **D** 0 **2. A** 0 **B** $\frac{3}{7}$ **C** $-\frac{5}{6}$ **D** $-\frac{1}{6}$
E $\frac{5}{8}$ **F** $-\frac{3}{8}$ **3. A** $-\frac{1}{7}$ **B** −1 **C** $\frac{6}{7}$ **D** $1\frac{2}{3}$ **E** $-\frac{1}{6}$ **F** $\frac{2}{3}$
4. A $\frac{4}{5}$ **B** $-\frac{2}{5}$ **C** $\frac{5}{8}$ **D** $-\frac{1}{6}$ **E** $3\frac{1}{5}$ **F** $-1\frac{1}{2}$ **G** 1
H $-1\frac{1}{4}$ **I** −9 **5. A** $1\frac{1}{3}$ **B** $-1\frac{1}{2}$ **C** −6 **D** $-2\frac{1}{3}$ **E** $\frac{1}{4}$
F $-\frac{1}{5}$ **G** $-\frac{3}{4}$ **H** $\frac{5}{7}$ **6. A** $\frac{3}{5}$ **B** $-\frac{1}{2}$ **C** $1\frac{1}{3}$ **D** $1\frac{2}{5}$ **E** $2\frac{1}{2}$
F $-\frac{1}{3}$ **G** −4 **H** $-1\frac{7}{9}$ **I** $-\frac{5}{8}$ **7. A** 16 **B** $\frac{11}{10}$ **C** $\frac{3}{7}$ **D** $\frac{1}{2}$
E $\frac{5}{9}$ **F** 31 **G** $\frac{1}{5}$ **H** $\frac{1}{2}$ **8. A** $\frac{1}{6}$ **B** $\frac{3}{5}$ **C** $-\frac{3}{10}$ **D** $-\frac{1}{2}$
E $-1\frac{1}{2}$ **F** $-1\frac{1}{3}$ **9.** $-\frac{3}{4} < x < \frac{1}{2}$ **10.** $99 **11.** $\frac{3}{4}, \frac{1}{4}, 0, -\frac{1}{4}$

UNIT F: MODULE 2

F-39 USING THE IDEAS

1. A $1/10^2$ **B** $1/10^4$ **C** $1/10^5$ **D** $1/10^{14}$ **E** $1/6^2$
F $1/5^3$ **G** $\frac{1}{28}$ **H** $\frac{1}{34}$ **I** $\frac{1}{73}$ **J** $1/8^5$ **2. A** 10^{-3} **B** 10^{-4}
C 7^{-5} **D** 3^{-3} **E** 10^{-1} **F** 10^{-7} **G** 10^{-3} **H** 10^{-4}
I 2^{-1} **J** 3^{-5} **3. A** 10^3 **B** 10^{-3} **C** 10^{-2} **D** 10^2
E 10^4 **F** 10^{-1} **G** 10^{-5} **H** 10^{-4} **I** 10^6 **J** 10^0
4. A $\frac{16}{25}$ **B** $\frac{703}{1000}$ **C** $3\frac{1}{2}$ **D** $1\frac{39}{100}$ **E** $807\frac{2}{5}$ **F** $\frac{5631}{10000}$
G $12\frac{37}{100}$ **H** $310\frac{3}{50}$ **5.** $7\frac{3}{4}$ **6.** *C*

Think

F-41 USING THE IDEAS

1. A hundreds, 3×10^2 **B** hundredths, 6×10^{-2}
C thousandths, 7×10^{-3} **2. B** 0.7 **C** 0.19 **D** 0.239
E 0.9 **F** 0.03 **G** 0.743 **H** 0.6197 **I** 0.39 **J** 0.007
K 0.0291 **L** 0.009 **3. B** 7.142 **C** 56.375 **D** 475.3
E 25.07 **F** 27.046 **G** 62.0539 **H** 3.0007 **4. A** $3 \cdot 10^0 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 7 \cdot 10^{-3}$ **B** $2 \cdot 10^2 + 9 \cdot 10^1 + 8 \cdot 10^0 + 4 \cdot 10^{-1} + 2 \cdot 10^{-2}$ **C** $6 \cdot 10^{-1} + 1 \cdot 10^{-2} + 2 \cdot 10^{-3} + 9 \cdot 10^{-4}$
D. $6 \cdot 10^{-3} + 7 \cdot 10^{-4}$ **5. A** 23.1 **B** 8.03 **C** 965.81
D 700.03 **6. A** 0.7 **B** 0.70 **C** 0.700 **7.** 0.750; 0.7500
8. A 72.435 **B** 8.020 **C** 0.074

Think **A** 5 **B** 3 **C** 10 **D** 7 **E** 5 **F** 3

F-43 USING THE IDEAS

1. A 111.01 **B** 1.541 **C** 1661.62 **D** 58.966
E 0.222 **2. A** 14.5 **B** 5.09 **C** 0.038 **D** 22.8
E 6.217 **3. A** 7.985 **B** 10.802 **C** 21.222 **D** 20.276
4. A 13.96 **B** 46.166 **C** 24.103 **D** 23.064
5. A 2.558 **B** 12.483 **C** 3.85 **D** 99.1224 **6. A** 5.6

B −1.191 **C** −15.9 **D** 0.0656 **E** −0.26 **F** −0.94 **G** 2.17 **H** −4.25

F-44 CELSIUS TEMPERATURES

1. A 1.3°C **B** 2.2°C **2.** 13.8 **3.** 3.6 **4.** 105.6 **5.** 146.1 **6.** 47.8 **7.** −273.16°C **8.** 194.66 **9.** no **10.** 11.8

F-45 SPEED RECORDS

1. 789.657 **2.** 39.83 **3.** 225.773 **4.** 0.97 **5.** 5.29 **6.** 1.77 **7.** 0.24

F-47 USING THE IDEAS

1. A hundredths **B** thousandths **C** hundred thousandths **2. A** 127.8 **B** 3.78 **C** 22.26 **D** 1.625 **E** 5250 **F** 14.744 **G** 27.82 **H** 4.7124 **I** 77.89012 **3.** 90¢ **4.** 1.2 cm **5.** 3.01 s **6.** $21 **7.** 1008 **8.** 300 000 km/s **9. A** −1.68 **B** −0.24 **C** 2.24 **D** −0.793 **E** −2.56 **F** 0.504 **G** −3.1714 **H** −0.0684 **I** 0.36519

Think 1

F-49 USING THE IDEAS

1. A 1 **B** 2 **C** −1 **D** −2 **E** 3 **F** 6 **G** 5 **H** −8 **I** 0 **2. A** 2.3×10^2 **B** 1.28×10^3 **C** 6.19×10^3 **D** 3.029×10^3 km **E** 4.0×10^4 **F** 4.44×10^6 **G** 8.7×10^9 **H** 3.0×10^{22} **3. A** 7000 **B** 0.062 **C** 0.00000545 **D** 90 300 000 **E** 6.43 **F** 0.000000000005 **G** 11 000 000 000 **H** 0.0000000337 **4. A** 0.02 **B** 0.35 **C** 0.000064 **D** 0.000053 **E** 0.000045 **F** 0.00000025 **G** 0.0000000106 **H** 0.000000000000000003 **5. A** 3.0×10^7 **B** 2.0×10^{23} **C** 2.0×10^{-4} **D** 5.6×10^{-2} **E** 1.0×10^8 **F** 1.0×10^2 **G** 4.2×10^{13} **H** 6.0×10^{-7} **I** 1.0×10^{-6}

F-51 USING THE IDEAS

1. A 2 **B** 3 **C** 2 **D** 3 **E** 4 **F** 5 **2. A** 500 cm **B** 0.00005 mm **C** 50 000 000 yr **D** 50 000 kg **C** 0.00000005 cm **F** 5 ml **G** 5000 m **H** 50 000 000 000 km **I** 0.005 cm **3. A** $\frac{1}{130}$ **B** $\frac{1}{144}$ **C** $\frac{1}{60}$ **D** $\frac{1}{250}$ **4. A** $(3.2 \pm .05) \times 10^{-2}$ **B** 2.14×10^{-3} **C** 4.0×10^3

F-53 USING THE IDEAS

1. A 3.8 **B** 0.57 **C** 0.063 **D** 3.56 **E** 0.0023 **F** 0.00015 **G** 5.294 **H** 0.74 **2. A** 35.7 **B** 38 **C** 6.4 **D** 0.062 **E** 0.41 **F** 7.7 **G** 1.7 **H** 0.036 **3.** 125 **4.** 40 **5.** 1.31 **6.** 192 **7.** 64 s **8.** 80 **9. B** −2 **B** −0.5 **C** −700 **D** 0.2 **E** 0.05 **F** 30

Think **1.** 3 **2.** 7 **3.** 3 **4.** 3

F-55 USING THE IDEAS

1. A 3.6×10^7 **B** 6×10^4 **C** 4.4×10^1 **D** 9×10^{-1} **E** 1.6×10^{-4} **F** 1.6×10^{-5} **G** 1.4×10^{-4} **H** 3.2×10^{-2} **I** 2.8×10^{-9} **2. A** 2.3 **B** 4.7 **C** 3.14 **D** 0.4 **E** 3.258 **F** 0.0037 **3. A** 10^3 **B** 10^4 **C** 10^{-2} **D** 6×10^3 **E** 3×10^8 **F** 2×10^0 **G** 4×10^{-3} **H** 2×10^{-4} **I** 2×10^2 **4. A** 2×10^2 **B** 2×10^3 **C** 3×10^1 **D** 3×10^{-3} **E** 2×10^3 **F** 5×10^{-8} **G** 1×10^5 **H** 2 **I** 3×10^2 **5. A** 0.002 **B** 500 **C** 14 **D** 0.27 **6. A** 8×10^{-9} **B** 8.1×10^{13} **C** 6.4×10^7 **D** 1.6×10^{-15} **E** 3.2×10^{-9} **F** 10^{30} **7.** 3×10^6

F-57 USING THE IDEAS

1. A 0.4; 0.6; 1.5 **B** 0.43; 0.62; 1.52 **C** 0.435; 0.622; 1.516 **2. A** 0.83 **B** 0.81 **C** 0.43 **D** 0.73 **E** 0.55 **3. A** 0.500 **B** 0.778 **C** 0.583 **D** 0.657 **E** 0.146 **4. A** 0.607 **B** 0.583 **C** 0.582 **D** 0.632 **5. A** *D* **B** *C* **6. A** 0.750 **B** 0.688 **C** 0.500 **D** 0.438 **E** 0.375

F-59 USING THE IDEAS

1. A $0.06\frac{1}{4}\%$ **B** $0.01\frac{1}{2}$ **C** $0.34\frac{3}{10}$ **2. A** $0.5\frac{5}{9}$, $0.58\frac{1}{3}$, $0.583\frac{1}{3}$ **B** $0.8\frac{1}{3}$, $0.83\frac{1}{3}$, $0.833\frac{1}{3}$ **C** $1.3\frac{1}{3}$, $1.33\frac{1}{3}$, $1.333\frac{1}{3}$ **D** $0.9\frac{3}{8}$, $0.93\frac{3}{4}$, $0.937\frac{1}{2}$ **E** $0.2\frac{6}{37}$, $0.21\frac{23}{37}$, $0.216\frac{8}{37}$ **F** $0.1\frac{23}{27}$, $0.18\frac{14}{27}$, $0.185\frac{5}{27}$ **3. A** $\frac{1}{8}$ **B** $\frac{1}{8}$ **C** $\frac{1}{12}$ **D** $\frac{3}{8}$ **E** $\frac{5}{8}$ **F** $\frac{5}{12}$ **4. A** $0.2\frac{11}{32}$, $0.23\frac{7}{16}$, $0.234\frac{3}{8}$, $0.2343\frac{3}{4}$, $0.23437\frac{1}{2}$, 0.234375 **B** $0.3\frac{73}{125}$, $0.35\frac{21}{25}$, $0.358\frac{2}{5}$, 0.3584 **C** $0.1\frac{7}{8}$, $0.18\frac{3}{4}$, $0.187\frac{1}{2}$, 0.1875 **D** $0.6\frac{4}{25}$, $0.61\frac{3}{5}$, 0.616 **E** $0.9\frac{3}{4}$, $0.97\frac{1}{2}$, 0.975 **F** $0.4\frac{7}{8}$, $0.48\frac{3}{4}$, $0.487\frac{1}{2}$, 0.4875

Think 15

F-61 USING THE IDEAS

1. A $0.\overline{3}$ **B** $0.\overline{7}$ **C** $0.\overline{15}$ **D** $0.\overline{324}$ **E** $0.\overline{56}$ **F** $0.\overline{9}$ **G** $0.\overline{024}$ **H** $0.\overline{3715}$ **I** $0.\overline{0426}$ **J** $0.\overline{507}$ **K** $0.\overline{62}$ **L** $0.\overline{1}$ **2. A** $5.2\overline{63}$ **B** $0.7\overline{4}$ **C** $3.\overline{28}$ **D** $526.1\overline{4}$ **E** $73.4\overline{6}$ **F** $77.77\overline{13}$ **G** $56.66\overline{15}$ **H** $0.0432\overline{1}$ **I** $0.7\overline{6243}$ **3. A** $0.\overline{3}$ **B** $0.\overline{6}$ **C** $0.\overline{1}$ **D** $0.\overline{3}$ **E** $0.\overline{8}$ **F** $0.\overline{09}$ **G** $0.\overline{27}$ **H** $0.\overline{81}$ **I** $0.1\overline{6}$ **J** $0.\overline{01}$ **K** $0.\overline{02}$ **L** $0.\overline{71}$ **M** $0.\overline{142857}$ **N** $0.\overline{571428}$ **4. A** 1 **B** 1 **C** 1 **D** 1 **E** 1 **F** 2 **G** 2 **H** 2 **I** 1 **J** 2 **K** 2 **L** 2 **M** 6 **N** 6 **5. A** $0.\overline{2}$ **B** $\frac{7}{9}$ **C** $0.\overline{09}$ **D** $\frac{10}{11}$ **E** $0.\overline{43}$ **F** $0.\overline{285}$ **G** $0.\overline{24}$ **H** 9 **I** $1.\overline{561}$ **J** 2 **6. A** $\frac{1}{3}$ **B** $\frac{71}{99}$ **C** $\frac{28}{99}$ **D** $\frac{17}{18}$ **C** $\frac{277}{333}$ **F** $\frac{127}{198}$ **G** $\frac{1741}{2475}$ **H** $\frac{23}{75}$

F-62 COMPUTING WITH REPEATING DECIMALS

1. A 0.666 . . . **B** 0.8787 . . . ; 1.902993 **2. A** 0.444 . . . **B** 0.4949 . . . **C** 0.0101 . . . **3. A** 0.666 . . . **B** 0.3232 . . . **C** 1.332 **4.** Since the divisor is 17 and if it repeats, it should do so after 17 places. **5. A** 0.9999 . . . **B** 0.9999 . . .

Think $\frac{2}{27} = 0.\overline{074}$

F-63 SPECIFIC GRAVITY

1. A 19 300 **B** 35.2 **C** 798 **D** 78.6 **E** 311.65 **F** 9000 **G** 90 000 **H** 1000 **2. A** 1.3 g **B** 1300 g **3.** 499.2 kg **4.** 13.55 **5.** 51.8

F-64 REVIEWING THE IDEAS

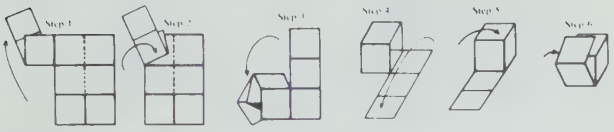
1. A 10^4 **B** 10^{-1} **C** 10^{-3} **2.** 8400.56 **3.** $5.10^2 + 6.10^1 + 3 \times 10^0 + 8.10^{-2} + 1.10^{-3}$ **4. A** hundreds **B** hundredths **5. A** 25.98 **B** 173.876 **C** 0.1665 **6. A** 3.72 **B** 4.71 **C** 5.47 **D** 96.921 **7.** 9.405 **8. A** 8.75 **B** 4080 **C** 9.715 **D** 0.0215 **E** 0.156 **F** 199.71 **9. A** 900 000 **B** 6300 **C** 0.053 **D** 0.00000388 **10. A** 2.5×10^3 **B** 4.5×10^4 **C** 5.9×10^{-2} **D** 3.1×10^{-4} **11.** 3.9×10^5 **12.** 5×10^2 **13. A** 0.14 **B** 14 **C** 14 **D** 0.00014 **E** 0.14 **F** 140 000 **14. A** 0.25 **B** 1.25 **C** 560 **D** −9.1 **15. A** 4.8×10^5 **B** 2.1 **C** 4.8×10^7 **D** 4×10 **16. A** 0.313 **B** 0.267 **C** 0.889 **17.** $0.63\frac{4}{11}$ **18. A** $0.\overline{75}$ **B** $0.3\overline{04}$ **19. A** $0.8\overline{3}$ **B** $0.5\overline{}$ **C** $0.\overline{285714}$

UNIT F: MODULE 3

F-69 USING THE IDEAS

1. A 4 to 8; 5 to 10; 6 to 12 **B** 12 : 8; 15 : 10; 18 : 12 **C** 12 to 16; 15 to 20; 18 to 24 **D** $\frac{20}{32}$, $\frac{25}{40}$, $\frac{30}{48}$ **E** 8 : 20; 10 : 25; 12 : 30 **F** $\frac{20}{24}$; $\frac{25}{30}$; $\frac{30}{36}$ **2. A** 6 **B** 15 **C** 20 **D** 9 **E** 50 **F** 21 **G** 15 **H** 10 **I** 60 **J** 18 **3. A** 1 : 2 **B** 5 : 4 **C** 7 : 4 **D** 14 : 17 **E** 2 : 13 **4. A** $\frac{3}{4}$ **B** $\frac{2}{5}$ **C** $\frac{9}{10}$ **D** $\frac{6}{7}$ **E** $\frac{8}{20}$ **5. A** 0.64 **B** 0.65 **C** 0.29 **D** 1.29 **E** 0.22 **6.** 39

Think



F-71 USING THE IDEAS

1. A $3 \cdot 3 = 1 \cdot 9$; $9 = 9$ **B** $5 \cdot 12 = 3 \cdot 20$; $60 = 60$ **C** $6 \cdot 10 = 4 \cdot 15$; $60 = 60$ **D** $3 \cdot 28 = 7 \cdot 12$; $84 = 84$ **E** $5 \cdot 60 = 3 \cdot 100$; $300 = 300$ **F** $90 \cdot 27 = 30 \cdot 81$; $2430 = 2430$ **2. A** 18 **B** 30 **C** 16 **D** 10 **E** 9 **F** 5 **G** 10 **H** 1 **I** 7 **J** 7 **K** 24 **L** 21 **3. A** \$1.17 **B** 20 **C** 175 km **D** 750 **4.** 26 km **5. A** $\frac{9}{8}$ **B** $3\frac{1}{9}$ **C** $4\frac{2}{3}$ **D** $7\frac{4}{3}$ **6. A** 24 **B** 12 **C** 6 **D** 15

F-72 RATIO AND UNITS OF MEASURE

1. A 700 **B** 7000 **C** 19 000 **D** 3 **E** 2 **F** 3 **G** 3.2 **H** 3 **I** 20 000 **J** 4 **K** 200 **L** 4.5 **M** 900 **N** 30 **O** 300 **P** 0.6 **Q** 200 **R** 25 200 **2. A** 60, 2 **B** 7, 200 **C** 60, 50 **3. A** $\frac{1}{50}$ **B** $\frac{5}{12}$ **C** $\frac{1}{200\,000}$ **D** $\frac{2}{5}$ **E** $\frac{17}{50}$

F-73 SPORTS RATIOS

1. A $\frac{79}{164}$ **B** 0.482 **2.** 0.625 **3.** 0.544 **4.** 1.95 **5.** 2.20 **6. A** 0.841 **B** 0.159 **7.** 0.574 **8.** 0.730 **9.** 0.318 **10. A** $\frac{11}{26}$ **B** $\frac{31}{78}$ **C** $\frac{7}{39}$

F-75 USING THE IDEAS

1. 24 **2.** 12 **3.** 4.25 **4.** 3.3 **5.** 24 **6.** 0.8 **7. A** 25 mm **B** 2.5 mm **C** 100 mm **D** 30 mm **8.** 3 times

F-77 USING THE IDEAS

1. 3.12 m **2.** 4.92 m **3.** about 47 m **4. A** 0.013 mm **B** 0.006 mm **5.** 1 : 25 000 000

Think

F-78 RATIOS AND ELECTRICITY

1. 3.6 **2. A** 1.6 **B** 6.25 **C** 5.0 **D** 1.40 **3.** 14.4 **4.** 0.17 **5.** 31.5

F-79 EXPANSION OF METALS

1. 5.6 **2.** 0.0156 **3.** 0.66 **4.** 0.021 **5.** 20.75

F-81 USING THE IDEAS

1. A 19% **B** 26% **C** 4% **D** 150% **E** 0.5% **F** 100% **G** 65% **H** 24% **2. A** 5% **B** 4% **C** $2\frac{1}{2}$% **D** $3\frac{1}{8}$% **E** $3\frac{1}{2}$% **F** $2\frac{5}{8}$% **G** 2.3% **H** 37.1% **I** $\frac{1}{2}$% **J** 0.75% **3. A** 40, $13\frac{1}{3}$ **B** 200, $66\frac{2}{3}$ **C** 500, $41\frac{2}{3}$ **D** 100, $12\frac{1}{2}$ **4. A** 58% **B** 72% **C** 83% **D** 75% **E** 69% **F** 6% **G** 154% **H** 88% **I** 57% **J** 96% **5. A** 58.3% **B** 7.8% **C** 24.1% **D** 362% **E** 23.9% **F** 0.7% **6.** 5% **7. A** $\frac{30}{41}$ is 73% not 75% **B** 4

Think 23421314 or 41312432

F-83 USING THE IDEAS

1. 0.735; 0.015 **2.** 0.87 **3.** 32% **4.** 71% **5.** 32.9% **6.** 4% **7.** 2% **8.** 53% **9.** 95% **10.** 81% **11.** 0.085 **12.** 2.2%

F-85 USING THE IDEAS

1. 90% **2.** 80% **3.** 40% **4.** 75% **5.** 25% **6.** 75% **7.** 10% **8.** 25% **9.** 80% **10.** $66\frac{2}{3}$% **11.** $58\frac{1}{3}$% **12.** 14% **13. A** 20% **B** $33\frac{1}{3}$% **C** $6\frac{2}{3}$%

Think 89, 98 & 99

F-87 USING THE IDEAS

1. A \$4.50 **B** \$13.50 **2. A** \$5.50 **B** \$5.78 **3. A** yes **B** no **4.** \$112.63 **5.** *B* **6.** Erton's: \$12; Sampson's: \$12 **7.** \$129.60 **8. A** \$126 **B** \$3.60 less

F-89 USING THE IDEAS

1. A 0.005 **B** 2.74 **C** 0.003 **D** 8 **E** 0.007 **F** 0.0025 **G** 0.0001 **H** 2.1 **I** 3.5 **J** 0.002 **K** 0.0005 **L** 0.00125 **2.** 0.007 m **3.** 0.6 cm **4.** 1.6 **5.** \$12,000 **6.** 95 400 **7.** \$1308 **8.** 0.0001%

F-91 USING THE IDEAS

1. A 6, $\frac{6}{84}$, 7% **B** 8, $\frac{8}{78}$, 10.3% **C** 9, $\frac{9}{80}$, 11.3% **D** 5, $\frac{5}{7c}$, 7% **2. A** 6 **B** 4.3% **3. A** 32 **B** 22.5% **4.** 6.6% **5.** 58.3% **6.** 16%

F-92 REVIEWING THE IDEAS

1. 4 : 10, 6 : 15, 8 : 20 **2. A** 1 : 4 **B** 3 to 8 **C** 2 to 3 **3.** 16, 9 **4.** 3 : 4 = 9 : 12 **5.** product of the extremes **6. A** 8 **B** 5 **C** 16 **D** 25 **E** 8 **F** 6 **7.** 60 km **8.** 0.09 mm **9.** 4 cm **10.** *C* **11.** 5 **12. A** 11% **B** 8% **C** 12% **D** 4% **13. A** 4% **B** 8% **C** 20% **D** $\frac{1}{2}$% **14. A** 60% **B** 17% **C** $66\frac{2}{3}$% **D** 125% **E** 0.2% **F** 2.25% **15.** 85% **16.** \$44 **17.** \$1.00 **18.** 25% **19.** \$960 **20.** 97%

F-94 CUMULATIVE REVIEW

1. 37.5% or $\frac{3}{8}$ **2.** $\frac{10}{12}$, $\frac{15}{18}$, $\frac{20}{24}$, $\frac{25}{30}$, $\frac{30}{36}$ **3. A** $\frac{2}{3}$ **B** $-\frac{5}{4}$ **4. A** $\frac{7}{9}$ **B** $\frac{1}{2}$ **C** $\frac{22}{69}$ **5. A** 2 **B** 9 **C** 81 **D** 64 **6. A** < **B** = **C** < **D** > **7. A** $14\frac{11}{15}$ **B** $1\frac{14}{15}$ **C** $5\frac{25}{28}$ **D** $\frac{55}{72}$ **8. A** commutative + **B** commutative + **C** one **D** distributive **9. A** $\frac{7}{24}$ **B** $2\frac{2}{27}$ **C** 1 **D** $\frac{1}{5}$ **10. A** $1\frac{1}{2}$ **B** $\frac{1}{4}$ **C** −9 **D** no **11. A** $\frac{1}{12}$ **B** $-\frac{5}{8}$ **C** $1\frac{2}{3}$ **D** $-\frac{37}{40}$ **12. A** $-\frac{1}{4}$ **B** $\frac{3}{10}$ **C** $\frac{45}{49}$ **D** $-\frac{6}{11}$ **13.** \$3.32 **14.** \$5200 **15. A** 0.47 **B** 0.32 **C** 0.4 **D** 0.375 **E** $0.16\frac{2}{3}$ **F** 0.2857142 **16. A** $\frac{2}{25}$ **B** $\frac{3}{10}$ **C** $\frac{33}{100}$ **D** $2\frac{2}{5}$ **E** $16\frac{11}{100}$ **F** $8\frac{7}{20}$ **17. A** 77.906 **B** 43.821 **18. A** 7.3×10^{-3} **B** 5.8×10^{4} **C** 8.2×10^{-5} **D** 9.8×10^{10} **19. A** 1.632 **B** 2.8038 **C.** 6.4 **D** 3.9

F-95

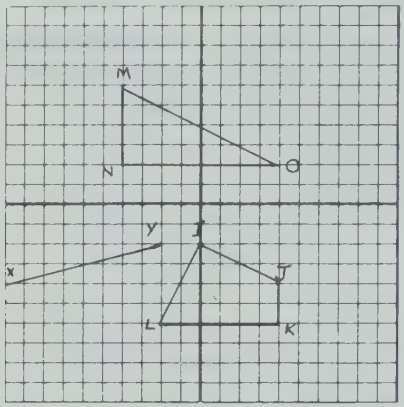
20. A 71 **B** 8.2 **C** 2160 **D** 0.217 **E** 6.39 **F** 0.0554 **21. A** 86.8 **B** 5.53 **C** 20.978 **D** 0.0216

22. A 8.14 **B** 62.8 **C** 9600 **D** 0.0283 **23. A** 19 230 **B** 2610.41 **C** 27 **D** 0.0719 **24. A** 28 **B** 1.8 **C** 1200 **D** 27 000 **25. A** 4 **B** 6 **C** 500 **26.** 3:2 **27. A** 16 **B** 18 **C** 7 **D** 64 **E** 8 **F** 9 **28.** 3 to 9 **29. A** 75% **B** 70% **C** 33% **D** 53% **E** 110% **F** 16% **30.** 84% **31.** 2% **32.** $9.80 **33. A** $4.56 **B** $80.51 **34.** 7.1% **35.** $500 **36.** 0.06

UNIT G: MODULE 1

G-3 USING THE IDEAS

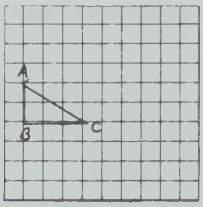
1.



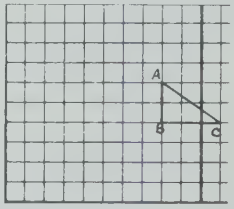
2. A triangle **B** segment **C** quadrilateral
3. A *DEF* **B** *D*, *E*, *F* **4. A** *xyz* **B** *x*, *y*, *z*
5. Same except for its opposite direction.

G-5 USING THE IDEAS

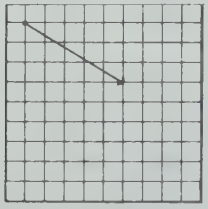
1. A



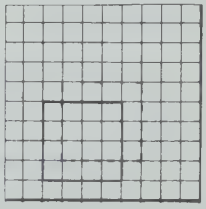
B



C



2.

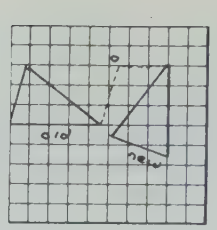


4. yes **5. A** right 1, down 1 **B** left 4, up 5 **C** left 2, down 7 **D** right 4, up 4 **E** down 3 **F** right 2 **6. C** no

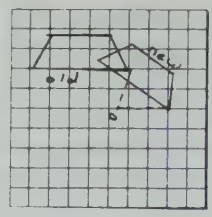
G-7 USING THE IDEAS

1. A yes **B** no **C** yes **D** yes **E** yes **F** yes
2. A *F*, *G*, *H* **B** 45° clockwise

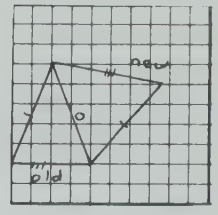
3. A



B



C



4. *D*, *E*, *A*, *B*

G-9 USING THE IDEAS

1. *A-H*, *S-Q*, *T-L*, *M-E* **4. C** no **D** yes **5. A** 4 **B** 2 **6. A** 3 **B** 2 **C** 0 **D** uncountable

G-13 USING THE IDEAS

1. A $\overline{DF}$ **B** $\overline{FE}$ **C** $\overline{DE}$ **D** $\angle D$ **E** $\angle F$ **F** $\angle E$
2. A $\overline{HI} \cong \overline{MN}$; $\overline{IJ} \cong \overline{NO}$; $\overline{JK} \cong \overline{OL}$; $\overline{KH} \cong \overline{LM}$ **B** $\angle I \cong \angle N$; $\angle J \cong \angle O$; $\angle K \cong \angle L$; $\angle H \cong \angle M$ **3. A** *I'* **B** $\angle A'AC$ **C** yes **D** $\angle A''A'B' \cong \angle A'AB$ **4. A** *RSTU* & *EFGD* **B** *RSTU* & *MLKJ* **C** *RSTU* & *YXWZ*
5. A $\overline{RS} \cong \overline{EF}$, $\overline{ST} \cong \overline{FG}$; $\overline{TU} \cong \overline{GD}$, $\overline{UR} \cong \overline{DE}$; $\angle R \cong \angle E$, $\angle S \cong \angle F$; $\angle T \cong \angle G$, $\angle U \cong \angle D$; *RSTU* $\cong$ *EFGD* **B** $\overline{RS} \cong \overline{ML}$, $\overline{ST} \cong \overline{LK}$; $\overline{TU} \cong \overline{KJ}$, $\overline{UR} \cong \overline{JM}$; $\angle R \cong \angle M$, $\angle S \cong \angle L$; $\angle T \cong \angle K$, $\angle U \cong \angle J$; *RSTU* $\cong$ *MLKJ* **C** $\overline{RS} \cong \overline{YX}$, $\overline{ST} \cong \overline{XW}$; $\overline{TU} \cong \overline{WZ}$, $\overline{UR} \cong \overline{ZY}$; $\angle R \cong \angle Y$, $\angle S \cong \angle X$; $\angle T \cong \angle W$, $\angle U \cong \angle Z$; *RSTU* $\cong$ *YXWZ*

G-15 USING THE IDEAS

1. A $\overline{D'E'}$ **B** $\overline{F'E'}$ **C** $\overline{D'F'}$ **D** $\angle D'$ **E** $\angle E'$ **F** $\angle F'$
2. A $\overline{MK}$ **B** $\overline{KL}$ **C** $\overline{ML}$ **D** $\angle M$ **E** $\angle K$ **F** $\angle L$
G $\triangle MKL$ **3.** $\overline{RS} \cong \overline{XW}$; $\overline{RT} \cong \overline{XY}$; $\overline{ST} \cong \overline{WY}$; $\angle R \cong \angle X$; $\angle S \cong \angle W$; $\angle T \cong \angle Y$ **4. A** $\triangle ABD \cong \triangle CBD$
B $\triangle EFH \cong \triangle GFH$ **C** $\triangle JKL \cong \triangle LMJ$
D $\triangle WXY \cong \triangle VZY$ **E** $\triangle POS \cong \triangle QOR$
F $\triangle DGH \cong \triangle HED$
Think 27

G-17 USING THE IDEAS

1. A *DEF*, *IHG* **B** *I* **C** *H* **D** *G* **2. A** *UVW*, *YXZ* **B** $\angle U \cong \angle Y$; $\angle V \cong \angle X$; $\angle W \cong \angle Z$ **3. A** $\overline{SQ}$; $\overline{SQ} \cong \overline{SQ}$, $\overline{SP} \cong \overline{SR}$; $\overline{PQ} \cong \overline{RQ}$; $\triangle SPQ \cong \triangle SRQ$ **B** $\overline{MQ}$; $\overline{MQ} \cong \overline{MQ}$, $\overline{MP} \cong \overline{MN}$; $\overline{PQ} \cong \overline{NQ}$; $\triangle PMQ \cong \triangle NMQ$ **C** $\overline{SU}$; $\overline{SU} \cong \overline{SU}$, $\overline{SR} \cong \overline{ST}$; $\overline{RU} \cong \overline{TU}$; $\triangle SRU \cong \triangle STU$
D $\overline{DB}$; $\overline{DB} \cong \overline{DB}$, $\overline{AD} \cong \overline{CD}$; $\overline{AB} \cong \overline{CD}$, $\overline{AB} \cong \overline{BC}$; $\overline{AD} \cong \overline{BC}$; $\triangle ADB \cong \triangle CDB$ **4.** $\overline{AD} \cong \overline{AD}$; $\overline{AB} \cong \overline{DC}$; $\overline{BD} \cong \overline{CA}$; $\triangle ABD \cong \triangle DCA$ **5. A** yes, *SSS* **B** yes

G-19 USING THE IDEAS

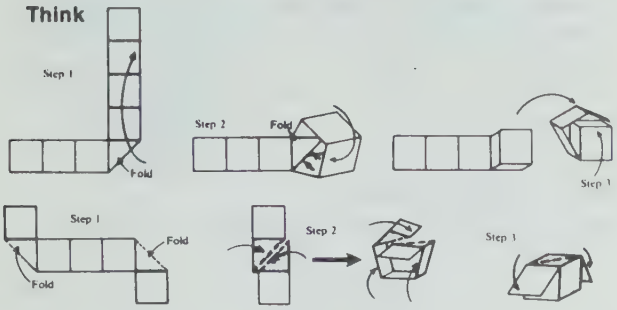
1. A $\triangle ABC \cong \triangle FDE$ **B** *SAS* **C** $\overline{FD}$, $\angle F$, $\angle D$
2. A $\overline{CD}$ **B** $\overline{AC}$ **C** $\angle 2$ **D** *CAB*; *SAS* **3. A** *SAS* **B** *SSS* **C** *SSS* **D** *SAS*

Think
$\triangle ACD \cong \triangle AED$; $\triangle ABG \cong \triangle AFG$; $\triangle BGC \cong \triangle FGE$; $\triangle GCD \cong \triangle GED$; $\triangle BCE \cong \triangle FEC$; $\triangle ACG \cong \triangle AEG$; $\triangle ACF \cong \triangle AEB$

G-21 USING THE IDEAS

1. A *SUT, ASA* **B** $\angle S$ **C** $\overline{TS}, \overline{SU}$ **2. A** *IHJ* **B** $\overline{ZX}, \overline{YX}$, $\angle I$ **3. A** *SSS* **B** *ASA* **C** *SAS* **D** *ASA*
4. A translation **B** reflection **C** rotation **D** translation

Think



G-23 USING THE IDEAS

3. yes **4.** $M \perp T$ **5.** $S \parallel T$ **6.** $A \parallel C$
Think the second car

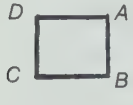
G-25 USING THE IDEAS

1. A 6 **B** *CEGA* **C** yes **2. A** yes **B** rectangle
3. A *AED* and *BFC* **B** *AEFB* and *DEFC*
4. A triangular **B** hexagonal

G-26 REVIEWING THE IDEAS

1. A *Z* **B** $\overline{XY}$ **2.** Answers will vary, however, the final image should be "left 2 and down 2" from the original position. **3. A** clockwise **B** *E* **C** $\triangle EDF$

4.



5. *RST* **6.** 2 **7.**

8. true **9. A** $\triangle MNP \cong \triangle QRS$ (*SAS*)
B $\triangle HIJ \cong \triangle GFE$ (*ASA*) **C** $\triangle ABD \cong \triangle CDB$ (*SSS*)
10. $P \parallel M$ **11. A** $\angle 1$, $\angle 8$, $\angle 5$ **B** $\angle 4$, $\angle 6$, $\angle 7$
12. A *ABC* and *DEF* **B** rectangles

UNIT G: MODULE 2

G-31 USING THE IDEAS

1. A $l(\overline{AB}) = 4$; $l(\overline{CD}) = 2$; $l(\overline{EF}) = 2$ **B** $l(\overline{AB}) = 4$; $l(\overline{CD}) = 2\frac{1}{2}$; $l(\overline{EF}) = 1\frac{1}{2}$ **C** 8 **2. A** $l(\overline{GH}) = 0.8$; $l(\overline{IJ}) = 0.7$; $l(\overline{KL}) = 0.4$ **B** 1.9 **3. A** 26 **B** 3.6 **C** 5.4 **D** 19.54
4. A 4 by 2 **B** 4 by $1\frac{1}{2}$ **C** 4.2 by 1.7 **5. A** 12 **B** 11 **C** 11.8

G-33 USING THE IDEAS

1. A $l(\overline{AB}) = 9$; $l(\overline{CD}) = 7$; $l(\overline{EF}) = 3$ **B** 9, 7.5, 3.5 **C** 88, 74, 33 **2. A** 5 **B** 3 **C** 1.2 **D** 2000 **E** 1600 **F** 2.27 **G** 470 **H** 1500 **I** 35.8 **J** 0.25 **K** 1 **L** 1
3. A 1.52 **B** 5.48 **C** 27.6 **D** 3.66 **E** 3.4 **4. A** 250 **B** 38 **C** 300 **D** 5 **5. A** 10.8 **B** 91 **C** 0.086

G-35 USING THE IDEAS

1. A 345 **B** 6.12 **C** 0.0567 **2.** 20.68 **3.** $247.75
4. A 322 **B** 23.04 **C** 8.51 **5. A** 23.4 **B** 40.32
6. A 423.36 **B** 1747.36 **C** 1324
Think 8

G-37 USING THE IDEAS

1. A 60 **B** 12 **C** 81 **2. A** 6.9 **B** 49.8 **C** 7.9625
3. A 28 **B** 20 **C** 48 **4.** 532.5 **5. A** 96 **B** 1182 **C** 71.04
Think 16, yes

G-39 USING THE IDEAS

1. A 153°7′ **B** 26°53′ **C** 74°10′ **2.** 19°41′
3. 54°21′45″ **4. A** 30° **B** 180° **C** 150° **D** 150° **E** yes **5.** 132°40′ **6. A** 57°13′ **B** 56°53′
7. A 110°55′50″ **B** 147°12′4″ **C** 180° **D** 126°40′13″
8. A 59°14′9″ **B** 51°42′20″ **C** 88°48′49″ **D** 35°45′8″

G-41 USING THE IDEAS

1. A 50° **B** 23° **C** 145° **D** 45° **2. A** 90° **B** 80° **C** 74° **D** 77° **E** 93°29′ **F** 66°45′ **G** 95°4′ **H** 83°55′30″ **3.** 60° **4.** 42° **5.** 144° **6.** 123°
7. A 540° **B** 108° **8. A** 720° **B** 120° **9. A** 1440° **B** 144° **10.** $M\angle H = 87°30'$; $M\angle G = 43°45'$; $M\angle I = 48°45'$
Think 180°

G-43 USING THE IDEAS

1. A 9, 9, 18 **B** 16, 1, 17 **2.** 45 **3.** 2 **4.** 41 **5.** 25 **6.** 1 **7.** 200 **8.** 6.25 **9.** 71.41 **10.** 223.49
11. 9,2; 6, 7

G-45 USING THE IDEAS

1. A 8.5 **B** 13 **C** 17 **2. A** 4 **B** 12 **C** 12
3. 10 cm **4.** 24 **5.** 500 **6. A** 24 **B** 20 **C** 36
7. A 20 **B** 0.5 **C** 25 **D** 9 **E** 16 **F** 10

G-47 USING THE IDEAS

1. A 15 **B** 7.5 **C** 15 **2. A** 337.5 **B** 279 **C** 27.225 **D** 0.875 **E** 3.5 **F** 93.75 **G** 840
3. A 1620 **B** 1.62 **4.** 24 **5.** 81
Think 6

G-49 USING THE IDEAS

1. A 560 **B** 24 **C** 600 **2. A** 8 **B** 1.1529 **C** 4.35132 **3. A** 560 g **B** 24 g **C** 600 g **4. A** 8 **B** 1.1529 **C** 4.35132 **5.** 1000 **6.** *C* **7. A** 80 cm^3 **B** 32 cm^3
Think mm

G-51 USING THE IDEAS

1. A 84 **B** $38\frac{1}{2}$ **C** 89.46 **D** 95.472 **2. A** 20 **B** 190 **C** 18 **3.** 2 592 276 m^3 **4. A** 49.5 **B** 113 850
5. 2500 cm^3 **6. A** 320 **B** 1408

G-52 REVIEWING THE IDEAS

1. A 3.7 **B** 0.37 **C** 0.037 **2. A** 55 **B** 80 **3. A** 75 **B** 53.94 **C** 11.3 **D** 20 **4.** 166.6667 m **5. A** 49.4 **B** 0.8075 **6. A** 72°29′36″ **B** 71°37′42″ **7.** 52°45′
8. 23° **9.** 360° **10.** 13 **11.** 5 m **12.** 3000
13. A 0.2 kg **B** 1 ℓ **14.** 14.4 **15. A** 480 **B** 480 **C** 160

UNIT G: MODULE 3

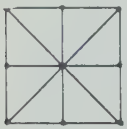
G-57 USING THE IDEAS

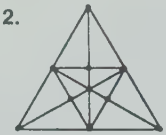
1. A yes **B** yes **2. A** 7 **B** 6 **C** 9 **3. A** 6 **B** 6 **C** 6 **4. A** 4 **B** 4.5 **C** 5 **D** 8 **E** 7.5 **F** 4.5 **5. A** 3 **B** 6 **C** 3.75 **D** 7.5

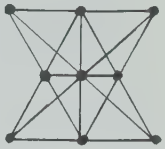
G-59 USING THE IDEAS

1. $l(\overline{DE}) = 6$; $l(\overline{EF}) = 15$; $l(\overline{DF}) = 12$ **2. A** yes **B** yes **C** yes **3.** yes **4.** 60° **5.** $m\angle P = 27°$; $m\angle Q = 90°$; $m\angle R = 63°$ **6.** 141 × 96 **7.** 60 **8.** 3 to 4

Think

1. 

2. 

3. 

G-61 USING THE IDEAS

1. $m\angle R = 35°$; $m\angle S = 120°$; $m\angle T = 25°$ **2.** 6 **3.** 5 **4.** 7.5 **5.** $1\frac{1}{7}$ **6.** $5\frac{2}{5}$ **7.** 1.32 **8.** $\frac{2}{3}$ **9.** 3.57 **10.** $1\frac{49}{51}$ **11.** 15 m **12.** 192 m **13.** 0.9

G-63 USING THE IDEAS

1. *AMN* **2.** w_2/y_2 **3.** $b/f = c/g = d/h$ **4. A** $\frac{1}{2}$ **B** $\frac{1}{2}$ **5. A** $\frac{4}{5}$ **B** $\frac{4}{5}$ **6.** $\frac{1}{2}$ **7.** $l(a) = 20$; $l(b) = 15$; $l(c) = 10$; $l(d) = 5$ **8.** 14.86 m

G-65 USING THE IDEAS

1. A 0.35 **B** 0.34 **C** 0.95 **2. A** 0.5 **B** 60° **3. A** 1 **B** 45° **4. A** 0.95 **B** 0.35 **C** 2.7 **5.** 5 **6.** 4.55 **7.** 3.8 **8. A** 70° **B** 4.1 m

G-67 USING THE IDEAS

1: A 8.55 **B** 12.3 **2. A** 18.75 **B** 35.29 **3. A** 0.42 **B** 0.38 **C** 0.92 **4. A** 0.225 **B** 0.220 **5. A** 5 **B** 0.6 **C** 0.8 **D** 0.75

G-69 USING THE TRIGONOMETRIC TABLE

1. A 407 **B** 1.036 **C** 0.574 **D** 0.602 **E** 6.314 **F** 0.454 **G** 0.035 **H** 0.574 **2. A** 75° **B** 27° **C** 70° **D** 4° **3.** 90° **4. A** 4.07 **B** 17.42 **C** 5.32 **D** 9.96 **5. A** 37° **B** 67° **C** 44° **D** 56° **6.** 3° **7.** 2355.6 km

G-70 REVIEWING THE IDEAS

1. 8 **2.** 7 **3. A** $\angle J \cong \angle P$; $\angle K \cong \angle Q$; $\angle L \cong \angle R$ **B** $\overline{JK}$ and $\overline{PQ}$; $\overline{KL}$ and $\overline{QR}$; $\overline{JL}$ and $\overline{PR}$ **4.** $4\frac{3}{8}$ **5.** $\angle S$ **6.** $\frac{8}{9}$ **7.** *A* and *E*, *C* and *D*, *B* and *F* **8. A** sin **B** cos **C** tan **9.** 37° **10. A** 0.454 **B** 0.788 **C** 0.344 **D** 0.326 **E** 1.111 **F** 0.993 **11. A** 70° **B** 9° **C** 30° **D** 60° **12.** $x = 23.8$; $y = 49.1$

UNIT G: MODULE 4

G-75 USING THE IDEAS

1. A 1 cm **B** chord **C** $\overset{\frown}{ACD}$ **2. A** interior **B** on **C** exterior **3.** 10, 15, 21, 28, 36, 45 **4. A** *OB* **B** no **C** yes

G-77 USING THE IDEAS

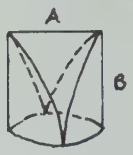
1. A circle **B** radius **C** Anywhere on the sphere **2.** no **3.** interior to sphere **4. A** hemisphere **B** spherical cap **5. A** $\overline{OB}$ **B** no **C** yes **6. A** 90° **B** sum = 180°

G-79 USING THE IDEAS

1. A 28 cm **B** 22 cm **2. A** *A* 6 cm; *B* 6 cm; *C* 6 cm; *D* 6 cm **B** *A* **C** *D* **3. A** *D* **B** *F* **C** *E*

4. A **B**

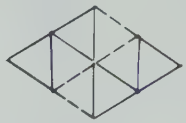
Think To see the circular shape, view from *A*. To see the triangular shape, view from *B*.



G-81 USING THE IDEAS

1. to **4.** Constructions left to the student. **5.** 3.14

Think



G-83 USING THE IDEAS

1. A 25.12 **B** 47.1 **C** 19.468 **D** 376.8 **E** 2.23 **F** 4.08 **G** 3.14 dm **H** 7.54 **2.** 26.5 **3. A** 31.4 **B** 42.7 **C** 232.36 **D** 5.02 **4.** 40 000 **5. A** 37.68 **B** 12 **6. A** 490 **B** 245 **7.** 3.140868

G-85 USING THE IDEAS

1. A 153.86 **B** 314 **C** 4.52 **D** 0.015 **2.** 5 **3. A** 16, 50.24, 200.96 **B** 2.5, 15.7, 19.625 **C** 2, 4, 12.56 **D** 10, 62.8, 314 **E** 2.25, 4.5, 15.90 **F** 0.24, 0.75, 0.045 **4. A** 1017.36 **B** 278.64 **5.** 169.18

G-86 CIRCUMFERENCE AND AREA PROBLEMS

1. A 11.304 **B** 5.652 **C** 5.652 **D** 10.1736 **E** 5.0868 **2.** 571.48 g **3.** 9.8 **4.** 400 **5.** 9555.3 **6.** 14.6 **7. A** 3.5325, 6.28, 9.81

G-89 USING THE IDEAS

1. A 113.04, 1130.4, 376.8 **B** 452.16, 2712.96, 452.16 **C** 1256, 75 360, 7536 **D** 0.785, 0.196, 0.79 **E** 2.96, 27.5, 148.7 **2.** 5.6 **3. A** 10, 113.04, 37.68, 301.44, 188.4 **B** 2, 4.52, 7.54, 2.41, 7.54 **C** 12, 78.5, 31.4, 314, 204.1 **4.** 1.96 m^3 **5. A** 616 **B** same

G-91 USING THE IDEAS

1. A 113.04, 113.04 **B** 904.32, 452.16 **C** 6202.7, 1632.3 **D** 2143.6, 803.8 **E** 4.19, 12.56 **2. A** 4082 g **3. A** 514 457 600 **B** 363 207 060 **C** 1 097 509 500 000 **4. A** 37 503 959 **B** 21 602 280 000 km^3 **5.** 4.19 **6. A** 7234.56 cm^3 **B** 6589.44 cm^3 **C** 1808.6 cm^2 **D** 1647.36 cm^2 **7.** $(\frac{4}{3} \cdot \pi \cdot r^3 / \pi \cdot r^2 \cdot h) = (\frac{4}{3}/2) = \frac{2}{3}$ **8.** $8x$, $4x$ **9.** $27x$, $9x$

G-92 REVIEWING THE IDEAS

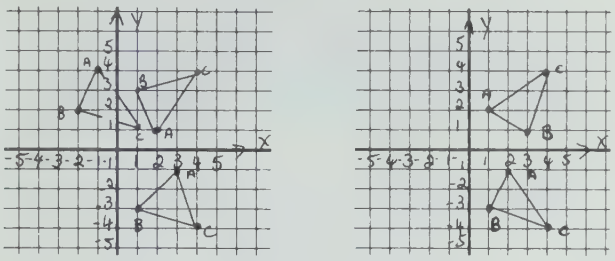
1. A *AB* **B** *OC* **C** *O* **D** $\overset{\frown}{BCD}$, or $\overset{\frown}{ABC}$ **2.** great circle **3.** *B* **4.** 3.14 **5.** $C = \pi d$ or $2\pi r$ **6.** 77.24 **7.** 2.8 **8.** πr^2 **9.** 117.6 **10.** 125.6 **11.** 784 **12.** 4.46

13. A 6 cm **B** 28.26 **C** 4.5 **14.** 5.19 **15.** 1.73
16. 305.3 **17. A** 38.8 **B** 55.4

G-94-95 CUMULATIVE REVIEW

1. $A'(-1, 4)$, $B'(-2, 2)$, $C'(2, 1)$ **2.** $A''(2, 1)$, $B''(1, 3)$, $C''(5, 4)$

3.



4. A $\angle Y$ **B** $\overline{XZ}$ **5.** C and D (ASA); B and F (SSS); A and E (SAS) **6.** $\angle 3$, $\angle 5$, $\angle 7$ **7. A** $A = 9.87$; $P = 13.6$ **B** $A = 24$; $P = 24$ **C** $A = 10.6$; $P = 15$ **D** $A = 30.59$; $P = 28.8$ **8.** $m\angle 1 = 114°$; $m\angle 3 = 24°$; $m\angle 4 = 66°$; $m\angle 5 = 90°$; $m\angle 6 = 66°$ **9. A** 110° **B** 70° **C** 50°
10. A 17 **B** 16 **C** 10 **D** 0.5 **11. A** 1134 **B** 105
12. 1380 **13.** 4 **14.** $g = 6$; $h = 5.1$ **15. A** $\frac{5}{13}$ **B** $\frac{12}{13}$ **C** $\frac{5}{12}$ **16.** 16.97 **17.** 3.14 to 1 **18.** 31.4 **19.** 113.04
20. A 785 **B** 314 **C** 471 **21. A** 96 **B** 75.36
22. A 33.5 **B** 50.24

UNIT H: MODULE 1

H-3 USING THE IDEAS

1. A 50 black; 30 orange; 20 white **B** 500, 0.5; 300, 0.3 200, 0.2 **2. A** 50 **B** 0.5, 0.5 **3. A** 6 **B** $\frac{1}{6}$ **4. A** 3 **B** $\frac{1}{3}$ **C** 20 **D** 100

Think

17	113	47
89	59	29
71	5	101

H-5 USING THE IDEAS

1. A $\frac{1}{2}$ **B** $\frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{6}$ **C** $\frac{1}{4}, \frac{1}{2}$ **D** $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ **2.** 1
3. A $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ **B** yes **C** $\frac{1}{12}$
4. A 25 **B** A: $\frac{18}{25}$; B: $\frac{6}{25}$; C: $\frac{1}{25}$ **C** A: 72; B: 24; C: 4

H-7 USING THE IDEAS

1. A $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\{2, 3\}$, $\frac{1}{2}$ **D** $\{1, 2, 3, 4\}$, 1 **E** $\varnothing$, 0
2. A $\frac{1}{2}$ **B** $\frac{1}{3}$ **C** $\frac{1}{6}$ **D** 1 **E** $\frac{2}{3}$ **3. A** $\frac{1}{3}$ **B** $\frac{2}{3}$ **C** $\frac{5}{6}$ **D** $\frac{1}{3}$ **E** $\frac{1}{2}$ **F** $\frac{2}{3}$ **4. B** $P(\{2, 4, 6, 8\}) = \frac{4}{9}$ **C** $P(\{4, 8\}) = \frac{2}{9}$ **D** $P(\{1, 2, 4\}) = \frac{1}{3}$ **E** $P(\{1, 4, 9\}) = \frac{1}{3}$ **F** $P(\{2, 3, 5, 7\}) = \frac{4}{9}$ **G** $P(\{1, 2, 3, 6\}) = \frac{4}{9}$ **H** $P(\{8, 9\}) = \frac{2}{9}$ **I** $P(\{7, 8, 9\}) = \frac{1}{3}$ **J** $P(\{1, 2, 3, 4, 5, 6\}) = \frac{2}{3}$ **K** $P(\{1, 2, 3, 4, 5, 6, 7\}) = \frac{7}{9}$ **L** $P(\{1, 2, 3, 4, 5, 6, 8\}) = \frac{7}{9}$ **M** $P(\{2, 4, 6\}) = \frac{1}{3}$ **N** $P(\{1, 2, 4\}) = \frac{1}{3}$

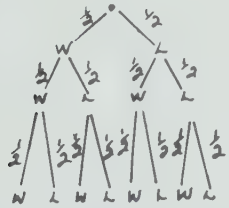
H-9 USING THE IDEAS

1. A $\{(H, H), (H, T), (T, H), (T, T)\}$ **B** $\frac{1}{4}$ **2. A** $\{(H, T)\}$ **B** $\{(H, H), (H, T)\}$ **C** $\{(H, T), (T, T)\}$ **D** $\{(H, H), (H, T), (T, H)\}$ **E** $\{(H, T), (T, H)\}$ **F** $\{(H, H), (T, T)$ **3. A** $\frac{1}{4}$ **B** $\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $\frac{3}{4}$ **E** $\frac{1}{2}$ **F** $\frac{1}{2}$
4. A $(D, 3)$ **B** $\{(A, 1), (A, 2), (A, 3), (B, 1), (B, 2), (B, 3), (C, 1), (C, 2), (C, 3), (D, 1), (D, 2), (D, 3)\}$ − 12 **C** $\frac{1}{12}$
5. A $\frac{1}{36}$ **B** $\frac{1}{9}$ **C** $\frac{1}{12}$ **D** $\frac{5}{6}$ **E** $\frac{5}{36}$ **F** $\frac{1}{6}$ **G** $\frac{1}{36}$ **H** $\frac{15}{36}$

H-11 USING THE IDEAS

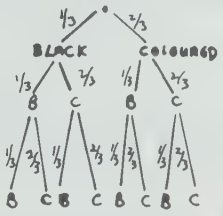
1. A $\frac{1}{4}$ **B** $\frac{1}{8}$ **C** $\frac{1}{8}$ **D** $\frac{1}{8}$ **E** $\frac{1}{16}$ **F** $\frac{1}{16}$ **G** $\frac{1}{8}$ **H** $\frac{1}{16}$ **I** $\frac{1}{16}$

2.



3. A $\frac{1}{4}$ **B** $\frac{1}{8}$ **C** $\frac{1}{8}$ **D** $\frac{1}{8}$ **E** $\frac{3}{8}$ **F** $\frac{1}{8}$

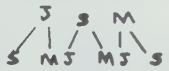
4.



5. A $\frac{2}{9}$ **B** $\frac{1}{9}$ **C** $\frac{4}{9}$ **D** $\frac{4}{27}$ **E** $\frac{4}{27}$ **F** $\frac{8}{27}$ **G** $\frac{1}{27}$ **H** $\frac{4}{27}$

H-13 USING THE IDEAS

1. A



B 6 **2.** 6 **3.** 24 **4.** 6 **5.** 120 **6.** 720
Think 10:16¢, 31¢, 36¢, 40¢, 56¢, 61¢, 65¢, 76¢, 80¢, 85¢

H-15 USING THE IDEAS

1. 5! = 120; 6! = 720; 7! = 5040; 8! = 40 320; 9! = 362 880; 10! = 3 628 800; 11! = 39 916 800; 12! = 479 001 600 **2.** 6 **3.** 362 880 **4.** 5040 **5.** 120 **6.** 720 **7. A** 10! = 3 628 800 **B** 7! = 5040 **C** 11! = 39 916 800 **D** 5 **E** 20 **F** 6 **G** 90 **H** 60 **I** 5 **J** 7! = 5040 **K** 4! = 24 **L** 6! = 720 **M** 72 **N** 5
Think *ETTF*

H-17 USING THE IDEAS

1. A 5 **B** 4 **C** 20 **2.** 60 **3.** 116 280 **4.** 8 **5. A** 9 **B** 10 **C** 90 **7.** 900 **8.** 15 600

H-19 USING THE IDEAS

1. 6 **2.** 4 **3. A** 4 **B** 1 **4.** 21 **5.** 45 **6. A** 3 **B** 15 **C** 5 **D** 120 **7.** 15 **8.** 22 100
Think 6

H-21 USING THE IDEAS

1. A (V, S), (V, B), (S, B) **B** 2 **C** $\frac{2}{3}$ **D** $\frac{1}{3}$ **2. A** 20 **B** 2 **C** $\frac{1}{10}$ **3. A** 10 **B** $\frac{1}{10}$ **4. A** 45 **B** 10 **C** $\frac{2}{9}$ **D** $\frac{2}{9}$ **E** $\frac{5}{9}$

H-23 USING THE IDEAS

1. A E: $\{2, 4, 6\}$, $P(E) = \frac{1}{2}$; F: $\{1, 2, 3\}$, $P(F) = \frac{1}{2}$ **B** $E \cap F = \{2\}$; $P(E \cap F) = \frac{1}{6}$ **C** $E \cup F = \{1, 2, 3, 4, 6\}$; $P(E \cup F) = \frac{5}{6}$ **D** $\frac{5}{6}$ **2. A** $\frac{3}{10}$ **B** $\frac{1}{2}$ **C** $\frac{1}{10}$ **D** $\frac{7}{10}$ **3. A** $\frac{1}{12}, \frac{1}{9},$

$\frac{5}{36}, \frac{1}{6}, \frac{5}{36}$ **B** $\frac{23}{36}$ **C** $\frac{1}{36}, \frac{1}{18}, \frac{1}{12}, \frac{1}{9}$ **D** $\frac{10}{36} = \frac{5}{18}$
E $\{2, 3, 4, 5, 6, 7, 8\}$ **F** $\frac{13}{18}$ **G** $\{4, 5\}$ **H** $\frac{7}{36}$ **I** $\frac{13}{18}$

H-24 REVIEWING THE IDEAS

1. 75 **2. A** $\frac{1}{4}$ **B** about 10 white; about 30 red
3. A $\{4, 5, 6\}$ **B** $\{1, 2, 3, 4\}$ **C** $\{2, 4, 6\}$ **D** $\{2, 3, 5\}$
4. A $\frac{1}{2}$ **B** $\frac{2}{3}$ **C** $\frac{1}{2}$ **D** $\frac{1}{2}$
5.

6. A $\frac{1}{4}$ **B** $\frac{1}{6}$ **C** $\frac{1}{18}$ **D** $\frac{1}{9}$ **E** $\frac{1}{12}$ **F** $\frac{1}{36}$ **7.** 24 **8.** 72
9. 6 **B** 20 **C** 5 **10.** 10 **11. A** 24 **B** 6 **C** 36
D 120 **12. A** $E = \{(B, 1), (B, 2), (B, 3)\}$, $P(E) = \frac{1}{4}$
B $F = \{(A, 2), (B, 2), (C, 2), (D, 2)\}$, $P(F) = \frac{1}{3}$ **C** $\frac{1}{12}$ **D** $\frac{1}{2}$

UNIT H: MODULE 2

H-29 USING THE IDEAS

1. *B* **2. A** 56.25 **B** 400 **C** 36 **D** 9 **E** 625 **F** 1
3. A 10 s **B** 25 m **4.** 9 m **5. A** 25 cm **B** 75 **C** 37

H-31 USING THE IDEAS

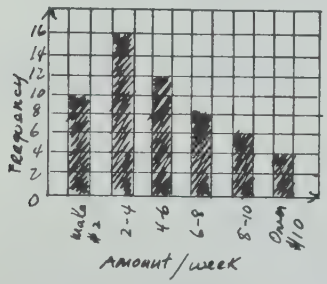
1. A 5 140 000 000 km **B** 1 700 888 800 km **C** Saturn
2. A 412, 442 **B** 137.3, 147.3; Mary **C** Linda 47; Mary 40 **3. A** 30 **B** 81.4 **4. A** 19.3 **B** 47.4 **5.** 92

H-33 USING THE IDEAS

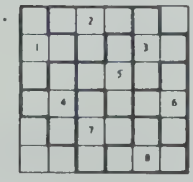
1. 77 **2. A** 40, 59, 65, 65, 65, 67, 68, 72, 76, 80, 100
B 67 **3. A** 12 **B** 12 **4. A** 7.4 **B** 7.96 **C** mean by 0.56 **5.** 4230 **6. A** 23 **B** 31.75
Think yes

H-35 USING THE IDEAS

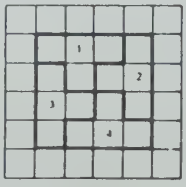
1. 13 **2.** 38 **3. A** 2 and 5 **B** 2 and 5 **4. A** $2–4
B



Think **1.**

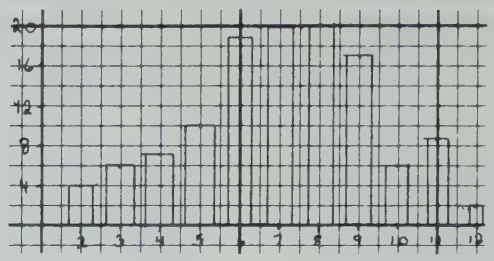


2.

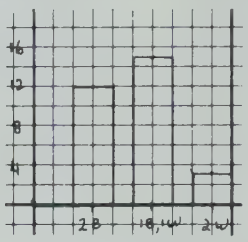


H-37 USING THE IDEAS

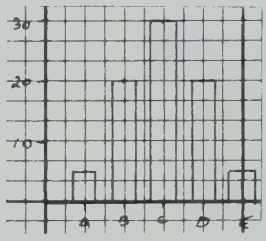
1. A 45 **B** 1, 4, 7, 10, 9, 6, 5, 3 **C** 22.2% **D** $6.6\overline{6}$
2.



3.

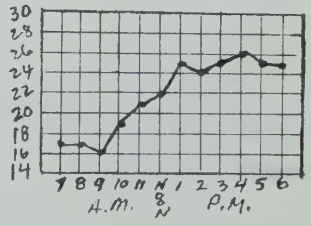


4. A *A*: $\frac{1}{16}$; *B*: $\frac{1}{4}$; *C*: $\frac{3}{8}$; *D*: $\frac{1}{4}$; *E*: $\frac{1}{16}$ **B** *A*: 5; *B*: 20; *C*: 30; *D*: 20; *E*: 5
C



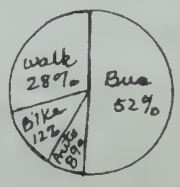
H-39 USING THE IDEAS

1. A *A*: 26; *B*: 22 **B** *A*: 27; *B*: 24.5 **C** *A*: 25.5; *B*: 22
D *A*: 1.5; *B*: 2.5 **2. A** 1600: 0.5 billion; 1700: 0.8 billion; 1800: 1 billion; 1900: 1.5 billion; 1950: 3 billion
B 4.5 billion **C** 10 billion **3. A** −10°C **B** 11°C
C 21°C **D** 17 **4.**



H-41 USING THE IDEAS

1. A basketball, $\frac{6}{15}$; other, $\frac{2}{15}$; tennis, $\frac{7}{12}$; golf, $\frac{1}{30}$; baseball, $\frac{3}{20}$; football, $\frac{1}{5}$ **B** basketball, 40%; other, 13.3%; tennis, 8.3%; golf, 3.3%; baseball, 15%; football, 20%
2. A $3.00 **B** $2.40 **C** 10%; $1.20 **3.** 437 by bus; 235 walk; 100 ride bicycles; 68 by car. **4.**



Think
A–B

H-43 USING THE IDEAS

1. A 13.8% **B** 15.884 **C** 10 801 **2. A** bias **B** unbias **C** bias **D** bias **E** bias **F** unbias **3. A** yes **B** undecided votes **D** *A*: 5 960 000, 4 760 000; *B*: 5 000 000, 3 800 000; *C*: 1 440 000, 240 000

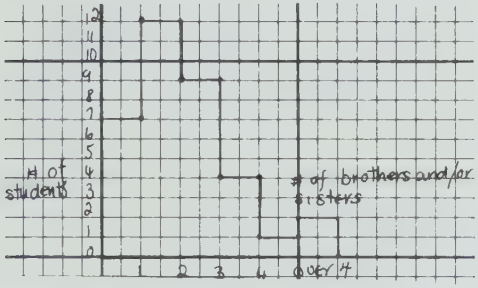
H-45 USING THE IDEAS

1. A 180 **B** 64 **C** 6 **D** 1 **E** no **2.** 0.2882634 **3. A** 73% **B** 876 **C** 324

H-48 REVIEWING THE IDEAS

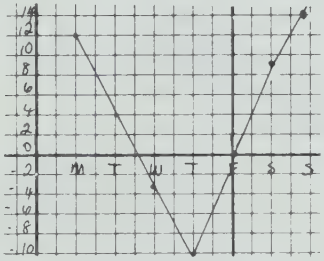
1. $13 **2. A** 18 **B** 87.4 **C** 88.5 **3. A** 1 **B** 1

4.



5. M: 820; T: 827; W: 830; T: 825; F: 837

6.



7. A 15% **B** 72° **8.** under 20: 12 000; 20–39: 7500; 40–60: 6000; over 60: 4500 · **9. A** no **B** no **10.** 625 000 **11.** 59

UNIT H: MODULE 3

H-53 USING THE IDEAS

1. A $\sqrt{3}$ **B** $-\sqrt{3}$ **C** $\sqrt{17}$ **D** $-\sqrt{17}$ **E** positive, 6 **F** negative, 6 **G** positive, 7 **H** $-\sqrt{47}$ **I** n **2. A** 2 **B** 5 **C** 6 **D** 15 **E** 100 **F** 225 **3. A** $\sqrt{9}$ **B** $\sqrt{7}$ **C** $\sqrt{11}$ **D** $\sqrt{16}$ **E** $\sqrt{17}$ **F** $\sqrt{19}$ **G** $\sqrt{63}$ **H** $\sqrt{64}$ **4.** $7 = \sqrt{49}$; $8 = \sqrt{64}$; $9 = \sqrt{81}$; $10 = \sqrt{100}$

H-55 USING THE IDEAS

1. $10, \sqrt{10}, \sqrt{10}, -\sqrt{10}$ **2.** $17, \sqrt{17}, \sqrt{17}, -\sqrt{17}$ **3.** $26, \sqrt{26}, \sqrt{26}, -\sqrt{26}$ **4.** $3, \sqrt{3}, \sqrt{3}, -\sqrt{3}$ **5.** $6, \sqrt{6}, \sqrt{6}, -\sqrt{6}$

H-57 USING THE IDEAS

1. A 3, 4 **B** 5, 6 **C** 12, 13 **2. A** *F* **B** *J* **C** *C* **D** *A* **E** *L* **F** *E* **G** *H* **H** *B* **I** *K* **J** *D* **K** *Q* **L** *G* **M** *M* **N** *P* **O** *O* **P** *N* **Q** *R* **R** *I* **3. A** 2.8; 2.9 **B** 7.2; 7.3 **C** 10.6; 10.7 **4. A** 2.8 **B** 7.2 **C** 10.6 **5.** 1, 2; 1.7, 1.8; 1.73, 1.74; 1.732, 1.733 **6.** 2, 3; 2.2, 2.3; 2.23, 2.24; 2.236, 2.237 **7.** 5, 6; 5.3, 5.4; 5.38, 5.39; 5.385, 5.386 **8.** 9, 10; 9.3, 9.4; 9.32, 9.33; 9.327, 9.328

H-59 USING THE IDEAS

1. A 26 **B** 79 **C** 3.5 **D** 0.18 **E** 6.7 **F** 5.3 **G** 57 **H** 11.8 **2. A** 4.1 **B** 9.4 **C** 12.8 **D** 4.6 **E** 13.1 **F** 20.9 **G** 8.3 **H** 14.1 **I** 0.4 **J** 11.8 **3. A** 9.27 **B** 24.96 **C** 3.61 **D** 4.36 **E** 5.65 **F** 7.27 **G** 2.81 **H** 54.77 **I** 0.63 **J** 1.02 **4.** *B* **5. A** 2 **B** 10 **C** 4 **D** 5 **E** −1 **6. A** 67; 667; 6667; 66667

Think **1.** 3 : 4 **2.** 7 : 6 **3.** engine 8 cm; dining car 12 cm; caboose 6 cm

H-61 USING THE IDEAS

1. A 2.449 **B** 7.746 **C** 3.606 **D** 11.402 **E** 4.243 **F** 13.416 **G** 3.317 **H** 10.488 **I** 14 **J** 13 **2. A** 2.828 **B** 4.472 **C** 3.464 **3. B** 11 **C** 150 **D** 90 **E** 15 **F** 15 **G** 3.162 **H** 11 **4. A** 2.121 **B** 1.225 **C** 3.240 **D** 4.183 **E** 2.915 **F** 4.278 **G** 4.405 **H** 4.025 **I** 3.688 **J** 3.847 **5.** 5.39 **6.** 4.47

H-63 USING THE IDEAS

1. A 3.16 **B** 4.24 **C** 5.385 **2.** 9.5 **3.** 10.4 **4.** yes **5.** 1.97 **6.** 6 km

H-65 USING THE IDEAS

1. A $0.\overline{6}$, 1 **B** $0.\overline{2857142}$, 7 **C** $0.625\overline{0}$, 1 **D** $0.\overline{384615}$, 6 **E** $0.8\overline{3}$, 1 **F** $0.0\overline{37}$, 3 **G** $0.\overline{81}$, 2 **2. B** $4.\overline{4}$ **C** $3.\overline{83}$ **D** $7.0\overline{7}$ **E** $842.\overline{842}$ **3. A** 4.444 . . . **B** 4 **C** $\frac{4}{9}$ **4. A** 27.2727 . . . **B** 27 **C** $\frac{3}{11}$ **5. A** 74.074074 . . . **B** 74 **C** $\frac{74}{999}$ **6. A** $\frac{5}{9}$ **B** $\frac{37}{99}$ **C** $2\frac{3}{11}$ **D** $\frac{5}{333}$ **E** $\frac{41}{333}$ **F** $\frac{11}{990}$ **G** $\frac{1}{18}$ **H** $4\frac{3}{110}$ **I** $3\frac{2}{9}$ **J** $\frac{1}{999}$ **7. A** $0.\overline{1}$ **B** $0.\overline{2}$ **C** $0.\overline{01}$ **D** $0.\overline{02}$ **E** $0.\overline{001}$ **F** $0.\overline{002}$ **8. A** $\frac{1}{9}, \frac{2}{9}$ **B** $\frac{1}{9}, \frac{1}{3}$ **C** $\frac{1}{9}, \frac{4}{9}$ **D** $\frac{1}{99}, \frac{2}{99}$ **E** $\frac{1}{99}, \frac{1}{11}$ **F** $\frac{1}{99}, \frac{10}{99}$ **G** $\frac{1}{99}, \frac{1}{9}$ **H** $\frac{1}{99}, \frac{4}{33}$ **I** $\frac{1}{99}, \frac{23}{99}$ **J** $\frac{1}{99}, \frac{68}{99}$ **K** $\frac{1}{999}, \frac{2}{999}$ **L** $\frac{1}{999}, \frac{1}{111}$ **M** $\frac{1}{999}, \frac{23}{999}$ **N** $\frac{1}{999}, \frac{467}{999}$

H-67 USING THE IDEAS

1. A rational **B** rational **C** irrational **D** rational **E** irrational **F** rational **G** rational **H** irrational **2. A** 4.414 **B** 2.828 **C** 0.414 **D** 2.828 **E** 0.2828 **F** $0.902\overline{3}$ **G** 0.7072 **H** 0.707 **3. A** 31.4 **B** 12.56 **C** 28.26 **D** 0.32 **E** 0.02 **F** 9.86 **G** 17.58 **H** 19.63 **4.** 1.62

Think **A** $\frac{355}{113}$ **B** 3.1415928

H-69 USING THE IDEAS

1. A rational **B** irrational **C** irrational **D** rational **2. A** 6 **B** 18 **C** 21 **D** 24 **E** 12 **F** 105 **3. A** $2 \cdot 3 = 6$ **B** $3 \cdot 5 = 15$ **C** $2 \cdot 7 = 14$ **D** $7 \cdot 5 = 35$ **4.** $\sqrt{6}$ **5.** 2.449 **B** 2.449 **6. A** $\sqrt{15}$ **B** $\sqrt{14}$ **C** $\sqrt{35}$ **D** $\sqrt{21}$ **E** $\sqrt{55}$ **F** $\sqrt{143}$ **G** $\sqrt{187}$ **H** $\sqrt{51}$ **I** 13 **7.** 6

H-70 REVIEWING THE IDEAS

1. A $\sqrt{5}$ **B** $-\sqrt{5}$ **C** positive, 13 **D** negative, 13 **2.** $\sqrt{10}$ **3.** −6 and −7 **4. A** 3, 4 **B** 5, 6 **C** 18, 19 **5. A** 3 **B** 5 **C** 18 **D** 4 **E** 16 **F** 13 **6. A** 3.6

B 7.28 **7.** 4.3 **8.** 12.77 **9. A** 3 **B** 10 **C** 52 **D** 7 **10.** 8.1 **11.** $0.\overline{857142}$ **12.** $\frac{23}{99}$ **13. B** irrational **14.** 14 **15.** 6.71 **16.** 33 **17.** 4.863 **18.** 16.97

UNIT H: MODULE 4

H-75 USING THE IDEAS

1. A 3 **B** 2 **C** 2 **D** 0 **E** 3 **F** 3 **G** 0 **H** 1 **I** 2 **J** 1 **2. A** 8 **B** −24 **C** 9 **D** −28 **E** −40 **F** 30 **G** −44 **H** −60 **3. A** true **B** true **C** false **D** true **E** true **F** false **4. A** 1 **B** 2 **C** 0 **D** 3 **E** 1 **F** 0 **G** 1 **H** 0 **I** 3 **5. A** 0 **B** 1 **C** 2 **D** 3 **6. A** 3 **B** 0 **C** 3 **D** 2 **E** 0 **F** 3 **G** 2 **H** 0

H-77 USING THE IDEAS

1.

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2. A 3 **B** 0 **C** 1 **D** 2 **E** 0 **F** 3 **3.** yes **4.** yes **5. A** 0 **B** 3 **C** 2 **D** 1 **6. A** 2 **B** 0 **7. A** 3 **B** 3 **C** 2 **D** 0 **E** 3 **8. A** $1 \oplus_4 2 = 3$ **B** $2 \oplus_4 1 = 3$ **C** $3 \oplus_4 3 = 2$ **D** $2 \oplus_4 2 = 0$ **E** $3 \oplus_4 0 = 3$

H-79 USING THE IDEAS

1.

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

2. A 2 **B** 0 **C** 3 **D** 3 **E** 0 **F** 2 **3.** yes **4.** 1, 3 **5.** 1 **6. A** no **B** 1 and 1; 3 and 3 **7. A** 2 **B** 2 **C** 1 **D** 1 **E** 2 **F** 2

Think The ratio of the areas.

H-81 USING THE IDEAS

1 and **2. A** $L \star L = U$ **B** $U \star R = L$ **C** $R \star R = U$ **D** $L \star R = I$ **3. A** *U* **B** *R* **C** *L* **D** *U* **E** *I* **F** *R* **G** *U* **H** *R* **I** *U* **4. A** *U* **B** *U* **C** *U* **D** *U* **E** *R* **F** *R* **G** *L* **H** *L* **5.** associativity **6.** *I* times any element is that element. **7. A** *I* **B** *L* **C** *U* **D** *R* **8.** yes, *I* and *I*, *L* and *R*, *R* and *L*, *U* and *U*

H-83 USING THE IDEAS

1. A $H \star H = I$ **B** $\not{Q} \star \not{\emptyset} = U$ **C** $U \star \not{Q} = \not{\emptyset}$ **D** $\not{\emptyset} \star U = \not{\emptyset}$

3.

☆	I	R	U	L	H	V	$\not{Q}$	$\not{\emptyset}$
I	I	R	U	L	H	V	$\not{Q}$	$\not{\emptyset}$
R	R	U	L	I	$\not{\emptyset}$	$\not{Q}$	H	V
U	U	L	I	R	V	H	$\not{\emptyset}$	$\not{Q}$
L	L	I	R	U	$\not{Q}$	$\not{\emptyset}$	V	H
H	H	$\not{Q}$	V	$\not{\emptyset}$	I	U	R	L
V	V	$\not{\emptyset}$	H	$\not{Q}$	U	I	L	R
$\not{Q}$	$\not{Q}$	V	$\not{\emptyset}$	H	L	R	I	U
$\not{\emptyset}$	$\not{\emptyset}$	H	$\not{Q}$	V	R	L	U	I

4. I **5.** not commutative in flip section of octic system; $R \star H = \not{\emptyset}$ while $H \star R = \not{Q}$ **6.** $I \star I = I$, $U \star U = I$, $L \star R = I$, $H \star H = I$, $V \star V = I$, $\not{\emptyset} \star \not{\emptyset} = I$ **7. A** I **B** I **C** V **D** V **E** R **F** R **8.** yes **9.** U

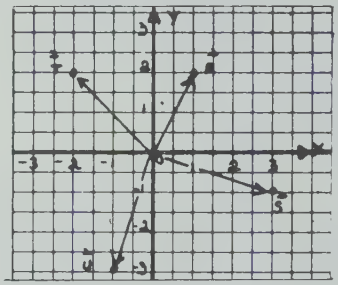
H-85 USING THE IDEAS

1. A (−3, 1) **B** (−1, −1) **C** (0, 3) **D** (4, 5) **E** (3, −4) **F** (4, 2) **2. A** $\sqrt{10}$ **B** $\sqrt{2}$ **C** 3 **D** $\sqrt{41}$ **E** 5 **F** $\sqrt{20}$ **4. A** $\sqrt{17}$ **B** $\sqrt{2}$ **C** 4 **D** 5 **E** 2 **F** $\sqrt{29}$ **G** $\sqrt{18}$ **H** 5 **5. A** *b* **B** *d* **C** *e* **D** *g*

Think 24

H-87 USING THE IDEAS

1.



2. A (5, 1) **B** (1, 1) **C** (−3, −1) **D** (2, −4) **E** (1, −1) **F** (0, 4) **3.** (−7, 3) **4.** (0, 0) **5. A** (0, 0) **B** (0, 0) **C** yes **6. A** (−6, 5) **B** yes **7. A** (−1, 4) **B** (1, −5) **C** (3, 1) **D** (3, 1) **8.** yes **9.** (−8, −11)

Think $65 = 1^2 + 8^2 = 4^2 + 7^2$; $85 = 2^2 + 9^2 = 6^2 + 7^2$; $125 = 2^2 + 11^2 = 5^2 + 10^2$

H-89 USING THE IDEAS

1.

$\otimes$	0	1
0	0	0
1	0	1

2. A 0 **B** 0 **C** 0 **D** 1 **3. A** closed **B** $1 \otimes 1 = 1$ **4. A** open **B** $0 \otimes 1 = 0$ **5. A** no **B** open **C** 0 **6. A** 0 **B** 0 **C** 0 **D** 1 **7.** $0 \otimes 1 = 0$, $1 \otimes 0 = 0$ **8.** yes

H-91 USING THE IDEAS

1. A 1 **B** 0 **C** 1 **D** 1 **E** 0 **F** 0

2. A $(1 \otimes 1) \oplus 0 = 1$ **B** $(0 \otimes 1) \oplus 0 = 0$ **C** $(0 \otimes 0) \oplus 1 = 1$ **D** $(1 \otimes 1) \oplus 1 = 1$ **E** $(0 \otimes 1) \oplus 0 = 0$ **F** $(0 \otimes 0) \oplus 0 = 0$

4. A 2 (eg. F and E) **6.** 7 (eg. D, E, J)

H-92 REVIEWING THE IDEAS

1.

$\oplus_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2. A 0 **B** 1 **C** 1 **D** 0

3.

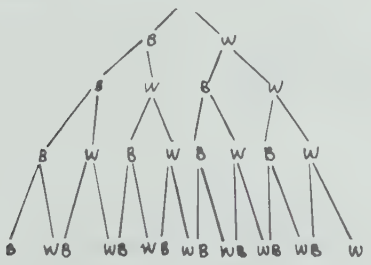
$\otimes_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

4. A *R* **B** *L* **C** *L* **D** *R* **E** *U* **F** *L* **5. A** D **B** D **C** *U* **D** *U* **E** *I* **F** *U* **6. A** (3, 4) **B** 5 **C** $(-3, -4)$
7. (8, 2) **8.** (5, 2) **9.** $(-6, 2)$ **10.** (4, 4)
11. A parallel **B** series **12.** yes **13. A** 0 **B** 1 **C** 1 **D** 1 **E** 0 **F** 1 **G** 1 **H** 1

H-94-95 CUMULATIVE REVIEW

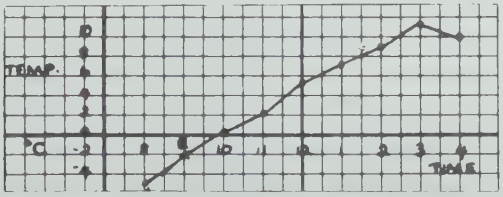
1. $\frac{2}{3}$ **2.** $\frac{1}{6}$ **3.** $\frac{5}{8}$ **4. A** $\frac{1}{4}$ **B** $\frac{1}{4}$ **C** $\frac{1}{2}$ **D** $\frac{3}{4}$

5.



6. 24 **7.** 20 **8.** 21 **9. A** $\frac{2}{3}$ **B** $\frac{1}{3}$ **C** 0 **D** 1
10. A 39 **B** 163 **11.** 88 **12. A** 4 **B** 6 **C** 0
13. A 2H, 1T **B** 3H **C** 24

14.



15. A 40% **B** 72° **16. A** 10 600 **B** 9400
17. $\sqrt{2}, -\sqrt{2}$ **18.** 1.4

19.

20. A 3, 4 **B** 6, 7 **C** 9, 10 **D** 24, 25 **21. A** 6.9 **B** 30.5 **C** 75 **22.** $0.\overline{5}$ **23. A** $\frac{8}{25}$ **B** $\frac{71}{1000}$ **C** $\frac{7}{9}$
24. A repeating **B** non-repeating **25. A** 15 **B** 14 **C** 105 **D** 6 **26. A** 3.65 **B** 3.16 **C** 0.82 **D** 1.58
27. A 0 **B** 1 **28. A** 0 **B** 0 **29.** U ☆ L = R **30.** U
31. (9, 6) **32.** (2, −6) **33.** C

SUPPLEMENTARY EXERCISES

S-2 SET 1

1. A 5^2 **B** 8^3 **C** 2^5 **D** 95^4 **E** 10^6 **F** 3^9 **2. A** 10^3 **B** 10^4 **C** 10^5 **D** 10^5 **E** 10^6 **F** 10^7 **G** 10^5 **H** 10^9
3. A 10^2 **B** 10^1 **C** 10^3 **D** 10^5 **E** 10^6 **F** 10^2 **G** 10^6 **H** 1 **4. A** $(3 \cdot 10) + 8$ **B** $(6 \cdot 10^2) + (2 \cdot 10) + 1$ **C** $(8 \cdot 10^2) + 2$ **D** $(1 \cdot 10^3) + (2 \cdot 10^2) + (5 \cdot 10) + 9$ **E** $(2 \cdot 10^3) + (3 \cdot 10) + 5$ **F** $(8 \cdot 10^3) + 1$ **G** $(1 \cdot 10^4) + (4 \cdot 10^3) + (3 \cdot 10) + 5$ **H** $(8 \cdot 10^4) + (1 \cdot 10^3) + (2 \cdot 10^2) + (5 \cdot 10) + 3$ **I** $(6 \cdot 10^5) + (8 \cdot 10^4) + (7 \cdot 10^3) + (5 \cdot 10^2) + (2 \cdot 10) + 4$ **J** $(3 \cdot 10^5) + (5 \cdot 10)$

S-2 SET 2

1. A 70 **B** 90 **C** 130 **D** 640 **E** 1250 **F** 7510 **G** 15 630 **H** 48 110 **I** 200 010 **J** 555 560 **2. A** 400 **B** 600 **C** 1300 **D** 6800 **E** 9100 **F** 17 900 **G** 24 000 **H** 105 100 **I** 843 700 **J** 503 500 **3. A** 1000 **B** 3000 **C** 6000 **D** 16 000 **E** 21 000 **F** 135 000 **G** 458 000 **H** 756 000 **4. A** 5 607 380 **B** 5 607 400 **C** 5 607 000 **D** 5 610 000 **E** 5 600 000 **F** 6 000 000

S-3 SET 3

1. A 400 **B** 1300 **C** 30 **D** 400 **E** 500 **F** 6000 **G** 42 000 **2. A** 800 **B** 500 **C** 5400 **D** 30 **E** 18 000 **F** 9800 **G** 1900 **H** 4 **I** 600 000 **J** 23 000 **K** 450 000 **L** 6 000 000

S-3 SET 4

1. A 89 **B** 92 **C** 207 **D** 844 **E** 2204 **F** 18 932 **G** 204 **H** 208 **I** 1500 **J** 1516 **K** 9814 **L** 88 087 **2. A** 196 **B** 262 **C** 247 **D** 3885 **E** 1266 **F** 700 **G** 1565 **H** 6862 **I** 367 **J** 19 461 **K** 41 070 **L** 6950 **M** 11 097 **N** 79 594 **O** 77 274 **P** 896 218 **Q** 1 590 423 **R** 13 090 851

S-4 SET 5

1. A 33 **B** 33 **C** 47 **D** 162 **E** 686 **F** 449 **G** 1229 **H** 991 **I** 59 **J** 555 **K** 3194 **L** 9409 **2. A** 66 **B** 81 **C** 178 **D** 284 **E** 625 **F** 628 **G** 99 **H** 6961 **I** 8890 **J** 898 **K** 49 908 **L** 28 587 **M** 27 000 **N** 66 005 **O** 96 081 **P** 90 000 **Q** 99 000 **R** 99 900 **S** 318 207 **T** 689

S-4 SET 6

1. A 312 **B** 584 **C** 783 **D** 516 **E** 3192 **F** 4515 **G** 3152 **H** 27 654 **I** 72 720 **J** 37 533 **K** 185 295 **L** 636 321 **2. A** 972 **B** 1742 **C** 1012 **D** 8855 **E** 11 000 **F** 45 980 **G** 82 800 **H** 429 786 **I** 349 932 **J** 270 000 **K** 853 576 **L** 4 507 125 **3. A** 35 979 **B** 204 240 **C** 292 521 **D** 431 210 **E** 175 910 **F** 858 945 **G** 369 342 **H** 1 780 731 **I** 4 269 687 **J** 5 252 065 **K** 5 771 500 **L** 4 066 540

S-5 SET 7

1. A 14, R4 **B** 15 **C** 15, R5 **D** 64 **E** 37, R1
F 63, R3 **G** 167 **H** 83, R4 **I** 137 **J** 253, R4
K 476, R1 **L** 4999, R1 **M** 4287, R4 **N** 5706, R1
O 8785, R4 **2. A** 4, R15 **B** 8, R12 **C** 4, R28
D 29, R2 **E** 83, R4 **F** 34, R11 **G** 28, R3 **H** 122, R50
I 98, R62 **J** 2700 **K** 876, R22 **L** 823, R88
3. A 22, R2 **B** 8, R31 **C** 9, R1 **D** 13, R30
E 9, R270 **F** 9, R821 **G** 46, R5 **H** 68, R11
I 83, R39 **J** 143, R4 **K** 112, R268 **L** 295, R285

S-5 SET 8

1. 1, 3, 5, 7 **2. A** 12 **B** 15 **C** 18 **D** 7 **E** 12
F 17 **G** 10 **H** 6 **I** 10 **J** 100 **3. A** 4 **B** 1 **C** 0
D 6 **E** 3

S-6 SET 9

1. A 7 **B** 4 **C** 9 **D** 9 **E** 6 **F** 4 **G** 5 **H** 9 **I** 7
J 2 **K** 0 **L** 5 **2. A** $\{0, 1, 2, 3, 4\}$ **B** $\{10\}$
C $\{0, 1, 2, 3, 4, 5, 6\}$ **D** $\{8, 9, 10\}$
E $\{3, 4, 5, 6, 7, 8, 9, 10\}$ **F** $\{0, 1, 2, 3, 4, 5\}$ **G** $\{0, 1, 2\}$
H $\{5, 6, 7, 8, 9, 10\}$ **I** $\{0, 1, 2, 3, 4\}$
J $\{0, 1, 2, 3, 4, 5\}$ **K** $\{0, 1, 2, 3, 4, 5, 6\}$ **L** $\{0, 1, 2, 3, 4\}$

S-6 SET 10

1. A 6 **B** 4 **C** 49 **D** 5 **E** 14 **F** 36 **G** 23 **H** 5
I 3 **J** 6 **K** 1 **L** 3 **M** 0 **N** 7 **O** 2 **2. A** 6, 2
B 12, 6 **C** 12, 3 **D** 8, 4 **E** 10, 2 **F** 21, 3 **G** 20, 5
H 0, 0

S-7 SET 11

1. A 0, 1 **B** 1, 2 **C** 0, 0 **D** 5, 9 **E** 2, 7 **F** 8, 3
2. A 2 **B** 3 **C** 3 **D** 4 **E** 2 **F** 3 **3. A** 5 **B** 5
C 6 **D** 3 **E** 4 **F** 1 **G** 2 **H** 7 **I** 9

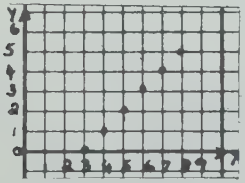
S-7 SET 12

1. 9 **2.** 6 **3.** 3 **4.** 86 **5.** 19 **6.** 29 **7. A** 23
B 10, 13

S-8 SET 13

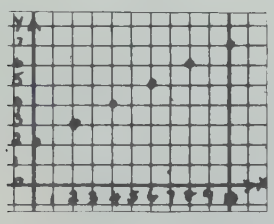
1. 4, 3, 2, 1, 0

2.

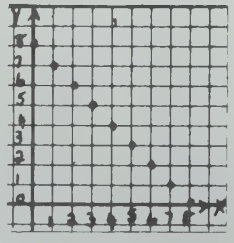


3. $2y$: 8, 10, 12, 14; x: 2, 4, 6, 8, 10

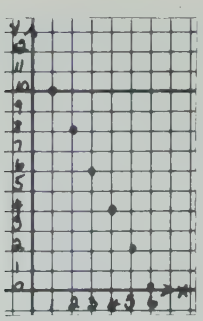
4.



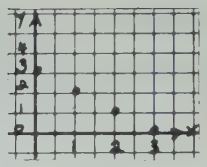
5. A



B



C



S-8 SET 14

1. A $2 \times 3 \times 5$ **B** $2 \times 2 \times 2 \times 7$ **C** $2 \times 2 \times 2 \times 3$
D $2 \times 3 \times 3$ **E** $2 \times 2 \times 2 \times 3 \times 3$ **F** $2 \times 2 \times 11$
G $3 \times 3 \times 3$ **H** $2 \times 3 \times 7$ **I** 3×17 **J** $3 \times 5 \times 7$
K 103 **L** $2 \times 5 \times 11$ **M** $5 \times 5 \times 5$ **N** $2 \times 2 \times 2 \times 17$
O $2 \times 3 \times 5 \times 5$ **P** 5×61 **Q** 3×163 **R** 29
2. A 3×5 **B** 2×3^2 **C** 3^3 **D** $2^2 \times 3^2$ **E** 2×3^3
F $2^2 \times 3 \times 5$ **G** 5^3 **H** $2^2 \times 5 \times 7$ **I** $2^4 \times 3 \times 7$
J $2^2 \times 3^2 \times 7$ **K** $2^2 \times 5^3$ **L** 2^9 **M** $2^2 \times 7 \times 11$
N $2^2 \times 5 \times 41$ **O** $2^6 \times 11$ **P** $2^2 \times 3^2 \times 17$ **Q** $2^3 \times 79$
R $2 \times 3^2 \times 11$ **S** $2 \times 3^4 \times 5$ **T** $2 \times 5^2 \times 7$
U $2 \times 3 \times 5^3$ **V** $2^2 \times 3^2 \times 5^2$ **W** $2^6 \times 3 \times 5$
X $2^2 \times 5 \times 7^2$

S-9 SET 15

1. A 6 **B** 1 **C** 5 **D** 6 **E** 5 **F** 12 **G** 7 **H** 6
I 20 **J** 7 **K** 14 **L** 10 **M** 4 **N** 25 **O** 24 **P** 30
Q 1 **R** 1 **S** 30 **T** 90 **2. A** 3 **B** 10 **C** 11 **D** 5
E 4 **F** 2 **G** 9 **H** 1 **I** 9 **J** 15 **K** 25 **L** 30 **M** 75
N 5 **O** 3 **P** 80 **Q** 10 **R** 25

S-9 SET 16

1. A 10 **B** 18 **C** 20 **D** 14 **E** 24 **F** 20 **G** 60
H 100 **I** 60 **J** 34 **K** 30 **L** 99 **M** 60 **N** 48
O 105 **P** 120 **Q** 130 **R** 90 **S** 48 **T** 180 **2. A** 30
B 30 **C** 60 **D** 60 **E** 30 **F** 12 **G** 28 **H** 36 **I** 30
J 120 **K** 70 **L** 30 **M** 24 **N** 60 **O** 48

S-10 SET 17

1. A −1 **B** −2 **C** −3 **D** 0 **E** −7 **F** −7 **G** 2
H 4 **I** −23 **J** −42 **K** 20 **L** 65 **M** −19 **N** −100
O 16 **P** 1 **Q** −198 **R** −48 **S** −212 **T** −935
2. A 3 **B** −5 **C** 9 **D** −3 **E** −9 **F** 9 **G** 11
H −2 **I** −3 **J** −97 **K** 60 **L** 0 **M** 0 **N** −60
O −120

S-10 SET 18

1. A −3 **B** −2 **C** 3 **D** 4 **E** 2 **F** −4 **G** 8
H −8 **I** −4 **J** 4 **K** −4 **L** −8 **M** −6 **N** 13
O −15 **P** 10 **Q** 20 **R** 2 **S** −15 **T** −9 **2. A** 9
B −8 **C** −19 **D** −21 **E** −10 **F** 3 **G** 5 **H** 10
I 45 **J** −35 **K** 3 **L** 17 **M** −3 **N** −32 **O** 53
P −65 **Q** 43 **R** 28 **S** 96 **T** −108 **U** −224 **V** 0
W 560 **X** −396

S-11 SET 19

1. A −20 **B** 24 **C** −60 **D** 48 **E** −150 **F** 72
G 0 **H** 53 **I** 90 **J** 0 **K** −280 **L** 120 **M** −43

N 1500 **O** 34 **P** −87 **Q** 87 **R** −870 **S** 870
T −870 **U** −1600 **V** 0 **W** 140 **X** −1232 **Y** 1350
2. A 27 **B** −27 **C** 27 **D** −27 **E** −27 **F** −20
G 90 **H** −800 **I** −180 **J** 144 **K** 400 **L** 400
M 400 **N** −400 **O** 400

S-11 SET 20

1. A 9 **B** −3 **C** −6 **D** −19 **E** 11 **F** −7 **G** 9
H 73 **I** 9 **J** 1 **K** −3 **L** −1 **M** −10 **N** 100 **O** 5
P 0 **2. A** −5 **B** 9 **C** 7 **D** −6 **E** −1 **F** −5
G −3 **H** 13 **I** 33 **J** 3 **K** −5 **L** 59 **M** 40 **N** 100
O −50 **P** −2 **Q** 3 **R** −2 **S** 2 **T** 6 **U** $-\frac{1}{2}$

S-12 SET 21

1. A 3 **B** 3 **C** 5 **D** 8 **E** 42 **F** 0 **G** 6 **H** 2
I 2 **J** 6 **K** 2 **L** 6 **2. A** 8 **B** 3 **C** 2 **D** 3 **E** 0
F 11 **G** 7 **H** 7 **I** 15 **J** 3, 7 **K** −36, 24 **L** −11, 29
3. A < **B** < **C** = **D** = **E** = **F** <

S-12 SET 22

1. A −4, −1 **B** −3, −1 **C** 16, 8 **2. A** −9, −3
B 20, 10 **C** −8, −2 **D** 3, 1 **E** 10, −4 **3. A** −5
B −9 **C** −39 **D** −78 **E** −10 **F** −2 **G** −3 **H** −9
I −9 **J** −5 **K** −9 **L** −6

S-13 SET 23

1. Many are possible.—Sample answers: **A** $\frac{2}{8}, \frac{4}{16}$
B $\frac{6}{8}, \frac{9}{12}$ **C** $\frac{2}{12}, \frac{3}{18}$ **D** $\frac{4}{6}, \frac{6}{9}$ **E** $\frac{6}{16}, \frac{9}{24}$ **F** $\frac{14}{20}, \frac{21}{30}$ **G** $\frac{4}{10}, \frac{6}{15}$
H $\frac{6}{4}, \frac{9}{6}$ **I** $\frac{10}{16}, \frac{15}{24}$ **J** $\frac{10}{12}, \frac{15}{18}$ **K** $\frac{14}{6}, \frac{21}{9}$ **L** $\frac{10}{24}, \frac{15}{36}$ **M** $\frac{20}{18}, \frac{30}{27}$
N $\frac{6}{40}, \frac{9}{60}$ **O** $\frac{11}{14}, \frac{12}{21}$ **P** $\frac{10}{18}, \frac{15}{27}$ **Q** $\frac{20}{22}, \frac{30}{33}$ **R** $\frac{3}{3}, \frac{6}{6}$ **S** $\frac{16}{10}, \frac{24}{15}$
T $\frac{2}{30}, \frac{3}{45}$ **U** $\frac{6}{32}, \frac{9}{48}$ **V** $\frac{2}{26}, \frac{3}{39}$ **W** $\frac{8}{42}, \frac{12}{63}$ **X** $\frac{8}{60}, \frac{12}{90}$ **2. A** $\frac{7}{8}$
B $\frac{5}{6}$ **C** 1 **D** $\frac{2}{7}$ **E** $\frac{3}{4}$ **F** $\frac{1}{6}$ **G** $\frac{9}{10}$ **H** $\frac{5}{8}$ **I** $\frac{7}{20}$ **J** $\frac{7}{9}$ **K** $\frac{12}{25}$
L $\frac{3}{7}$ **M** $\frac{2}{9}$ **N** $\frac{5}{7}$ **O** 3 **P** $\frac{4}{15}$ **Q** $\frac{2}{5}$ **R** $\frac{5}{6}$ **S** $\frac{12}{25}$ **T** $\frac{2}{15}$
U $\frac{4}{5}$ **V** $\frac{9}{11}$ **W** $\frac{3}{8}$ **X** $\frac{19}{30}$

S-13 SET 24

1. A no **B** yes **C** no **D** no **2. A** ≠ **B** ≠ **C** =
D = **E** = **F** = **G** ≠ **H** ≠ **I** = **J** = **K** ≠ **L** ≠
M ≠ **N** ≠ **O** ≠ **3. A** 12 **B** 24 **C** 21 **D** 3 **E** 4
F 42 **G** 81 **H** 80 **I** 39 **J** 125

S-14 SET 25

1. A $\frac{8}{9}$ **B** $\frac{5}{8}$ **C** $\frac{7}{8}$ **D** $1\frac{2}{3}$ **E** $1\frac{1}{8}$ **F** $1\frac{1}{3}$ **G** $-1\frac{5}{18}$ **H** $1\frac{1}{6}$
I $\frac{3}{20}$ **J** $-\frac{1}{12}$ **K** $-\frac{13}{14}$ **L** $\frac{17}{18}$ **M** $-\frac{1}{10}$ **N** $1\frac{15}{56}, -\frac{9}{20}$
2. A $39\frac{7}{16}$ **B** $45\frac{3}{4}$ **C** $80\frac{1}{2}$ **D** $23\frac{1}{3}$ **E** $101\frac{1}{10}$ **F** $427\frac{9}{10}$
G $87\frac{1}{12}$ **H** $24\frac{44}{45}$ **I** $54\frac{13}{100}$ **J** $65\frac{29}{30}$ **K** $70\frac{25}{36}$ **L** $357\frac{11}{20}$

S-14 SET 26

1. A $\frac{3}{8}$ **B** $\frac{5}{12}$ **C** $\frac{1}{6}$ **D** $\frac{3}{10}$ **E** 0 **F** $\frac{13}{20}$ **G** $\frac{7}{20}$ **H** $\frac{8}{21}$
I $\frac{4}{5}$ **J** 0 **K** $\frac{1}{9}$ **L** $\frac{31}{36}$ **M** $\frac{1}{6}$ **N** $\frac{14}{45}$ **O** $\frac{16}{21}$ **2. A** $2\frac{1}{6}$ **B** $3\frac{1}{6}$
C $7\frac{1}{4}$ **D** $7\frac{7}{12}$ **E** $56\frac{17}{40}$ **F** $41\frac{19}{20}$ **G** $43\frac{14}{15}$ **H** $29\frac{3}{8}$ **I** $98\frac{91}{100}$
J $383\frac{2}{3}$ **K** $411\frac{1}{6}$ **L** $585\frac{91}{100}$

S-15 SET 27

1. A $\frac{1}{20}$ **B** $\frac{3}{8}$ **C** $\frac{1}{10}$ **D** 1 **E** $\frac{2}{5}$ **F** $\frac{1}{3}$ **G** $\frac{3}{4}$ **H** 1 **I** $\frac{1}{2}$
J 30 **K** $2\frac{5}{8}$ **L** 0 **M** $\frac{11}{12}$ **N** $\frac{7}{40}$ **O** $\frac{4}{15}$ **2. A** $-\frac{1}{6}$ **B** $-\frac{1}{15}$
C $\frac{1}{21}$ **D** $-\frac{1}{9}$ **E** $-\frac{1}{2}$ **F** $\frac{1}{3}$ **G** $-\frac{4}{15}$ **H** $\frac{3}{20}$ **I** $-\frac{1}{12}$ **J** −21
K $\frac{1}{9}$ **L** $-\frac{1}{2}$ **M** $-2\frac{7}{10}$ **N** 0 **O** $-\frac{1}{9}$ **3. A** $\frac{5}{12}$ **B** $17\frac{1}{2}$
C $-1\frac{1}{5}$ **D** $-1\frac{5}{8}$ **E** $1\frac{2}{3}$ **F** $-12\frac{1}{2}$ **G** $1\frac{13}{36}$ **H** $2\frac{1}{2}$ **I** 15
J $-38\frac{1}{3}$ **K** −78 **L** −6 **M** $-27\frac{5}{8}$ **N** $-121\frac{1}{2}$ **O** $\frac{81}{100}$

S-15 SET 28

1. A $1\frac{1}{2}$ **B** $3\frac{1}{2}$ **C** $\frac{2}{3}$ **D** $1\frac{1}{3}$ **E** $\frac{5}{12}$ **F** $-1\frac{2}{3}$ **G** $6\frac{2}{3}$ **H** $\frac{6}{7}$
I $\frac{17}{32}$ **J** $-\frac{1}{12}$ **K** $2\frac{2}{15}$ **L** $-1\frac{7}{8}$ **M** $\frac{1}{16}$ **N** $-2\frac{2}{5}$ **O** $2\frac{13}{16}$
2. A 14 **B** $10\frac{2}{3}$ **C** $\frac{3}{10}$ **D** $\frac{1}{14}$ **E** $\frac{2}{3}$ **F** $2\frac{4}{5}$ **G** $-5\frac{1}{2}$
H −3 **I** −6 **J** −3 **K** $2\frac{2}{3}$ **L** $-\frac{1}{5}$ **M** $-\frac{1}{4}$ **N** $-\frac{9}{10}$
O −6

S-16 SET 29

1. 18 **2.** $7.50 **3.** $70+ **4.** $1\frac{5}{12}$ hr **5.** 75 km/h− **6.** 8−

S-16 SET 30

1. A 7.93 **B** 12.791 **C** 246.85 **D** 9.007 **E** 6.746
F 251.248 **G** 39.375 **L** 70.7105 **I** 862.943
J 62.0379 **K** 2.7409 **L** 410.563 **2. A** 2.58 **B** 6.36
C 10.9 **D** 0.277 **E** 29.35 **F** 4.081 **G** 8.42
H 12.665 **I** 25.34 **J** 4.57 **K** 5.818 **L** 281.32
M 21.28 **N** 434.63 **O** 651.67 **P** 4506.029 **3. A** 1.26
B 3.6 **C** −13.43 **D** −32.08 **E** −1.55 **F** 0.166
G 4.83 **H** −0.0524 **I** 14.95 **J** 84.55 **K** 4.95
L 43.704

S-17 SET 31

1. A 28.32 **B** 12.8 **C** 3.01 **D** 156.91 **E** 220.92
F 1.544 **G** 25.976 **H** 5.369 **I** 13.086 **J** 0.90611
K 171.038 **L** 1.36279 **M** 13.3668 **N** 195.038
O 33.8355 **P** 414.15136 **Q** 0.32812 **R** 3688.626
S 12.04924 **T** 0.418573 **2. A** −0.27 **B** −12.6
C −2.8 **D** 4.25 **E** −3.569 **F** 6.585 **G** −10.132
H −848.16 **I** 29.96856 **J** −48824 **K** −1767.633
L 3523.455 **M** −53.049651 **N** −0.07425 **O** 202.4891

S-17 SET 32

1. A 4×10^2 **B** 3×10^3 **C** 6.3×10^3 **D** 2×10^4
E 5.4×10^4 **F** 3.5×10^5 **G** 9×10^6 **H** 4.56×10^{10}
I 8.03×10^9 **J** 7.905×10^{12} **K** 5×10^{-1}
L 8.49×10^0 **M** 1.4×10^{-1} **N** 7.0×10^{-2}
O 3.0×10^{-3} **P** 7.02×10^{-2} **Q** 9.0×10^{-5}
R 4.53×10^{-4} **S** 8.0×10^{-7} **T** 8.002×10^{-4}
2. A 42 000 **B** 561 **C** 3 700 000 **D** 92.2 **E** 405 000
F 3 200 000 000 **G** 9.55 **H** 107 000 000 **I** 90 050
J 35 400 000 **K** 0.6 **L** 0.008 **M** 0.15 **N** 0.034
O 0.000691 **P** 0.00000007 **Q** 0.00000000044
R 0.0000039 **S** 0.000000801 **T** 0.00000000001392

S-18 SET 33

1. A 2.4 **B** 63.4 **C** 9.3 **D** 3.8 **E** 3.51 **F** 0.32
G 0.2535384 **H** 0.93 **I** 63.4 **J** 3.61 **K** 6.5 **L** 0.37
M 23.5 **N** 0.36 **O** 0.99 **P** 0.114 **2. A** 0.82 **B** 5.15
C 0.345 **D** 1.05 **E** 0.062 **F** 115.5 **G** 40 **H** 4
I 32.28 **J** 460 **K** 0.15 **L** 135 **M** 45 **N** 0.55
O 6.5 **P** 4.1 **Q** 0.73 **R** 0.53 **S** 3.4 **T** 7.5110963

S-18 SET 34

1. A 3.2×10^7 **B** 2.4×10^5 **C** 1.2×10^4
D 2.8×10^{-1} **E** 4.9×10^3 **F** 1 **G** 4.8×10^4
H 1.5×10^5 **I** 3.5×10^2 **J** 2.7×10^2 **K** 2.4×10^3
L 3×10^7 **M** 2.8 **N** 35 **O** 4.5×10^{-5} **2. A** 5
B 200 **C** 20 **D** 15 000 **E** 0.3 **F** 200 **G** 50
H 1200 **I** 0.5 **J** 20 **K** 2 **L** 0.3

S-19 SET 35

1. A 0.5 **B** 0.53 **C** 0.533 **2. A** 0.88 **B** 0.38
C 0.11 **D** 0.36 **E** 0.39 **F** 0.87 **3. A** 0.688 **B** 0.889

C 0.938 **D** 0.711 **E** 0.577 **F** 0.974

S-19 SET 36

1. A 24 **B** 45 **C** 20 **D** 9 **E** 27 **F** 42 **G** 6 **H** 15 **2. A** $1.29 **B** 30 **C** 184 **D** 91 **E** 160 km

S-20 SET 37

1. A 12 cm **B** 24 cm **C** 1.5 cm **D** 22.5 cm **2.** 279 km **3.** 18 cm **4.** 4 cm by 7.5 cm

S-20 SET 38

1. A 0.18 **B** 0.33 **C** 0.1 **D** 0.09 **E** 0.05 **F** 0.7 **G** 1.0 **H** 0.01 **I** 0.02 **J** 0.75 **K** 0.075 **L** 0.04 **M** 0.015 **N** 0.333 **2. A** 38% **B** 65% **C** 20% **D** 3% **E** 150% **F** 15% **G** 130% **H** 205% **I** 9.3% **J** 550% **K** 1630% **L** 17.8% **M** 1.7% **N** 30.9% **3. A** $\frac{37}{100}$ **B** $\frac{1}{2}$ **C** $\frac{2}{5}$ **D** $\frac{3}{10}$ **E** $\frac{3}{4}$ **F** $\frac{1}{20}$ **G** $\frac{1}{50}$ **H** $\frac{1}{100}$ **I** $\frac{3}{20}$ **J** $\frac{4}{25}$ **K** $\frac{7}{25}$ **L** $\frac{2}{25}$ **M** $5\frac{1}{5}$ **N** $\frac{14}{25}$ **4. A** 59% **B** 3% **C** 1% **D** 75% **E** 34% **F** 28% **G** 36% **H** 56% **I** 39% **J** 42% **K** 71% **L** 30% **M** 60% **N** 93%

S-21 SET 39

1. A $3.15 **B** $12.60 **2.** 80% **3. A** $169 **B** $481 **4. A** $9.30 **B** $164.30 **5.** $45 600 **6.** *A* by $1

S-21 SET 40

1. A 2.7 **B** 8.5 **C** 10 **D** 7.95 **E** 12.5 **F** 35 **G** 0.003 **H** 0.007 **I** 0.005 **J** 0.0035 **K** 0.0006 **L** 0.00125 **2. A** 1950 **B** 1700 **C** 2720 **D** 6300 **E** 2250 **F** 72 240 **G** 4.5 **H** 4.94 **I** 2.56 **J** 1.2 **K** 2.25 **L** 6.25 **3. A** 172 125 **B** 0.0075 m

S-22 SET 41

1. A 3 **B** 1.3 **C** 5000 **D** 15 000 **E** 4 **F** 0.74 **G** 4600 **H** 400 **I** 0.53 **J** 100 **K** 10 **L** 1 **M** 0.001 **N** 0.0001 **O** 3 **2. A** 9.8 **B** 15 **C** 8.2 **D** 10.8

S-22 SET 42

1. A 14.82 **B** 16.17 **C** $16\frac{3}{20}$ **D** 2086.5 **E** 27.0375 **2. A** 9.03 **B** 5.225 **C** 2161.08 **D** 2709.63 **E** 5.2768

S-23 SET 43

1. A 49 **B** 154 **C** $56\frac{7}{8}$ **D** 928.05 **2. A** 6975 **B** 20.8 **C** $18\frac{11}{40}$ **D** 43.26 **3. A** 10.66 **B** 75 **C** 175.5 **D** 1995

S-23 SET 44

1. A 65°48′56″ **B** 57°48′22″ **C** 99°11′49″ **D** 95°8″ **E** 130°42′1″ **F** 113°25′33″ **G** 90° **H** 183°51′10″ **2. A** 22°26′6″ **B** 27°23′47″ **C** 50°44′11″ **D** 27°20′47″ **E** 61°45′38″ **F** 24°54″ **G** 25°48′49″ **H** 46°7′13″

S-24 SET 45

1. A 10 **B** 25 **C** 17 **D** 13 **2. A** 24 **B** 20 **C** 12 **D** $7\frac{1}{2}$ **3. A** 25 **B** 26 **C** 30 **D** 35 **E** 12.7 **F** 20 **G** 30 **H** 3.3 **I** 3.2

S-24 SET 46

1. A 180 **B** 75.516 **C** 884.4 **D** 1 155 000 **E** 297.1 **2. A** 57.5 **B** 105 **C** 42.02 **D** 10.04

S-25 SET 47

1. A 11.66 **B** 441 **C** 3725 **D** 8450 **2. A** 28.7 **B** 700 **C** 1916.7 **D** 13

S-25 SET 48

1. A 3 **B** 8 **C** 32 **D** 7 **E** 39 **F** 2.1 **G** 3 **H** 0.45 **2. A** 4 **B** 12 **C** 9 **D** 5 **E** 48 **F** 30 **G** 3.9 **H** 4

S-26 SET 49

1. A *E* **B** *e*/*d* **C** *EDF* **2. A** $\overline{RS}/\overline{RQ} = \overline{TU}/\overline{UV}$ **B** $\angle P \cong \angle V$ **C** $\triangle SRQ \cong \triangle TUV$ **3. A** $\frac{2}{5}$ **B** $\frac{2}{5}$ **4. A** 3 **B** 4

S-26 SET 50

1. A 5.84 **B** 9.88 **2. A** 14.74 **B** 16.28 **3.** $t = 19.05$; $x = 25$ **4. A** 0.53 **B** 0.47 **C** 0.88

S-27 SET 51

1. A 15.7 **B** 10.04 **C** 43.96 **D** 72.2 **E** 973.4 **F** 5.02 **G** 54.3 **H** 2.8 **2. A** 25.1 **B** 21.98 **C** 138.2 **D** 2763 **E** 329.7 **F** 52.1 **G** 2.20 **H** 38.62

S-27 SET 52

1. A 113.04 **B** 176.6 **C** 1256 **D** 452.2 **E** 7.07 **F** 30.2 **G** 32.2 **H** 0.50 **2. A** 379.94 **B** 632.06 **3. A** 520.96 **B** 298.98

S-28 SET 53

1. A 863.5 **B** 1161.8 **C** 368.95 **D** 6280 **E** 6782.4 **F** 156.9 **G** 23.25 **2. A** 150.7 **B** 942 **C** 211.95 **D** 196.25 **E** 100 480 **F** 4.57 **G** 1.21

S-28 SET 54

1. A $\frac{1}{6}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **2. A** $\frac{4}{9}$ **B** $\frac{5}{9}$ **C** $\frac{1}{3}$ **D** $\frac{2}{9}$ **3. A** $\frac{1}{6}$ **B** $\frac{5}{6}$ **C** $\frac{1}{2}$ **D** $\frac{1}{6}$ **E** $\frac{2}{3}$ **F** $\frac{2}{3}$ **G** $\frac{1}{3}$ **H** $\frac{1}{2}$ **I** 0

S-29 SET 55

1. A 24 **B** 720 **C** 5040 **D** 48 **E** 720 **F** 120 **2.** 24 **3.** 120 **4.** 40 320 **5.** 362 880 **6. A** 7! **B** 10 **C** 7·6 **D** 4! **E** 4! **F** 7! **G** 9! **H** 6·5·4 **I** 6

S-29 SET 56

1. A 6 **B** 6 **C** 36 **2.** 24 **3.** 15 **4.** 56 **5. A** 6 **B** 20 **C** 21 **D** 56 **E** 35 **F** 84

S-30 SET 57

1. A $\frac{1}{3}$ **B** $\frac{1}{2}$ **C** $\varnothing$, 0 **D** $\{1, 2, 3, 5, 6\}$, $\frac{5}{6}$ **2. A** $\frac{3}{8}$ **B** $\frac{1}{2}$ **C** $\frac{1}{4}$ **D** $\frac{5}{8}$ **3. A** $\frac{2}{5}$ **B** $\frac{3}{5}$ **C** 1

S-30 SET 58

1. A 27 **B** 67 **C** 241 **D** 120 **E** 218 **F** 4539 **2. A** 51 **B** 313 **C** 140 **D** 697 **E** 1291 **F** 88 **G** 55 **H** 53

S-31 SET 59

1. A 49 **B** 86 **C** 125 **D** 40 **E** 1350 **F** 10 **G** 19 **H** 313.5 **I** 1555 **J** 76 **2.** 75 **3.** 125 **4. A** 56.05 **B** 55.7 **C** mean

S-31 SET 60

1. **2.** 2–3 **3.** 16–20

S-32 SET 61

1. A 110 **B** 27.3% **C** 63.6% **2. A** 27°C **B** 22°C **C** 4 **3. A** $150 **B** $90 **4. A** 30% **B** 15%

S-32 SET 62

1. A 13 **B** 23 **C** 21 **D** 16 **E** 34 **F** 71 **G** 54 **H** 47 **I** 56 **J** 86 **K** 83 **L** 94 **M** 9.9 **N** 8.8 **O** 8.4 **P** 6.1 **Q** 6.8 **R** 3.8 **S** 0.5708765 **T** 0.43 **2. A** 3.7 **B** 4.8 **C** 8.3 **D** 12.2 **E** 20.6 **F** 22.0 **G** 22.4 **H** 95.4 **I** 54.8 **J** 3.5 **K** 5.1 **L** 6.4 **M** 7.2 **N** 0.7 **O** 0.3

DATE DUE SLIP

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F255

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